MATH 115, SUMMER 2012
WORKSHEET FOR LECTURE 20

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"Form" on this worksheet always means integral binary quadratic form.

(1) Let $f(x, y) = ax^2 + bxy + cy^2$ $(a, b, c \in \mathbb{Z})$. Let g be the gcd of a, b, c. Let S be the set of integers represented by f. Prove that only multiples of g can be in S[1]. Is S an ideal? If yes, say why. If no, provide a counterexample by writing down some particular form.

(2) Let f be the form $f(x, y) = x^2 + 5y^2$. Find all positive integers less than 200 that can be represented by f[2]. Which of them can be properly represented? Which primes are represented? Can you make a guess in general as to which primes can be represented by this form[3]? We'll be able to prove the answer to this in a few days!

It may help you to use the following list of primes < 200:

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199$

(3) In this problem, the vector symbol $\mathbf{x}$ always means the column vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, with $x, y \in \mathbb{Z}$.

(a) If A is any real $n \times n$ matrix, show that there is a unique symmetric matrix B and skew-symmetric[4] such that $A = B + C$.

(b) If $f(x, y)$ is a form given by

$$f(x, y) = \mathbf{x}^T A\mathbf{x},$$

for some 2×2 matrix A, and if $f(x, y) = 0$ for all $(x, y) \in \mathbb{Z}^2$, show that A is skew-symmetric.

(c) Use parts (a) and (b) to show that any form $f(x, y)$ can be written in terms of a matrix:

$$f(x, y) = \mathbf{x}^T A\mathbf{x},$$

where A is *symmetric*.

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[1] This *doesn't* say, however, that every multiple of g is in S.

[2] Hint: draw a picture of the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. Next to each lattice point, write the value of f at that point. This may speed up your computations.

[3] Hint: how do they relate to multiples of 20?

[4] B is symmetric means $B = B^T$, the transpose of B. C is skew-symmetric means $C = -C^T$.