MATH 115, SUMMER 2012
HOMEWORK 6
DUE TUESDAY, AUGUST 7TH

JAMES MCIVOR

(1) Show that the equation $12x^2 + 13y^2 = 227$ has no solutions in integers.

(2) Show that the curve $17x^2 + 24y^2 = 221z^2$ contains no rational points besides $(0, 0, 0)$.

(3) Consider the projective plane curve given by the equation $4x^2 + y^2 = z^2$. Draw pictures of this curve in the three subsets U_0, U_1, U_2 of $\mathbb{P}^2$ where x, y, and z, respectively, are nonzero.

(4) Consider the curve C given in the xy-plane by the equation $f(x, y) = y - x^2 = 0$. Let L be the line in the xy-plane $x = 0$. The line L intersects C in only one point in the xy-plane, and it is not tangent to C, either.

In this problem, we show that L and C actually meet in two points if we regard both in the projective plane. So now let C be the *projective* plane curve given by the homogeneous equation $yz - x^2 = 0$, and let L as before be defined by the equation $x = 0$. By considering separately the cases $x = 1$, $y = 1$, and $z = 1$, find the two intersection points of C and L in $\mathbb{P}^2$.

(5) Prove the assertion made in class that $\mathbb{P}^2$ can be written as

$$\mathbb{P}^2 = \mathbb{R}^2 \cup \mathbb{R} \cup \{\text{point}\}.$$

Be sure to specify which point it is on the right hand side (it may depend on your choice of decomposition into $\mathbb{R}^2$ and $\mathbb{R}$).

[**Hint**: First consider those points $[x : y : z]$ where $z \neq 0$. Show that this set of points is in bijection wih $\mathbb{R}^2$. Now consider the set of points $[x : y : 0]$.]

(6) For this problem, we need the following definition:

Definition 1. If C is a projective plane curve given by a homogeneous polynomial equation $F(x, y, z) = 0$, then we say that the point $[a : b : c]$ is a **singular point** of C if $F(a, b, c) = 0$ (i.e., the point lies on C) and also all partial derivatives of F vanish at $[a : b : c]$, i.e.,

$$\frac{\partial F}{\partial x}(a, b, c) = \frac{\partial F}{\partial y}(a, b, c) = \frac{\partial F}{\partial z}(a, b, c) = 0$$

If C has no singular points, then we say C is **nonsingular**

Now consider the curve C given by the equation

$$F(x, y, z) = y^2 z - x^3 + ax^2 z + bz^3.$$

Show that C is nonsingular if $\Delta \neq 0$, where

$$\Delta = 16(4a^3 - 27b^2).$$

[Hint: Assume $P = [x : y : z]$ is a singular point of C, and show that this forces $\Delta = 0$.]