MATH 115, SUMMER 2012
HOMEWORK 5
DUE TUESDAY, JULY 31ST

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(1) (NZM 3.5.1) Find a reduced form equivalent to $7x^2 + 25xy + 23y^2$.
(2) (NZM 3.5.4) Show that a binary quadratic form f properly represents an integer n if and only if there is a form equivalent to f in which the coefficient of x^2 is n.
(3) Find all reduced positive definite primitive forms of discriminant -7.
(4) Find all reduced positive definite primitive forms of discriminant -8.
(5) Find all reduced positive definite primitive forms of discriminant -27.
(6) Determine which prime numbers are represented by the form $2x^2 + 3y^2$.
(7) Determine which prime numbers are represented by the form $x^2 + 7y^2$.
(8) Determine which prime numbers are represented by the form $x^2 + 8y^2$.
(9) Prove that if $a = 0$, the form $ax^2 + bxy + cy^2$ is not definite.
(10) Prove that if $f(x, y) = ax^2 + bxy + cy^2$ is a reduced positive definite form, then the smallest positive integer represented by f is a.

[Hint: Suppose f represents k, where $0 < k < a$, and consider separately the four cases:
 (a) $x = 0$
 (b) $y = 0$
 (c) x, y both nonzero but $|x| \leq |y|$.
 (d) x, y both nonzero but $|y| \leq |x|$.
]
(11) (NZM 5.2.2) For what integers a, b, c does the system

$$x_1 + 2x_2 + 3x_3 + 4x_4 = a$$
$$x_1 + 4x_2 + 9x_3 + 16x_4 = b$$
$$x_1 + 8x_2 + 27x_3 + 64x_4 = c$$

have a solution in integers? What are the solutions if $a = b = c = 1$?
(12) (NZM 5.3.2) Prove that if x, y, z is a Pythagorean triple then at least one of x, y is divisible by 3 and at least one of x, y, z is divisible by 5.
(13) (NZM 5.3.12) Show that if x, y satisfy $x^4 - 2y^2 = 1$, then $x = \pm 1$, $y = 0$. [Hint: Imitate the proof of the Pythagorean Triples Theorem]

More Hints:
 (a) Write the equation as $2y^2 = x^4 - 1 = (x^2 + 1)(x^2 - 1)$
 (b) Show that the gcd of $x^2 - 1$ and $x^2 + 1$ is 2. You may want to consider $x^2 + 1$ mod 4 to see this
 (c) Show that $\frac{x^2+1}{2}$ and $x^2 - 1$ are both square integers.
 (d) Thus $x^2 - 1 = r^2$ for some r. Use this to figure out what x could be, then figure out y.