**Math 115, Summer 2012**
**Homework 2**
**Due Tuesday, July 3rd**

*NZM a.b.c refers to a problem in our text, 5th edition - these may differ slightly from the problems appearing in other editions, so use the version printed here to be safe).*

(1) (NZM 2.1.5) Write a single congruence that is equivalent to the pair of congruences $x \equiv 1 \mod 4$ and $x \equiv 2 \mod 3$.
(2) (NZM 2.1.22) Prove that $n^{6k} - 1$ is divisible by 7 if $(n, 7) = 1$. Here $k$ is any positive integer.
(3) Find an integer $a$ such that $\{a, a^2, a^3, a^4\}$ is a reduced residue system mod 5.
(4) Find the smallest positive integer $x$ which is congruent to $32^{412}$ mod 7.
(5) (NZM 2.1.33) Show that $\{1^2, 2^2, 3^2, \ldots, m^2\}$ is not a complete residue system mod $m$ if $m > 2$.
(6) (NZM 2.1.35) If $n$ is a composite positive integer, show that $(n-1)! + 1$ is not a power of $n$.
(7) (NZM 2.1.40) If $m$ is odd, show that the sum of all the elements of $\mathbb{Z}/m$ is equal to zero.
(8) (NZM 2.1.48) If $r_1, \ldots, r_p$ and $r'_1, \ldots, r'_p$ are any two complete residue systems mod $p$, where $p$ is a prime greater than 2, show that the set $\{r_1r'_1, r_2r'_2, \ldots, r_pr'_p\}$ is not a complete residue system mod $p$.
(9) (NZM 2.2.5d,e) Find all solutions of the congruences $57x \equiv 87 \mod 105$ and $64x \equiv 83 \mod 105$.
(10) (NZM 2.3.2) Find all integers $x$ that satisfy all three congruences

$$\begin{aligned} x &\equiv 1 \mod 3 \\ x &\equiv 1 \mod 5 \\ x &\equiv 1 \mod 7. \end{aligned}$$

(11) (NZM 2.3.7) Determine whether the congruences $5x \equiv 1 \mod 6$ and $4x \equiv 13 \mod 15$ have a common solution. Find the solutions, if any exist.
(12) (NZM 2.3.19) Let $m_1, \ldots, m_r$ be relatively prime in pairs. Assuming that each of the congruences $b_i x \equiv a_i \mod m_i$ has a solution, prove that the congruences have a simultaneous solution (i.e., one $x$ that satisfies all congruences at once).
(13) (NZM 2.3.37) Let $a_1 = 3$, $a_{i+1} = 3^{a_i}$. Describe this sequence mod 100.
(14) Which of the following are ring homomorphisms?
   (a) $f\colon \mathbb{Z} \to \mathbb{Z}/2$ given by $f(n) = 0$ if $n$ is even and 1 if $n$ is odd.
   (b) $g\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = nx$, where $n$ is some fixed integer.
   (c) $E_3\colon \mathbb{Z}[x] \to \mathbb{Z}$ which sends a polynomial $f(x) \in \mathbb{Z}[x]$ to the integer $f(3)$.
   (d) Let $M_2(\mathbb{Z})$ be the set of all $2 \times 2$ matrices with integer entries. $h\colon M_2(\mathbb{Z})$ is the trace map, which sends a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $a + d$.
(15) Let $R = \mathbb{Z}[\sqrt{5}]$ be the ring consisting of elements of the form $a + b\sqrt{5}$, where $a$ and $b$ are integers. Let $S$ be the ring consisting of matrices of the form $\begin{pmatrix} a & b \\ 5b & a \end{pmatrix}$. Prove that $R$ is isomorphic to $S$.
(16) Find all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}/12$.