Math 115, Summer 2012
Homework 1
Due Tuesday, June 26th

NZM a.b.c refers to a problem in our text, 5th edition - these may differ slightly from the problems appearing in other editions, so use the version printed here to be safe).

(1) Calculate the gcd of 312 and 45.

(2) Find a pair of integers x, y such that $221x + 77y = 1$

(3) Find a pair of integers x, y such that $46x + 32y = -4$

(4) Explain why there are no integers x, y satisfying $198x + 780y = 9$.

(5) (NZM 1.2.6) Prove that the product of three consecutive integers is divisible by 6, and that the product of four consecutive integers is divisible by 24

(6) (NZM 1.2.11) Prove that $4 \nmid (n^2 + 2)$ for any n.

(7) (NZM 1.2.15) Prove that if x and y are odd, then $x^2 + y^2$ is even but not divisible by 4.

(8) (NZM 1.2.24) Prove that no integers exist satisfying $x + y = 100$ and $(x, y) = 3$.

(9) (NZM 1.2.29) Let g and l be two positive integers, prove that the equations $(x, y) = g$, $[x, y] = l$ have a solution if and only if $g | l$.

(10) (NZM 1.2.52) Suppose that $2^n + 1 = xy$, where $x, y > 1$ are integers and $n > 0$. Show that for any positive integer a, $2^a | (x - 1)$ if and only if $2^a | (y - 1)$.

(11) (NZM 1.3.6) Show that every positive integer n has a unique expression of the form $n = 2^r m$, with $r \geq 0$ and m an odd positive integer.

(12) (NZM 1.3.10) Prove that every positive integer of the form $3k + 2$ has a prime factor of the form $3m + 2$ for some m.

(13) (NZM 1.3.14) If a and b are integers, p is a prime, and $(a, p^2) = p$, $(b, p^3) = p^2$, compute (ab, p^4) and $(a + b, p^4)$.

(14) (NZM 1.3.18) If $(a, b) = c$, prove that $(a^2, b^2) = c^2$.

(15) (NZM 1.3.24) Prove that for any composite positive integer n, at least one of its prime factors must be $\leq \sqrt{n}$.

(16) (NZM 1.3.36) Let $S = \{1, 2, \ldots, n\}$. Let 2^k be the largest power of 2 occurring in S. Prove that 2^k does not divide any other number in S, and then use this to prove that

$$\sum_{i=1}^{n} \frac{1}{i}$$

is not an integer, for any $n > 1$.

(17) (NZM 1.3.42) If, for some integer $n > 0$, $2^n + 1$ is prime, prove that n is actually a power of 2 (in which case we call $2^n + 1$ a "Fermat prime").

(18) Let R be the ring[1] $\mathbb{Z}[\sqrt{-11}]$, which consists of numbers of the form $a + b\sqrt{-11}$, where a and b are integers. Exhibit two distinct nontrivial factorizations (meaning neither of the factors is ± 1) of 15 in this ring. Thus this ring does not have unique factorization.

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[1]You don't need to know what a ring is to do this problem.