
These problems are practice for the second exam, on rings and fields.

1. True or False
   
   (a) The canonical homomorphism \( \pi : R \to R/I \) is surjective.
   (b) Every homomorphism of rings is injective.
   (c) The element \( x \) is a unit in \( \mathbb{Q}[x]/(x^4 + 1) \).
   (d) There exists a homomorphism \( \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \).
   (e) If \( R \) is a unique factorization domain and \( I \) a proper ideal of \( R \), then \( R/I \) is a unique factorization domain.
   (f) If \( \sigma \in \text{Gal}(L : K) \), and \( \alpha \in L \) is a root of \( f \in K[x] \), then \( \sigma(\alpha) \) is a root of \( f \).
   (g) If \( R \) and \( S \) are domains, then \( R \times S \) is a domain.
   (h) Every algebraic field extension is finite.
   (i) \( \mathbb{Q}(i - \sqrt{7}) = \mathbb{Q}(i, \sqrt{-7} + 1) \).
   (j) The minimal polynomial of the extension \( \mathbb{Q} \subset \mathbb{Q}(e^{2\pi i/3}) \) is \( x^3 - 1 \).
   (k) If \( F \) is any field, there exists a homomorphism \( F \to \mathbb{C} \).
   (l) If \( K \subset L \) is a normal field extension of degree 4, then there exists exactly one intermediate subfield \( F \neq K, L \).
   (m) The polynomial \( 3x^4 - 30x^2 + 10x + 15 \) is irreducible over \( \mathbb{Z} \).
   (n) If \( f : R \to S \) is a surjective ring homomorphism, and \( m \) a maximal ideal in \( S \), then \( f^{-1}(m) \) is a maximal ideal in \( R \).
   (o) There exists a homomorphism \( \mathbb{Q}[x]/(x^2 + 2x + 1) \to \mathbb{C} \).

2. (a) If \( R \) is a ring, say what it means for an element \( r \in R \) to be irreducible.
   (b) Give an example of an irreducible polynomial of degree larger than 2 in the ring \( \mathbb{Q}[x] \).
   (c) Let \( R \) be a domain and \( I = (f) \) a nonzero ideal. Prove that if \( I \) is prime, then \( f \) is irreducible.

3. (a) Let \( \alpha = \sqrt[3]{\sqrt{2} + \sqrt{3}} \), and consider the field extensions

   \[
   \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2} + \sqrt{3}) \subset \mathbb{Q}(\alpha) \subset \mathbb{Q}(i, \alpha) = K
   \]

   Given that \([K : \mathbb{Q}] = 24\), determine \([\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{2} + \sqrt{3})]\). Justify your answer.

   (b) Let \( \omega = \frac{-1 + i\sqrt{3}}{2} \), a cube root of 1. Consider the extensions \( \mathbb{Q} \subset \mathbb{Q}(\omega) \subset \mathbb{Q}(\sqrt[3]{7}, \omega) \). Let \( f \) and \( g \) be the automorphisms of \( \mathbb{Q}(\sqrt[3]{7}, \omega) \) defined by

   \[
   f: \begin{cases} 
   \sqrt[3]{7} \mapsto \omega \sqrt[3]{7} \\ 
   \omega \mapsto \omega 
   \end{cases}
   g: \begin{cases} 
   \sqrt[3]{7} \mapsto \sqrt[3]{7} \\ 
   \omega \mapsto \omega^2 
   \end{cases}
   
   Show that \( f \in \text{Gal}(\mathbb{Q}(\sqrt[3]{7}, \omega) : \mathbb{Q}(\omega)) \).

   (c) Find an element \( x \in \mathbb{Q}(\sqrt[3]{7}, \omega) \) such that \( f(g(x)) \neq g(f(x)) \).
(d) Using (d), and given that $[Q(\sqrt[3]{7}, \omega) : Q] = 6$, prove that $\text{Gal}(Q(\sqrt[3]{7}, \omega) : Q) \cong S_3$.

(e) Prove that $\text{Gal}(Q(\sqrt[3]{7}, \omega) : Q(\omega)) \cong \mathbb{Z}/3\mathbb{Z}$.

4. (a) For each of the following rings say, whether they are a field; domain; principal ideal domain; euclidean domain; unique factorization domain

i. $\mathbb{Z}[x]$

ii. $\mathbb{Q}[x]/(x^2 + x + 1)$

iii. $\mathbb{C}[x, y]$

(b) Define a principal ideal domain.

(c) Prove that if $R$ is a principal ideal domain and $I$ a prime ideal of $R$, then $R/I$ is a principal ideal domain.

(d) Let $f : \mathbb{Z}[x] \to \mathbb{Z}/2\mathbb{Z}$ be $f = g \circ h$, where $h : \mathbb{Z}[x] \to \mathbb{Z}$ is the evaluation map at $-1$ and $g : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the quotient map. Prove that $\ker f = (x + 1, x^2 + 1)$.

5. (a) State the (first) isomorphism theorem for rings.

(b) Consider the map $\phi : \mathbb{C}[x, y] \to \mathbb{C}[y]$ given by $\phi(p(x, y)) = p(y^2, y^3)$. Compute $\phi(x^2 + xy + y^2)$.

(c) Prove that $\text{im } \phi = \mathbb{C}[y^2, y^3] \subset \mathbb{C}[y]$.

(d) Prove that $\ker \phi$ is a prime ideal in $\mathbb{C}[x, y]$.

(e) Is $\text{im } \phi$ a unique factorization domain?