MATH 110, SUMMER 2013
MIDTERM EXAM 2 SOLUTION
THURSDAY, AUGUST 1ST

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(1) (a) Let $T\colon V \to V$ be a linear map such that $T^3 = I$. Prove that T is an isomorphism.

(b) If $S\colon V \to V$ is a linear map such that $S^3 = S$, can we conclude that S is an isomorphism? Explain why or why not.

Solution:

(a) T^2 is the inverse to T, since $T^2 \circ T = T^3 = I$.
(b) No, we cannot say whether S is an isomorphism. For example, S could be the zero map, which is not an isomorphism, or S could be the identity map, which is an isomorphism.

(2) (a) Verify that x^2+1 is an eigenvector of the operator $T \in \mathcal{L}(P_2(\mathbb{R}))$ given by $Tp(x) = p(1)x^2+p''(x)$.
(b) Find the eigenvalue(s) and eigenvectors, if any, of the operator $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ on $\mathbb{R}^2$.

Solution:

(a) $T(x^2+1) = 2x^2+2 = 2(x^2+1)$, so x^2+1 is an eigenvector with eigenvalue 2.
(b) The matrix is upper triangular, so the eigenvalues are on the diagonal. Thus 2 is the only eigenvalue. We now solve $\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$, in other words, the system of equations

$$\begin{aligned} 2x + 4y &= 2x \\ 2y &= 2y, \end{aligned}$$

which gives $y = 0$ (from the first equation), and x arbitrary. Thus the eigenspace for the only eigenvalue, 2, is the x-axis.

(3) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(a) Prove that the subspace $W = \left\{ \begin{pmatrix} x \\ -x \\ x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$ is T-invariant.

(b) Find a two-dimensional subspace of $\mathbb{R}^3$ that is invariant under T.

Solution:

(a) We pick any $\begin{pmatrix} x \\ -x \\ x \end{pmatrix} \in W$ and compute

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ -x \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is still in W because W is a subspace.

(b) From (a), it looks like W is the nullspace of T (our argument above actually just showed that W is contained in the nullspace). So we guess that the range of T is 2-D, and we also know that the range is always invariant. So let's calculate the range:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ x+2y+z \\ x+y \end{pmatrix} = (x+y)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (y+z)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

so $\operatorname{Range} T$ is the span of $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and hence is 2-D. It's invariant since the range of any operator is invariant.

(4) Let V be a real inner product space and u, v two vectors in V. Prove that if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, then u and v are orthogonal. Make sure to point out where we used the fact that V is over $\mathbb{R}$. (this is the *converse* to the pythagorean theorem proved in class)

Solution:

Assume that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$. Then since $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$, we get

$$\langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle = \langle u,u\rangle + \langle v,v\rangle,$$

which means

$$\langle u,v\rangle + \langle v,u\rangle = 0,$$

and since $\langle u,v\rangle = \langle v,u\rangle$ (as we work over $\mathbb{R}$), this means $2\langle u,v\rangle = 0$, so $\langle u,v\rangle = 0$. i.e., u and v are orthogonal.

(5) Let V be an inner product space, and $T\colon V \to V$ be an operator which has at least one eigenvalue, and for which $\langle Tv, v\rangle = 0$ for every vector $v \in V$. Prove that T is not an isomorphism.

Solution: We will show that T is not injective. Then it cannot be an isomorphism. To show T is not injective, we show its nullspace is nonzero. But the nullspace is the same as the eigenspace for 0, so if we can show that 0 is an eigenvalue, then we know it has a nonzero eigenvector with eigenvalue zero, and hence a nonzero vector in its nullspace.

We know that T has at least one eigenvalue, call it λ. Then we may pick a nonzero eigenvector v for λ, so $Tv = \lambda v$. Then for this v, by our assumption, $0 = \langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v,v\rangle$. So $0 = \lambda\langle v,v\rangle$. Since v is nonzero, $\langle v,v\rangle \neq 0$, so this means $\lambda = 0$. Thus the *only* possible eigenvalue of T is 0, and we're done.