(1) (a) Let $T : V \rightarrow V$ be a linear map such that $T^3 = I$. Prove that $T$ is an isomorphism.

(b) If $S : V \rightarrow V$ is a linear map such that $S^3 = S$, can we conclude that $S$ is an isomorphism? Explain why or why not.

Solution:

(a) $T^2$ is the inverse to $T$, since $T^2 \circ T = T^3 = I$.

(b) No, we cannot say whether $S$ is an isomorphism. For example, $S$ could be the zero map, which is not an isomorphism, or $S$ could be the identity map, which is an isomorphism.

(2) (a) Verify that $x^2 + 1$ is an eigenvector of the operator $T \in \mathcal{L}(P_2(\mathbb{R}))$ given by $T p(x) = p(1)x^2 + p''(x)$.

(b) Find the eigenvalue(s) and eigenvectors, if any, of the operator $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on $\mathbb{R}^2$.

Solution:

(a) $T(x^2 + 1) = 2x^2 + 2 = 2(x^2 + 1)$, so $x^2 + 1$ is an eigenvector with eigenvalue 2.

(b) The matrix is upper triangular, so the eigenvalues are on the diagonal. Thus 2 is the only eigenvalue. We now solve $\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$, in other words, the system of equations

\[ \begin{align*}
2x + 4y &= 2x \\
2y &= 2y,
\end{align*} \]

which gives $y = 0$ (from the first equation), and $x$ arbitrary. Thus the eigenspace for the only eigenvalue, 2, is the $x$-axis.

(3) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(a) Prove that the subspace $W = \left\{ \begin{pmatrix} x \\ -x \\ x \end{pmatrix} \mid x \in \mathbb{R} \right\}$ is $T$-invariant.

(b) Find a two-dimensional subspace of $\mathbb{R}^3$ that is invariant under $T$.

Solution:

(a) We pick any $\begin{pmatrix} x \\ -x \\ x \end{pmatrix} \in W$ and compute

\[ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ -x \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \]
which is still in \(W\) because \(W\) is a subspace.

(b) From (a), it looks like \(W\) is the nullspace of \(T\) (our argument above actually just showed that \(W\) is contained in the nullspace). So we guess that the range of \(T\) is 2-D, and we also know that the range is always invariant. So let’s calculate the range:

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
y + z \\
x + 2y + z \\
x + y
\end{pmatrix}
= (x + y)
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
+ (y + z)
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix},
\]

so \(\text{Range} \ T\) is the span of \(\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}\) and \(\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}\), and hence is 2-D. It’s invariant since the range of any operator is invariant.

(4) Let \(V\) be a real inner product space and \(u, v\) two vectors in \(V\). Prove that if \(\|u + v\|^2 = \|u\|^2 + \|v\|^2\), then \(u\) and \(v\) are orthogonal. Make sure to point out where we used the fact that \(V\) is over \(\mathbb{R}\). (this is the converse to the pythagorean theorem proved in class)

**Solution:**

Assume that \(\|u + v\|^2 = \|u\|^2 + \|v\|^2\). Then since \(\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle\), we get

\[
\langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle = \langle u, u \rangle + \langle v, v \rangle,
\]

which means

\[
\langle u, v \rangle + \langle v, u \rangle = 0,
\]

and since \(\langle u, v \rangle = \langle v, u \rangle\) (as we work over \(\mathbb{R}\)), this means \(2\langle u, v \rangle = 0\), so \(\langle u, v \rangle = 0\). i.e., \(u\) and \(v\) are orthogonal.

(5) Let \(V\) be an inner product space, and \(T: V \to V\) be an operator which has at least one eigenvalue, and for which \(\langle Tv, v \rangle = 0\) for every vector \(v \in V\). Prove that \(T\) is not an isomorphism.

**Solution:** We will show that \(T\) is not injective. Then it cannot be an isomorphism. To show \(T\) is not injective, we show its nullspace is nonzero. But the nullspace is the same as the eigenspace for 0, so if we can show that 0 is an eigenvalue, then we know it has a nonzero eigenvector with eigenvalue zero, and hence a nonzero vector in its nullspace.

We know that \(T\) has at least one eigenvalue, call it \(\lambda\). Then we may pick a nonzero eigenvector \(v\) for \(\lambda\), so \(Tv = \lambda v\). Then for this \(v\), by our assumption, \(0 = \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle\). So \(0 = \lambda \langle v, v \rangle\). Since \(v\) is nonzero, \(\langle v, v \rangle \neq 0\), so this means \(\lambda = 0\). Thus the only possible eigenvalue of \(T\) is 0, and we’re done.