MATH 110, SUMMER 2013
MIDTERM EXAM 1 SOLUTION

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(1) (a) Is $U_1 = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \,\middle|\, z = \overline{w} \right\}$ a subspace of the complex vector space $\mathbb{C}^2$? Explain briefly why or why not.

Solution: U_1 is not a subspace because it is not closed under (complex) scalar multiplication. For example, the vector $\begin{pmatrix} -i \\ i \end{pmatrix}$ is in U_1, but $i \begin{pmatrix} -i \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is not.

(b) Is the function $T \colon P_2(\mathbb{F}) \to P_4(\mathbb{F})$ given by $Tp(x) = x^2 p(x)$ a linear map? Explain why or why not.

Solution: T is linear. It is homogeneous since $T(cp)(x) = x^2(cp(x)) = cx^2 p(x) = cTp(x)$, and additive since $T(p+q)(x) = x^2(p+q)(x) = x^2 p(x) + x^2 q(x) = Tp(x) + Tq(x)$.

(2) Prove that $(1 - x, x - x^2, 2x^2)$ is a basis for $P_2(\mathbb{F})$.

Proof: First off, we know the dimension of $P_2(\mathbb{F})$ is three, so since this list has length 3, it is sufficient to show either that it spans or it is independent. I'll show both anyway, though. For independence, suppose $a(1-x) + b(x - x^2) = c(2x^2) = 0$. Rearranging we get that $a + (b-a)x + (2c-b)x^2 = 0$. Since we already know that $(1, x, x^2)$ is an independent list, the coefficients $a, (b-a)$, and $(2c-b)$ must all be zero, which implies that $a = b = c = 0$. Therefore the given list is independent.

For spanning, let $p(x) = a + bx + cx^2$ be an arbitrary polynomial in $P_2(\mathbb{F})$. Then we can write it as

$$p(x) = a + bx + cx^2 = a(1-x) + (a+b)(x - x^2) + \frac{1}{2}(a+b+c)(2x^2),$$

which shows that every polynomial is a linear combination of $1-x$, $x-x^2$, and $2x^2$, so they span $P_2(\mathbb{F})$.

(3) Let $T \colon V \to V$ be a linear map.

(a) Prove that $\operatorname{Null} T \subseteq \operatorname{Null} T^2$ (where $T^2 = T \circ T$).

Proof: Pick $v \in \operatorname{Null} T$. Then compute $T^2(v) = T(Tv) = T(0) = 0$, where the second equality is because $v \in \operatorname{Null} T$ and the third is because T is linear. This shows that $v \in \operatorname{Null} T^2$.

(b) Prove that if T^2 is injective, then T is injective (you may use part (a), even if you didn't prove it).

Proof: By (a), $\operatorname{Null} T$ is contained in $\operatorname{Null} T^2$, which is the zero space since T^2 was assumed injective. Since the only subspace of the zero space is the zero space itself, $\operatorname{Null} T$ must also be the zero space, so T is injective, too.

(4) Let $T\colon P_2(\mathbb{F}) \to P_3(\mathbb{F})$ be the map $Tp(x) = xp(x) + x^3p(1)$. Find the matrix of T with respect to the bases $B_1 = (1, x, x^2)$ for $P_2(\mathbb{F})$ and $B_2 = (x^3, x^2, x, 1)$ for $P_3(\mathbb{F})$. In other words, find $M(T, B_1, B_2)$.

Solution: We apply T to each of the input basis vectors and write the answers in terms of the output basis vectors (be careful with the order of the vectors):

$$T1 = x + x^3 = 1 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + 0 \cdot 1$$
$$Tx = x^2 + x^3 = 1 \cdot x^3 + 1 \cdot x^2 + 0 \cdot x + 0 \cdot 1$$
$$Tx^2 = x^3 + x^3 = 2 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 0 \cdot 1$$

The coefficients in each row give us the respective columns of the desired matrix, which is therefore

$$M(T, B_1, B_2) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(5) Let $P\colon V \to V$ be a linear map such that $P^2 = P$.

(a) Prove that $\operatorname{Null} P \cap \operatorname{Range} P = \{0\}$

Proof: Pick $v \in \operatorname{Null} P \cap \operatorname{Range} P$. We wish to show that $v = 0$. Since $v \in \operatorname{Range} P$, $v = Pw$ for some $w \in V$. Since $v \in \operatorname{Null} P$, $Pv = 0$. But $Pv = P(Pw) = P^2w = Pw = v$, so $v = 0$.

(b) Prove that $V = \operatorname{Null} P \oplus \operatorname{Range} P$ (you can use the result of part (a) even if you didn't manage to prove it).

Proof: Using part (a), all that's left is to show that $V = \operatorname{Null} P + \operatorname{Range} P$. So pick any $v \in V$. We must write it as a sum $v = u_1 + u_2$, where u_2 is from $\operatorname{Range} P$ and u_1 is from $\operatorname{Null} P$. Well, Pv is in $\operatorname{Range} P$, so let's try $u_2 = Pv$. Then what is the piece u_1 from $\operatorname{Null} P$? It must be $u_1 = v - Pv$, since that's the only way the equation $v = u_1 + u_2$ will be true. So we just check that $v - Pv$ is actually in $\operatorname{Null} P$: $P(v - Pv) = Pv - P(Pv) = Pv - P^2v = Pv - Pv = 0$, so indeed $v - Pv$ is in $\operatorname{Null} P$. Thus we have shown that for an arbitrary vector $v \in V$, we can write

$$v = (v - Pv) + Pv,$$

with $v - Pv \in \operatorname{Null} P$ and $Pv \in \operatorname{Range} P$.