MATH 110, SUMMER 2013
FINAL EXAM SOLUTION
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(1) Prove that the map $T: P_2(\mathbb{R}) \to \mathbb{R}^3$ given by $T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$ is an isomorphism.

Solution: It's enough to show that $\operatorname{Null} T = \{0\}$, since both the domain and codomain have the same dimension. Suppose $p(x) = ax^2 + bx + c \in \operatorname{Null} T$. Then $p(0) = p(1) = p(2) = 0$, so $c = 0$, $a + b + c = 0$, and $4a + 2b + c = 0$. These equations imply that $a = b = c = 0$, so $p(x) = 0$, hence T has trivial null space, so it's an isomorphism.

(2) Consider the subspaces $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x + y + z = 0 \right\} \subset \mathbb{R}^3$, and $W = \left\{ \begin{pmatrix} x \\ 2x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \subset \mathbb{R}^2$. Find a linear map $T: \mathbb{R}^3 \to \mathbb{R}^2$ whose null space is U and whose range is W.

Solution: Pick a basis $u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $u_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ for U, and extend it to basis for $\mathbb{R}^3$ by adding a third independent vector, say e_3. The construction theorem says that we can define T merely by specifying Tu_1, Tu_2, and Te_3. Set

$$Tu_1 = 0, \quad Tu_2 = 0, \quad Te_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This defines a map T with the desired null space and range.

(3) Let T be the map $P_2(\mathbb{F}) \to P_2(\mathbb{F})$ given by $Tp(x) = p(1)p'(x)$. Find the matrix of T with respect to the standard basis $(1, x, x^2)$.

Solution: We first apply T to these basis vectors, giving

$$T1 = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$Tx = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$Tx^2 = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

The coefficients in each row of the calculation give us the columns of the matrix, namely $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

(4) Let T be an operator on V and $v_1, \ldots, v_k$ eigenvectors of T (not necessarily a basis for V) with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$. Show that the subspace $U = \operatorname{Span}(v_1, \ldots, v_k)$ is invariant under T.

Solution: Let $v = a_1v_1 + \cdots + a_kv_k$ be an arbitrary element of U. We must show Tv is still in U. Compute

$$Tv = T(a_1v_1 + \cdots + a_kv_k) = a_1Tv_1 + \cdots + a_kTv_k = a_1\lambda_1v_1 + \cdots + a_k\lambda_kv_k,$$

which is in U since it is a linear combination of the v_is. Thus U is T-invariant.

(5) Fill in the blanks in the following table with YES or NO, according to whether the given map has the various properties. You do not need to justify your answers.

Operator	Normal?	Self-Adjoint?	Isometry?
$T: \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	NO	NO	NO
Reflection across the xz-plane in $\mathbb{R}^3$	YES	YES	YES
$(1 + 2i)I \in \mathcal{L}(V)$, where V is a complex vector space	YES	NO	NO

(6) Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $T^k = I$ for some $k > 2$. Prove that $T^2 = I$.

Solution: Since T is self-adjoint, it is also normal, so by the spectral theorem, we can find an ONB B of eigenvectors, with respect to which the matrix for T becomes

$$M(T,B) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \text{ so } \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

since $T^k = I$. Since T is self-adjoint, all the eigenvalues λ_i are real numbers. Then $\lambda_i^k = 1$ for each $i = 1, \ldots, n$ means the only possibilities are $\lambda_i = 1$ or -1. Since $(\pm 1)^2 = 1$, $\lambda_i^2 = 1$ for all i, hence $T^2 = I$.

(7) Let A be the real matrix $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$. (a) Find all eigenvectors of A. (b) Find the Jordan normal form of A. You must justify your answer in some way.

Solution: We see that the only eigenvalue of A is 2. For both parts, it will be useful to first calculate the various powers of $A - 2I$:

$$A - 2I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A-2I)^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A-2I)^3 = 0,$$

where that last 0 is a lazy way of saying the 3×3 zero matrix. For (a), the eigenvectors of A are the nullspace of $A - 2I$, which we can see is the span of e_1. The null space of $(A-2I)^2$ is spanned by e_1, e_2, and the nullspace of $(A-2I)^3$ is all of $\mathbb{R}^3$.

So the answer to (a) is $\text{Span}(e_1)$.

For (b), here's the **short solution**: since there is only one independent eigenvector, there can only be one Jordan block; it's a 3×3 block since A is 3×3.

The **long solution**: We construct the basis of Jordan chains. We pick an element of $\text{Null}(A-2I)^3$ which is not in $\text{Null}(A-2I)^2$, say e_3 (this isn't the only possible choice, of course). Then we get two other vectors by applying $A - 2I$:

$$(A-2I)e_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \quad (A-2I)^2 e_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Writing these in reverse order gives our Jordan chain

$$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which is a list of length 3, so we get one 3×3 Jordan block, hence the matrix of A in this basis is the JNF of A is $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

(8) Recall that an operator P is called a *projection* if it satisfies $P^2 = P$.
  (a) If P is a projection, show that $I - P$ is a projection also (I means the identity map).
  (b) Now let P be a projection, and $T \in \mathcal{L}(V)$ some other operator. Prove that $PT = TP$ if and only if both $\text{Range}\, P$ and $\text{Range}(I - P)$ are invariant under T.

Solution:
  (a) $(I-P)^2 = I^2 - IP - PI + P^2 = I - 2P + P^2 = I - P$, where we used that I commutes with P (it commutes with everything!) and $P^2 = P$.

  (b) This is the hardest problem on the test. First recall from class that the range of a projection is its 1-eigenspace, so $Pw = w$ whenever w is in $\text{Range}\, P$, and similarly for $I - P$. Let's first assume that $TP = PT$. We want to show that $\text{Range}\, P$ and $\text{Range}\, I - P$ are T-invariant. Pick $w \in \text{Range}\, P$, so $Pw = w$. Then $Tw = TPw = PTw \in \text{Range}\, P$, so $\text{Range}\, P$ is T-invariant. Now pick $w \in \text{Range}(I-P)$. Then $Tw = T(I-P)w = TIw - TPw = ITw - PTw = (I-P)Tw \in \text{Range}\, I - P$, so $\text{Range}\, I - P$ is T-invariant.[1]
  For the other direction, we assume that $\text{Range}\, P$ and $\text{Range}(I - P)$ are T-invariant. We want to show $TPv = PTv$ for all vectors $v \in V$. The key observation is that $\text{Range}(I-P) = \text{Null}\, P$. Reason: $v \in \text{Null}\, P$ iff $Pv = 0$ iff $v - Pv = v$ iff $(I-P)v = v$ iff $v \in \text{Range}\, P$.
  So pick any $v \in V$. Since P is a projection, we can write it as $v = w + u$, where $w \in \text{Range}\, P$ and $u \in \text{Null}\, P = \text{Range}(I-P)$. By our assumptions, then, $Tw \in \text{Range}\, P$ and $Tu \in \text{Null}\, P$, since both spaces are T-invariant. Now we compute

$$PTv = PT(w+u) = PTw + PTu = P(Tw) + 0 = Tw$$
$$TPv = TP(w+u) = TPw + TPu = Tw + T(0) = Tw$$

  Thus the two maps agree, so $PT = TP$.

[1] More generally, you can show that whenever two operators commute with each other, their null spaces and ranges are invariant under each other. So this direction does not actually require the fact that these maps are projections The other direction does.