Practice Midterm 2 Solutions

① We solve $Tp(x) = \lambda p(x)$ for λ and nonzero $p(x)$.

Write $p(x) = ax^3 + bx^2 + cx + d$, so

$$6ax^3 + 2bx^2 + 0x + 0 = \lambda ax^3 + \lambda bx^2 + \lambda cx + \lambda d$$

$$\Rightarrow \quad \begin{aligned} 6a &= \lambda a \\ 2b &= \lambda b \\ 0 &= \lambda c \\ 0 &= \lambda d \end{aligned}$$

If $\lambda = 0$, then c, d can be arbitrary, but $a = b = 0$,

so $E_0 = \mathrm{Span}(1, x)$

If $\lambda \neq 0$, then $c = d = 0$. In the 2nd equation, $b = 0$ or $\lambda = 2$. If $\lambda = 2$ then b can be anything, but $a = 0$, so $E_2 = \mathrm{Span}(x^2)$.

If $b = 0$, then $a \neq 0$, since we want $p(x) \neq 0$, so $\lambda = 6$, and $E_6 = \mathrm{Span}(x^3)$.

② Assume $V \neq \{0\}$, or else there are no eigenvalues.

Then, if λ is an e-value, and v a nonzero e-vector we have $Tv = \lambda v$, so $T^n v = \lambda^n v \Rightarrow 0 = \lambda^n v$, and since $v \neq 0$, $\lambda^n = 0$, so $\lambda = 0$.

This shows: <u>if</u> T has an e-value, it must be zero.

We must prove λ actually <u>is</u> an e-value. But $E_0 = \mathrm{Null}\, T$, and $\mathrm{Null}\, T \neq \{0\}$ since T is not invertible. If it were, $T^n = 0 \Rightarrow (T^{-1})^n T^n = (T^{-1})^n \cdot 0 \Rightarrow I = 0$, contradiction (again using $V \neq \{0\}$).

(3) Assume λ is an e-value, w/ nonzero e-vector v.

Then $0 \le \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2$

so $0 \le \lambda \|v\|^2$. Since $\|v\|^2 > 0$ $(v \neq 0)$,

we get $0 \le \lambda$.

(4) Obviously $\{0\}$ and $\mathbb{R}^3$ are invariant.

Looking at the factorization, we see that T is a $90°$ rotation around the z-axis, followed by projection to the xy-plane.

Thus any invariant subspace must be either the z-axis or contained in the xy plane.

But the rotation in the xy plane has no invariant subspaces besides the xy plane and $\{0\}$.

So the invariant subspaces are:

0-D	1-D	2-D	3-D
$\{0\}$	z-axis	xy-plane	$\mathbb{R}^3$

(5) (a) - A rotation through $45°$ in $\mathbb{R}^2$ has no e-values

- $Tp(x) = xp(x)$ on $P(\mathbb{F})$ has no e-values

- The zero map on the zero space has no e-values

(b) The matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ all

have e-values, but cannot be diagonalized

(c) See the last problem of HW 5.

(6) It didn't make sense as originally written.

Should write the basis as $(e_0, e_1, \dots, e_n)$.

Pick any $v, w \in V$, and write

$$v = a_0 e_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n e_n$$
$$w = b_0 e_0 + b_1 e_1 + \cdots + b_{n-1} e_{n-1} + b_n e_n$$

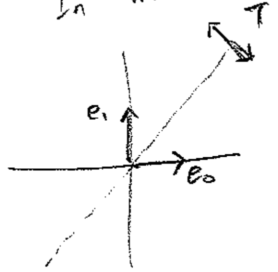
Then

$$\begin{aligned}
\langle Tv, Tw \rangle &= \langle a_0 e_n + a_1 e_{n-1} + \cdots + a_{n-1} e_1 + a_n e_0,\ b_0 e_n + \cdots + b_n e_0 \rangle \\
&= \langle a_0 e_n, b_0 e_n \rangle + \cdots + \langle a_0 e_n, b_n e_0 \rangle \\
&\quad + \langle a_1 e_{n-1}, b_0 e_n \rangle + \cdots + \langle a_1 e_{n-1}, b_n e_0 \rangle \\
&\quad + \cdots \\
&\quad + \langle a_n e_0, b_0 e_n \rangle + \cdots + \langle a_n e_0, b_n e_0 \rangle \\
&= a_0 b_0 \cancelto{1}{\langle e_n, e_n \rangle} + a_0 b_1 \cancelto{0}{\langle e_n, e_{n-1} \rangle} + \cdots + a_0 b_n \cancelto{0}{\langle e_n, e_0 \rangle} \\
&\quad + a_1 b_0 \cancelto{0}{\langle e_{n-1}, e_n \rangle} + a_1 b_1 \cancelto{1}{\langle e_{n-1}, e_{n-1} \rangle} + \cdots + a_{n-1} b_n \cancelto{0}{\langle e_{n-1}, e_0 \rangle} \\
&\quad + \cdots \\
&\quad + a_n b_0 \cancelto{0}{\langle e_0, e_n \rangle} + \cdots + a_n b_n \cancelto{1}{\langle e_n, e_n \rangle} \\
&= a_0 b_0 + a_1 b_1 + \cdots + a_n b_n.
\end{aligned}$$

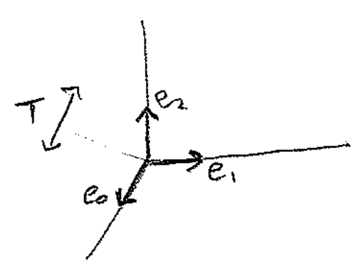
Compute similarly $\langle v, w \rangle = \cdots = a_0 b_0 + \cdots + a_n b_n$.

Intuition:

In $\mathbb{R}^2$:



In $\mathbb{R}^3$



T is a reflection, so it doesn't change lengths or angles, hence it preserves the inner product!

(a)

⑦ 2 cases:

① If $v = 0$,

Then $\phi_v(u) = \langle u, 0 \rangle = 0$,

so ϕ_v is the zero map $V \to \mathbb{F}$,

and $\text{Null } \phi_v = V$, so $\dim \text{Null } \phi_v = n$

② If $v \neq 0$, then $\phi_v(v) = \langle v, v \rangle = \|v\|^2 \neq 0$,

so ϕ_v is surjective, hence

$$\dim \text{Null } \phi_v = \dim V - \dim \text{Range } \phi_v$$
$$= n - \dim \mathbb{F}$$
$$= n - 1$$

(b) Pick $w \in U^\perp$, so $w \perp u$ $\forall u \in \text{Span } v$.

In particular, $w \perp v$, so $\phi_v|_{U^\perp}(w) = \langle w, v \rangle = 0$.

Thus $\phi_v|_{U^\perp}$ is the zero map, since w was an arbitrary element of $U^\perp$

⑧ Since T is an operator, T is invertible iff T is injective. To show it's injective,

pick $v \in \text{Null } T$, so $Tv = 0$.

Then $\|v\| = \|Tv\| = \|0\| = 0$

so $\|v\| = 0$, hence $v = 0$, so $\text{Null } T = \{0\}$