We solve \( T \mathbf{p}(x) = \lambda \mathbf{p}(x) \) for \( \lambda \) and \( \mathbf{p}(x) \).

Write \( \mathbf{p}(x) = ax^3 + bx^2 + cx + d \), so

\[
6ax^3 + 2bx^2 + 0x + 0 = \lambda ax^3 + \lambda bx^2 + \lambda cx + \lambda d
\]

\[
\Rightarrow \quad 6a = \lambda a \quad \Rightarrow \quad 2b = \lambda b \quad \Rightarrow \quad 0 = \lambda c \quad \Rightarrow \quad 0 = \lambda d
\]

If \( \lambda = 0 \), then \( a, b, c, d \) can be arbitrary, but \( a = b = 0 \),

so \( E_0 = \text{Span}(1, x) \).

If \( \lambda \neq 0 \), then \( a = d = 0 \). In the 2nd equation,

\[ b = 0 \quad \text{or} \quad \lambda = 2. \]

If \( \lambda = 2 \), then \( b \) can be anything, but \( a = 0 \), so \( E_2 = \text{Span}(x^2) \).

If \( b \neq 0 \), then \( a \neq 0 \), since we want \( \mathbf{p}(x) \neq 0 \), so \( \lambda = 6 \),

and \( E_6 = \text{Span}(x^3) \).

Assume \( V \neq 0 \), or else there are no eigenvalues.

Then, if \( \lambda \) is an eigenvalue and \( \mathbf{v} \) a nonzero eigenvector,

we have \( T \mathbf{v} = \lambda \mathbf{v} \), so \( T \mathbf{v} = \lambda \mathbf{v} \Rightarrow \mathbf{0} = \lambda \mathbf{v} \) and

since \( \mathbf{v} \neq \mathbf{0} \), \( \lambda = 0 \).

This shows: if \( T \) has no eigenvalues, \( \mathbf{v} \) must be zero.

We must prove \( \lambda \) actually is an eigenvalue. But \( E_0 = \text{Null}(T) \),

and \( \text{Null}(T) \neq 0 \), since \( T \) is not invertible. If \( \mathbf{v} \)

were, \( T \mathbf{v} = \mathbf{0} \Rightarrow (T^{-1})^n \mathbf{T}^n = (T^{-1})^n \mathbf{0} \Rightarrow I = \mathbf{0} \), contradiction

(again using \( \mathbf{v} \neq \mathbf{0} \)).
Assume \( \lambda \) is an eigenvalue, with nonzero eigenvector \( \mathbf{v} \).

Then \( 0 \leq \langle T\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle = \lambda \langle \mathbf{v}, \mathbf{v} \rangle = \lambda \| \mathbf{v} \|^2 \)

so \( 0 \leq \lambda \| \mathbf{v} \|^2 \). Since \( \| \mathbf{v} \|^2 > 0 \) (\( \mathbf{v} \neq \mathbf{0} \)),

we get \( 0 \leq \lambda \).

Obviously \( \mathbb{R}^3 \) and \( \mathbb{R}^3 \) are invariant.

Looking at the factorization, we see that \( T \) is a 90° rotation around the z-axis, followed by projection to the xy-plane.

Thus any invariant subspace must be either the z-axis or contained in the xy-plane.

But the rotation in the xy-plane has no invariant subspaces besides the xy-plane and \( \mathbb{R}^3 \).

So the invariant subspaces are:

\[
\begin{align*}
0-D & \quad 1-D & \quad 2-D & \quad 3-D \\
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \quad z-axis & \quad xy-plane & \quad \mathbb{R}^3
\end{align*}
\]

(a) A rotation through 45° in \( \mathbb{R}^3 \) has no eigenvalues.

(b) \( \mathbb{R}^2 \) has no eigenvalues.

(c) The zero map on the zero space has no eigenvalues.

(b) The matrices

\[
\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

have eigenvalues, but cannot be diagonalized.

(c) See the last problem of HW 5.
If didn't make sense as originally written.
Shall write the basis as \((e_0, e_1, \ldots, e_n)\).

Pick any \(v, w \in V\) and write

\[
v = a_0 e_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n e_n
\]
\[
w = b_0 e_0 + b_1 e_1 + \cdots + b_{n-1} e_{n-1} + b_n e_n
\]

Then

\[
\langle Tv, Tw \rangle = \langle a_n e_n + a_{n-1} e_{n-1} + \cdots + a_1 e_1 + a_0 e_0, b_n e_n + \cdots + b_0 e_0 \rangle
\]

\[
= \langle a_n e_n, b_n e_n \rangle + \cdots + \langle a_1 e_1, b_1 e_1 \rangle + \langle a_0 e_0, b_0 e_0 \rangle
\]

\[
+ \langle a_{n-1} e_{n-1}, b_{n-1} e_{n-1} \rangle + \cdots + \langle a_0 e_0, b_0 e_0 \rangle
\]

\[
= a_0 b_0 \langle e_0, e_0 \rangle + a_1 b_1 \langle e_1, e_1 \rangle + \cdots + a_n b_n \langle e_n, e_n \rangle
\]

\[
= a_0 b_0 + a_1 b_1 + \cdots + a_n b_n
\]

Compute similarly \(\langle v, v \rangle = \cdots = a_0 b_0 + \cdots + a_n b_n\).

**Intuition:**

In \(\mathbb{R}^2\):

\(T\) is a reflection, so it doesn't change lengths or angles, hence it preserves the inner product. In \(\mathbb{R}^3\):
(a) \[
\begin{align*}
\text{2. \ cases:} & \\
\text{(i) \ if } & v = 0, \\
& \phi_v(u) = \langle u, 0 \rangle = 0,
\end{align*}
\]
so \( \phi_v \) to the zero map \( \text{dim } V \to \mathbb{F} \),
and \( \text{Null } \phi_v = V \), so \( \text{dim } \text{Null } \phi_v = n \).

\( \text{(ii) if } v \neq 0, \text{ then } \phi_v(u) = \langle v, u \rangle = ||v||^2 \neq 0, \)
so \( \phi_v \) is surjective, hence
\[ \text{dim } \text{Null } \phi_v = \text{dim } V - \text{dim } \text{Range } \phi_v \]
\[ = n - \text{dim } \mathbb{F} = n - 1. \]

(b) Pick \( w \in U^\perp \), so \( w \perp u \) \( \forall u \in \text{Span } v. \)
In particular, \( w \perp v \), so \( \phi_v\big|_{U^\perp}(w) = \langle w, v \rangle = 0. \)
Thus \( \phi_v\big|_{U^\perp} \) is the zero map, since \( w \) was an arbitrary element of \( U^\perp \).

(c) Since \( T \) is an operator, \( T \) is invertible if and only if \( T \) is injective.
To show it’s injective, pick \( v \in \text{Null } T \), so \( Tv = 0. \)
Then \( ||v|| = ||Tv|| = ||0|| = 0. \)

So \( ||v|| = 0 \), hence \( v = 0 \), so \( \text{Null } T = \{0\}. \)