MATH 110, SUMMER 2013
PRACTICE PROBLEMS FOR 2ND MIDTERM
MONDAY, JULY 29TH

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(1) Find all eigenvalues and eigenvectors of the operator $T\colon P_3(\mathbb{F}) \to P_3(\mathbb{F})$ given by $Tp(x) = x^2 p''(x)$.

(2) Suppose an operator $T \in \mathcal{L}(V)$ satisfies $T^n = 0$ for some $n \in \mathbb{N}$. Prove that 0 is the only eigenvalue of T.

(3) Let T be an operator on V which satisfies $\langle Tv, v\rangle \geq 0$ for all $v \in V$. Prove that any eigenvalues of T must be nonnegative real numbers.

(4) Let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map given by multiplication by the matrix $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Find all invariant subspaces for S. Possible hint: use the factorization

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to try to understand the map S geometrically.

(5) Give examples of the following:
  (a) An operator with no eigenvalues.
  (b) An operator which has eigenvalues but is not diagonalizable.
  (c) An orthonormal basis for $P_2(\mathbb{F})$.

(6) If $(e_1, \ldots, e_n)$ is an orthonormal basis for an inner product space V, and $T \in \mathcal{L}(V)$ is the operator defined by $T(e_i) = e_{n-i}$, prove that $\langle Tv, Tw\rangle = \langle v, w\rangle$ for all $v, w \in V$.

(7) Let V be an n-dimensional inner product space, and fix a vector $v \in V$. Define a map $\phi_v\colon V \to \mathbb{F}$ by $\phi_v(u) = \langle u, v\rangle$.
  (a) What is the dimension of $\operatorname{Null} \phi_v$? Does it depend on our choice of v?
  (b) Now let $U = \operatorname{Span}(v)$, for this same choice of v above. Prove that $\phi_v\big|_{U^\perp}$ is the zero map.

(8) If T is an operator on the inner product space V such that $\|Tv\| = \|v\|$ for all $v \in V$, prove that T is invertible.