(1) Find all eigenvalues and eigenvectors of the operator $T: P_3(\mathbb{F}) \to P_3(\mathbb{F})$ given by $T p(x) = x^2 p''(x)$.

(2) Suppose an operator $T \in \mathcal{L}(V)$ satisfies $T^n = 0$ for some $n \in \mathbb{N}$. Prove that 0 is the only eigenvalue of $T$.

(3) Let $T$ be an operator on $V$ which satisfies $\langle Tv, v \rangle \geq 0$ for all $v \in V$. Prove that any eigenvalues of $T$ must be nonnegative real numbers.

(4) Let $S: \mathbb{R}^3 \to \mathbb{R}^3$ be the map given by multiplication by the matrix $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Find all invariant subspaces for $S$. Possible hint: use the factorization $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ to try to understand the map $S$ geometrically.

(5) Give examples of the following:
   (a) An operator with no eigenvalues.
   (b) An operator which has eigenvalues but is not diagonalizable.
   (c) An orthonormal basis for $P_2(\mathbb{F})$.

(6) If $(e_1, \ldots, e_n)$ is an orthonormal basis for an inner product space $V$, and $T \in \mathcal{L}(V)$ is the operator defined by $T(e_i) = e_{n-i}$, prove that $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$.

(7) Let $V$ be an $n$-dimensional inner product space, and fix a vector $v \in V$. Define a map $\phi_v: V \to \mathbb{F}$ by $\phi_v(u) = \langle u, v \rangle$.
   (a) What is the dimension of $\text{Null} \phi_v$? Does it depend on our choice of $v$?
   (b) Now let $U = \text{Span}(v)$, for this same choice of $v$ above. Prove that $\phi_v|_{U^\perp}$ is the zero map.

(8) If $T$ is an operator on the inner product space $V$ such that $\|Tv\| = \|v\|$ for all $v \in V$, prove that $T$ is invertible.