MATH 110, SUMMER 2013
PRACTICE MIDTERM EXAM 1
SOLUTION!

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PLEASE NOTE - my justifications below are just the ideas - in some cases you would be expected to say more on a test to receive full marks. Nevertheless, I hope the explanations give you the right idea if you're stuck.

(1) Which of the following are subspaces of the vector space $P_2(\mathbb{F})$? Explain why or why not.

(a) $\{p(x) \,|\, (p(x))^2 = cx^4 \text{ for some } c \in \mathbb{F}\}$
Solution: It's a subspace, namely the set of all multiples of x^2. We can write it as the span of the polynomial x^2, and we know that the span of any set is a subspace.

(b) $\{p(x) \,|\, p(x-1) = p(x)\}$

Solution: It's a subspace - namely the set of constant polynomials, equivalently the span of the polynomial 1.

(2) Which of the following functions are linear maps? Explain why or why not.

(a) $T\colon \mathbb{C} \to \mathbb{C}$ given by $Tz = z - \overline{z}$. (here $\mathbb{C}$ is regarded as a one-dimensional complex space).
Solution: Nope, not homogeneous (use an imaginary scalar for a counterexample).

(b) $T\colon \mathbb{F}^\infty \to \mathbb{F}^\infty$ given by $T(a_1, a_2, a_3, \ldots) = (a_1 + a_1, a_1 + a_2, a_1 + a_3, \ldots)$.

Solution: Yes, it's linear.

(3) Let $V = P_1(\mathbb{F})$ and $U = \{p(x) \in P_1(\mathbb{F}) \,|\, p(1) = 0\}$. Find a subspace W of $P_1(\mathbb{F})$ such that $V = U \oplus W$ and prove your choice is correct.

Solution: First off, notice that U is two-dimensional, with basis, for instance, the polynomials $x - 1$ and $x^2 - 1$. You should be able to prove this. So the W we're looking for should be 1-D. In fact, you can take any (nonzero) polynomial not in U, and let W be its span. So for example $W = \text{Span}(x)$ will work.

(4) Prove that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

Solution: Use the (contrapositive of) the linear dependence lemma, which says that for independence, you just need to check that each vector is not in the span of the prior ones. The second is clearly not a multiple of the first, and you can check that the third is not a linear combination of the first two. Thus they're independent, and since there are 3 of them, it's a basis.

(5) Let $T\colon V \to V$ be a linear map with the property that $T^2 = I$ (where $T^2 = T \circ T$ and I means the identity map). Prove that $\operatorname{Null} T = \{0\}$.

Proof: Pick $v \in \operatorname{Null} T$. Then $Tv = 0$, and applying T to both sides gives $T^2 v = T(0) = 0$. But $T^2 v = Iv = v$, so $v = 0$. Thus the only vector in $\operatorname{Null} T$ is the zero vector.

(6) If $T\colon \mathbb{C}^3 \to \mathbb{C}^2$ is the map $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ x - iy \end{pmatrix}$, find a basis for $\operatorname{Null} T$ and for $\operatorname{Range} T$.

Solution: If a vector (x, y, z) is in $\operatorname{Null} T$, we must have $z = 0$ and $x - iy = 0$, so $x = iy$. Thus we can rewrite the vector as $(iy, y, 0)$. So every vector in the null space is a multiple of the vector $(i, 1, 0)$, which is therefore a basis for $\operatorname{Null} T$. Also, by rank-nullity we now know that the range of T must be two-dimensional. But the range must be a subspace of $\mathbb{C}^2$, which is itself 2-dimensional, so $\operatorname{Range} T = \mathbb{C}^2$. For a basis we can use, say, the standard basis.

(7) Consider the map $T\colon P_2(\mathbb{F}) \to \mathbb{F}^2$ given by $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$. Write down the matrix of T with respect to the basis $(1, x, x^2)$ for $P_2(\mathbb{F})$ and the standard basis (e_1, e_2) for $\mathbb{F}^2$.

Solution: We compute $T(1) = (1, 1)$, $T(x) = (1, 2)$, and $T(x^2) = (1, 4)$. These vectors are already expressed with respect to the standard basis, so they are the three columns of the matrix we're looking for.

(8) Suppose that V and W are vector spaces with bases $B_1 = (v_1, v_2)$ and $B_2 = (w_1, w_2, w_3)$, respectively. Prove that if

$$M(T, B_1, B_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then T must be injective.

Proof: We must show that $\operatorname{Null} T = \{0\}$. So pick $v \in \operatorname{Null} T$; we can write it as $a_1v_1 + a_2v_2$. Then $Tv = a_1Tv_1 + a_2Tv_2$ by linearity, which is in turn equal to

$$Tv = a_1(w_1 + w_3) + a_2w_2,$$

by the given matrix. But since v is in the null space of T, this is equal to zero, so $a_1w_1 + a_2w_2 + a_1w_3 = 0$ and by independence of the w_i we get that $a_1 = a_2 = a_3 = 0$, so $v = 0$. Done!

(9) (harder) Prove that "every subspace is a nullspace". More precisely, let V be a finite-dimensional vector space, and U a subspace of V. Prove that there is a linear map $T\colon V \to V$ such that $\operatorname{Null} T = U$. (hint: "construct" such a map)

Proof Idea: Pick a basis $(u_1, \ldots, u_m)$ of U, extend it to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_k)$ for V. We can define our map T by saying what happens to each basis vector (this is the construction theorem) Send each of the u_i to zero, and send each v_i to itself. This is the map you're looking for (although there are others), and you should prove that its null space is exactly U. This map is actually the "projection onto U" possibly familiar from a previous HW assignment...