MATH 110, SUMMER 2013
PRACTICE MIDTERM EXAM 1
THURSDAY, JULY 11TH

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(1) Which of the following are subspaces of the vector space $P_2(\mathbb{F})$? Explain why or why not.
   (a) $\{p(x) \,|\, (p(x))^2 = cx^4 \text{ for some } c \in \mathbb{F}\}$
   (b) $\{p(x) \,|\, p(x-1) = p(x)\}$

(2) Which of the following functions are linear maps? Explain why or why not.
   (a) $T\colon \mathbb{C} \to \mathbb{C}$ given by $Tz = z - \overline{z}$. (here $\mathbb{C}$ is regarded as a one-dimensional complex space).
   (b) $T\colon \mathbb{F}^\infty \to \mathbb{F}^\infty$ given by $T(a_1, a_2, a_3, \ldots) = (a_1 + a_1, a_1 + a_2, a_1 + a_3, \ldots)$.

(3) Let $V = P_1(\mathbb{F})$ and $U = \{p(x) \in P_1(\mathbb{F}) \,|\, p(1) = 0\}$. Find a subspace W of $P_1(\mathbb{F})$ such that $V = U \oplus W$ and prove your choice is correct.

(4) Prove that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

(5) Let $T\colon V \to V$ be a linear map with the property that $T^2 = I$ (where $T^2 = T \circ T$ and I means the identity map). Prove that $\operatorname{Null} T = \{0\}$.

(6) If $T\colon \mathbb{C}^3 \to \mathbb{C}^2$ is the map $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ x - iy \end{pmatrix}$, find a basis for $\operatorname{Null} T$ and for $\operatorname{Range} T$.

(7) Consider the map $T\colon P_2(\mathbb{F}) \to \mathbb{F}^2$ given by $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$. Write down the matrix of T with respect to the basis $(1, x, x^2)$ for $P_2(\mathbb{F})$ and the standard basis (e_1, e_2) for $\mathbb{F}^2$.

(8) Suppose that V and W are vector spaces with bases $B_1 = (v_1, v_2)$ and $B_2 = (w_1, w_2, w_3)$, respectively. Prove that if

$$M(T, B_1, B_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then T must be injective.

(9) (harder) Prove that "every subspace is a nullspace". More precisely, let V be a finite-dimensional vector space, and U a subspace of V. Prove that there is a linear map $T\colon V \to V$ such that $\operatorname{Null} T = U$. (hint: "construct" such a map)