MATH 110, SUMMER 2013
PRACTICE FINAL SOLUTION

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(1) Find a linear map $T\colon P_2(\mathbb{F}) \to P_1(\mathbb{F})$ whose null space is the set of polynomials such that $p(1) = 0$ and whose range is the constant polynomials in $P_1(\mathbb{F})$.

Solution: Let $U = \{p(x)\,|\,p(1) = 0\} \subset P_2(\mathbb{F})$ and $W =$ the set of constant polynomials $= \text{Span}(1)$. We find a basis for U, and extend it to a basis of the domain $P_2(\mathbb{F})$. Since U is 2D (we imposed one condition on a 3D space), and $1-x$ and $1-x^2$ are independent elements of U, we have that $1-x, 1-x^2$ form a basis for U. Adding in any polynomial not in U, say 1, gives a basis for $P_2(\mathbb{F})$, namely $(1-x, 1-x^2, 1)$. By the construction theorem, it suffices to define $T(1-x), T(1-x^2)$, and $T(1)$. Set

$$T(1-x) = 0$$
$$T(1-x^2) = 0$$
$$T(1) = 1$$

to get the desired null space and range.

(2) Let V be a complex vector space and define a map $\text{Tr}\colon \mathcal{L}(V) \to \mathbb{C}$ by $\text{Tr}\,T =$ the sum of the eigenvalues of T (counted with multiplicities). Prove that Tr is surjective. Prove that if T is injective, then V must be one-dimensional.

Solution: To prove Tr is surjective, pick any $c \in \mathbb{C}$. Pick a basis B for V and consider the operator S whose matrix with respect to this basis is diagonal, with c on the first diagonal entry, and zeroes everywhere else. Then its only eigenvalues are c and 0, so $\text{Tr}\,S = c$. Thus each c in $\mathbb{C}$ has a pre-image, so Tr is surjective. If in addition, we are given that T is injective, then we know it must be an isomorphism, which implies $\dim \mathcal{L}(V) = \dim \mathbb{C} = 1$. But if $\dim V = n$, then $\dim \mathcal{L}(V) = n^2$, so $n^2 = 1$ forces $n = 1$.

(3) Consider the operator $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax-y \\ y-ax \\ az \end{pmatrix}$ on $\mathbb{R}^3$, where a is some unspecified real number. Prove that $\left\{\begin{pmatrix} x \\ ax \\ 0 \end{pmatrix} \Big| x \in \mathbb{R}\right\} \subseteq$ Null T. For which a is this *not* an equality?

Pick any $\begin{pmatrix} x \\ ax \\ 0 \end{pmatrix}$. Then $T\begin{pmatrix} x \\ ax \\ 0 \end{pmatrix} = \begin{pmatrix} ax-ax \\ ax-ax \\ 0 \end{pmatrix}$, so it's in Null T. For the second part, let's just calculate the null space explicitly. Setting $\begin{pmatrix} ax-y \\ y-ax \\ az \end{pmatrix}$ equal to zero gives $y = ax$ and $az = 0$. If $a \neq 0$, then the second equation forces $z = 0$ and we get the subspace from before. But if $a = 0$ then we do not have equality: the nullspace is $\left\{\begin{pmatrix} x \\ ax \\ z \end{pmatrix} \Big| x \in \mathbb{R}\right\}$

(4) Fill in the blanks in the following table with YES or NO, according to whether the given map has the various properties. You do not need to justify your answers.

Operator	Normal?	Self-Adjoint?	Isometry?
A reflection in $\mathbb{R}^2$	yes	yes	yes
$T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$	no	no	no
Rotation through $\pi/2$ around the x-axis in $\mathbb{R}^3$	yes	no	yes
Orthogonal projection onto the plane $x+y+z=0$ in $\mathbb{R}^3$	yes	yes	no

(5) Let $T \in \mathcal{L}(V)$ be a normal operator, all of whose eigenvalues are *purely imaginary* (meaning they are of the form ci for some $c \in \mathbb{R}$). Prove that $T = -T^*$.

Solution: Write $\lambda_j = c_j i$ for each eigenvalue λ_j of T, where $c_j \in \mathbb{R}$. Using Spectral Thm, pick an ONB and use it to write T as the matrix

$$\begin{pmatrix} c_1 i & 0 & \cdots & 0 \\ 0 & c_2 i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_k i \end{pmatrix}$$

Then using the conjugate transpose trick for adjoints (OK, since we're using an ONB), we get

$$T^* = \begin{pmatrix} -c_1 i & 0 & \cdots & 0 \\ 0 & -c_2 i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -c_k i \end{pmatrix} = - \begin{pmatrix} c_1 i & 0 & \cdots & 0 \\ 0 & c_2 i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_k i \end{pmatrix} = -T$$

Strictly, speaking, I'm being sloppy there - the matrix and the map are not the same thing. But you get the idea...

(6) Find the matrix of the map $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+y \end{pmatrix}$ with respect to the basis $B = (e_1 + e_2, e_1 - e_2)$ for $\mathbb{R}^2$. Use this matrix to say what the eigenvalues and eigenvectors of T are.

Solution: $T(e_1+e_2) = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3(e_1+e_2)$ and $T(e_1 - e_2) = T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -(e_1 - e_2)$, so the given basis is a basis of eigenvectors, with eigenvalues 3 and -1, and the matrix with respect to this basis is

$$\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

(7) Suppose T is normal and $T^3 = T^2$. Prove that $T^2 = T$. Is it also true that $T = I$? Prove it or give a counterexample.

Solution: As usual, since we have a normal operator, we should pick an ONB and write T as a diagonal matrix. Then we are given that

$$\begin{pmatrix} \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \lambda_2^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^3 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^2 \end{pmatrix}$$

which implies that $\lambda_i^3 = \lambda_i^2$, so $\lambda_i = 0$ or 1, hence $\lambda_i^2 = \lambda_i$ (since $0^2 = 0$ and $1^2 = 1$). This in turn implies $T^2 = T$. We cannot conclude that $T = I$, since some of the λ_is might be zero. For instance the zero map satisfies $T^3 = T^2$, but is not the identity map (unless of course V is the zero space).

(8) Find a basis for $\mathbb{R}^2$ with respect to which the matrix for the map $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x+2y \end{pmatrix}$ is in Jordan normal form. Bonus: Use this to calculate T^n, for any $n > 0$. Double bonus: use this to calculate T^n, with a PICTURE!

Solution: Write T as a matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$. First find the eigenvectors and eigenvalues. The only eigenvalue is 1, with a 1D null space, spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Now look for generalized eigenvectors - there's only one extra dimension's worth, so any non-zero vector that's not an eigenvector will do - take e_2 for instance. Then our Jordan chain is $(T-I)e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, e_2$. And in terms of this basis the matrix of T becomes $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let S be the matrix whose columns are our basis of eigenvectors, namely

$$S = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},$$

so we have $A = SJS^{-1}$. Then $A^n = SJ^nS^{-1}$. We saw in class once that $J^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, so

$$A^n = SJ^nS^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1-n & -n \\ n & n+1 \end{pmatrix}$$

where we used that actually S is its own inverse. For the geometrical point of view, see picture linked from course website.

(9) If $T \in \mathcal{L}(V)$ is self-adjoint, prove that $\operatorname{Null} T \perp \operatorname{Range} T$.

Solution: Pick $v \in \operatorname{Null} T$ and $u \in \operatorname{Range} T$. We must show $\langle v, w\rangle = 0$. Since $w \in \operatorname{Range} T$, we can write $w = Tu$ for some $u \in V$. Then

$$\langle v, w\rangle = \langle v, Tu\rangle = \langle v, t^* u\rangle = \langle Tv, u\rangle = 0.$$

(10) Let $\dim V = n$ and $T \in \mathcal{L}(V)$. Assuming that $T^n = 0$ (the zero map), but $T^{n-1} \neq 0$, prove that there exists a vector $v \in V$ such that $(v, Tv, \ldots, T^{n-1}v)$ is a basis for V. Compute the matrix for T with respect to this basis.

Solution: We claim that we can take as our v any vector not in the nullspace of T^{n-1}. It is possible to find such a v since by our assumption T^{n-1} is not the zero map. So fix such a v. It's nonzero, of course, since it's not in the subspace $\operatorname{Null} T^{n-1}$, and the zero vector *is* in that subspace. Since the list $(v, Tv, \ldots, T^{n-1}v)$ has length $n = \dim V$, to show it's a basis it suffices to check independence. So suppose

$$a_0 v + a_1 Tv + \cdots + a_{n-1} T^{n-1} v = 0.$$

Apply T^{n-1}. Then since $T^n = 0$, this gives $a_0 T^{n-1} v = 0$. Since $T^{n-1} v \neq 0$, we get $a_0 = 0$. So our equation now reads

$$a_1 Tv + \cdots + a_{n-1} T^{n-1} v = 0.$$

Now apply T^{n-2}, giving $a_1 T^{n-1} v = 0$, which similarly forces $a_1 = 0$. Continuing in this fashion, we show that each a_i is 0, so the list is independent, hence a basis for V. To find the matrix for T in this basis, we apply T to each vector in the list and write in terms of that same list:

$$\begin{aligned} Tv &= 0v + 1Tv + 0T^2 v + \cdots \\ T(Tv) &= 0v + 0Tv + 1T^2 v + 0T^3 v + \cdots \\ T(T^2 v) &= 0v + 0Tv + 0T^2 v + 1T^3 v + 0T^4 v + \cdots \\ &\vdots \end{aligned}$$

From this we see that the matrix of T in this basis is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$