MATH 110, SUMMER 2013
PRACTICE FINAL EXAM

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(1) Find a linear map $T\colon P_2(\mathbb{F}) \to P_1(\mathbb{F})$ whose null space is the set of polynomials such that $p(1) = 0$ and whose range is the constant polynomials in $P_1(\mathbb{F})$.

(2) Let V be a complex vector space and define a map $\operatorname{Tr}\colon \mathcal{L}(V) \to \mathbb{C}$ by $\operatorname{Tr} T =$ the sum of the eigenvalues of T. Prove that Tr is surjective. Prove that if T is injective, then V must be one-dimensional.

(3) Consider the operator $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax - y \\ y - ax \\ az \end{pmatrix}$ on $\mathbb{R}^3$, where a is some unspecified real number. Prove that $\left\{ \begin{pmatrix} x \\ ax \\ 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\} \subseteq \operatorname{Null} T$. For which a is this *not* an equality?

(4) Fill in the blanks in the following table with YES or NO, according to whether the given map has the various properties. You do not need to justify your answers.

Operator	Normal?	Self-Adjoint?	Isometry?
A reflection in $\mathbb{R}^2$			
$T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$			
Rotation through $\pi/2$ around the x-axis in $\mathbb{R}^3$			
Orthogonal projection onto the plane $x + y + z = 0$ in $\mathbb{R}^3$			

(5) Let $T \in \mathcal{L}(V)$ be a normal operator, all of whose eigenvalues are *purely imaginary* (meaning they are of the form ci for some $c \in \mathbb{R}$). Prove that $T = -T^*$.

(6) Find the matrix of the map $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \end{pmatrix}$ with respect to the basis $B = (e_1 + e_2, e_1 - e_2)$ for $\mathbb{R}^2$. Use this matrix to say what the eigenvalues and eigenvectors of T are.

(7) Suppose T is normal and $T^3 = T^2$. Prove that $T^2 = T$. Is it also true that $T = I$? Prove it or give a counterexample.

(8) Find a basis for $\mathbb{R}^2$ with respect to which the matrix for the map $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x + 2y \end{pmatrix}$ is in Jordan normal form. Bonus: Use this to calculate T^n, for any $n > 0$. Double bonus: use this to calculate T^n, with a PICTURE!

(9) If $T \in \mathcal{L}(V)$ is self-adjoint, prove that $\operatorname{Null} T \perp \operatorname{Range} T$.

(10) Let $\dim V = n$ and $T \in \mathcal{L}(V)$. Assuming that $T^n = 0$ (the zero map), but $T^{n-1} \neq 0$, prove that there exists a vector $v \in V$ such that $(v, Tv, \ldots, T^{n-1}v)$ is a basis for V. Compute the matrix for T with respect to this basis.