MATH 110 WORKSHEET, JULY 3RD - SOLUTION

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(1) (Axler 2.10) If V is an n-dimensional space, prove that there exist one-dimensional subspaces $U_1, \ldots, U_n$ such that

$$V = U_1 \oplus \cdots \oplus U_n$$

Proof: [Basic idea: we have to construct these subspaces - they should each be the span of one of the basis vectors]

Let $(v_1, \ldots, v_n)$ be a basis for V, and set, for each $i = 1, \ldots, n$, $U_i = \mathrm{Span}(v_i)$. Then each U_i is one-dimensional (it has v_i as a basis), and we must check that $V = U_1 \oplus \cdots \oplus U_n$. To show it's a sum, pick any v in V, and write it in terms of the basis: $v = c_1 v_1 + \cdots + c_n v_n$. Since each $c_i v_i$ is in U_i, this shows that $V = U_1 + \cdots + U_n$. To show the sum is direct, suppose that we can write $0 = u_1 + \cdots + u_n$, with $u_i \in U_i$. We must show each $u_i = 0$. But each $u_i = c_i v_i$ for some $c_i \in \mathbb{F}$, by the definition of the subspaces U_i. Then $0 = c_1 v_1 + \cdots + c_n v_n$ and independence of the v_i forces each $c_i = 0$, so all $u_i = 0$, and we're done.

(2) True or false: If $V = U \oplus W$, and (v_1, v_2, v_3) is a basis for V, and (v_1, v_2) is a basis for U, then (v_3) must be a basis for W.

Solution: This is false. For example, let $V = \mathbb{R}^3$, and take U to be the xy-plane, and W the z-axis. Then we certainly have $V = U \oplus W$. But consider the basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for $\mathbb{R}^3$. It's true that the first two form a basis for U, but the third vector isn't even in W! So it definitely can't be a basis for W.

(3) Consider $V = \mathbb{C}^2$ as a real vector space. We saw yesterday that it is 4-dimensional as a real vector space. Let

$$U = \left\{ \begin{pmatrix} a \\ bi \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

(a) Prove that U is a subspace of V.

Proof: Taking $a, b = 0$ shows that the zero vector is in U. Now let $\begin{pmatrix} a \\ bi \end{pmatrix}, \begin{pmatrix} c \\ di \end{pmatrix}$ be two elements in U. Their sum can be written as $\begin{pmatrix} a + c \\ (b+d)i \end{pmatrix}$, so it's in U also. Similarly, if we take a (real) scalar x, we have $x \begin{pmatrix} a \\ bi \end{pmatrix} = \begin{pmatrix} ax \\ (bx)i \end{pmatrix}$, and since ax and bx are both real, this is still in U. So U satisfies the subspace test.

(b) Find the dimension of U by finding a basis for U.

Solution: The most obvious basis for U is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix}$, but of course there are (infinitely many) other choices. In any case, it shows that the dimension of U is two.

(c) If we consider $\mathbb{C}^2$ as a 2-dimensional complex vector space instead (i.e., now we allow complex numbers for scaling), is the set U a subspace? If so, what is its dimension?

Solution: If we use complex scalars, U is no longer a subspace! The reason is that scaling a vector in U by an imaginary number causes it to leave U, so U is not closed under complex scaling. For a concrete example, $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is in U, but if we scale it by i, we get $\begin{pmatrix} i \\ -1 \end{pmatrix}$, which no longer satisfies the condition to be in U. Note that since U is not a subspace, it doesn't even make sense to speak of its dimension.