SOLUTION

MATH 110 WORKSHEET, AUGUST 7TH

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(1) For each of the following operators, say whether it is normal, self-adjoint, or an isometry, or any combination of these. Where possible, try to understand the map geometrically.

(a) T on $\mathbb{R}^3$ given by the matrix $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (in standard bases).

(b) T on $\mathbb{R}^3$ given by the following rule: reflect a vector across the xy-plane, and then multiply it by 2.

(c) T on $\mathbb{R}^2$ given by the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ (your answer will depend on a and b - say for which values of a and b it has the properties.

(2) Let $T\colon V \to W$ be a linear map and $\dim V = n$, $\dim W = m$. Prove that $\dim \operatorname{Null} T - \dim \operatorname{Null} T^* = n - m$.

(3) **Definition:** If $S, T \in \mathcal{L}(V)$, we say S is a *square root* of T if $S^2 = T$.

(a) Find a square root of the identity operator on $\mathbb{R}^2$.

(b) Prove that the 2×2 zero matrix has infinitely many square roots.

(c) Prove that any normal operator on a complex space has a square root. [hint: use the spectral theorem]

(4) Let $T \in \mathcal{L}(V)$, with V a real vector space. Suppose T is ~~unitary~~ an isometry, self-adjoint, and has positive eigenvalues. Prove that T is the identity map on V.

(5) Challenge: Prove that every normal operator on a complex space is a linear combination of orthogonal projection operators.

① (a) It is not self-adjoint, since it's not equal to its transpose.

Since $TT^* = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = I$, T is an isometry, hence also normal.

(b) The matrix for T is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, so it's self adjoint (hence also normal), but not an isometry.

(c) $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

So T is normal for all a,b.

T is self-adjoint iff $b=0$.

T is an isometry iff $a^2+b^2=1$.

(2) We know $\text{Null}\, T^* = (\text{Range}\, T)^\perp$, and

$$\dim (\text{Range}\, T)^\perp = \dim W - \dim \text{Range}\, T$$

so

$$\begin{aligned}
\dim \text{Null}\, T - \dim \text{Null}\, T^* &= \dim \text{Null}\, T - (m - \dim \text{Range}\, T) \\
&= \dim \text{Null}\, T + \dim \text{Range}\, T - m \\
&= \dim V - m \quad \text{(by rank-nullity)} \\
&= n - m
\end{aligned}$$

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(3)
(a) The ~~matrix~~ operator $-I$ is a square root of I, on any space V

(b) Any matrix $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ has $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and since there are infinitely many choices for $a \in \mathbb{R}$, we get infinitely many square roots.

(c) Let T be normal. Pick ONB of e-vectors (possible by spectral thm), call it B.

Then $M(T,B) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ Since $\lambda_i \in \mathbb{C}$, each λ_i has a square root, $\sqrt{\lambda_i}$. (Indeed, it has 2, unless $\lambda_i = 0$).

Then the matrix $\begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$ is a square root of $M(T,B)$,

so the operator S defined by $Sv_i = \sqrt{\lambda_i}\, v_i$ is a square root of T.

(4) Since T is an isometry, $T^*T = I$

Since T is self-adjoint, $T^* = T$,

so $T^2 = I$.

Therefore the only e-vals of T are ± 1.

Since T has only positive e-vals, the only possible e-val is $\lambda = 1$.

By the real spectral thm, since T is self-adjoint, it is orthogonally diagonalizable, so there is a basis with respect to which the matrix of T is

$$\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = I$$

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so $T = I$.

(5) By spectral thm, since T is normal, can find OND of e-vectors $(v_1, \dots, v_n)$ w/ e-vals $\lambda_1, \dots, \lambda_n$.

Let $U_i = \mathrm{Span}(v_i)$. Then $V = U_1 \oplus \cdots \oplus U_n$,

and $U_i \perp U_j$ (for $i \neq j$).

Let $P_i : V \to V$ be the orthogonal projection onto U_i,

and consider the operator $P = \lambda_1 P_1 + \cdots + \lambda_n P_n$. ← a linear combination of orth. projections

Then $Pv_i = \lambda_1 P_1 v_i + \cdots + \lambda_i P_i v_i + \cdots + \lambda_n P_n v_i$

$= \lambda_i v_i$ since $v_i \in U_i$ and $v_i \in U_j^\perp$ for $j \neq i$.

But $Tv_i = \lambda_i v_i$, so T and P do the same thing to each of the basis vectors, hence are the same map, by linearity. ✓