(1) For each of the following operators, say whether it is normal, self-adjoint, or an isometry, or any combination of these. Where possible, try to understand the map geometrically.

(a) $T$ on $\mathbb{R}^3$ given by the matrix
\[
\begin{pmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}
\]
(in standard bases).

(b) $T$ on $\mathbb{R}^3$ given by the following rule: reflect a vector across the $xy$-plane, and then multiply it by 2.

(c) $T$ on $\mathbb{R}^2$ given by the matrix
\[
\begin{pmatrix}
a & -b \\
b & a \\
\end{pmatrix}
\]
your answer will depend on $a$ and $b$ - say for which values of $a$ and $b$ it has the properties.

(2) Let $T: V \to W$ be a linear map and $\dim V = n$, $\dim W = m$. Prove that $\dim \text{null } T - \dim \text{null } T^* = n - m$.

(3) Definition: If $S, T \in \mathcal{L}(V)$, we say $S$ is a square root of $T$ if $S^2 = T$.

(a) Find a square root of the identity operator on $\mathbb{R}^2$.

(b) Prove that the $2 \times 2$ zero matrix has infinitely many square roots.

(c) Prove that any normal operator on a complex space has a square root. [hint: use the spectral theorem]

(4) Let $T \in \mathcal{L}(V)$, with $V$ a real vector space. Suppose $T$ is symmetric, self-adjoint, and has positive eigenvalues. Prove that $T$ is the identity map on $V$.

(5) Challenge: Prove that every normal operator on a complex space is a linear combination of orthogonal projection operators.

\[\textbf{1} (a) \text{ It is not self-adjoint, since it's not equal to its transpose.} \]
\[
\text{Since } T T^* = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} = I, \text{ T is an isometry, hence also normal.}\
\]

\[\textbf{b) The matrix for T is } \begin{pmatrix}
2 & 0 \\
0 & 2 \\
\end{pmatrix}, \text{ so it's self-adjoint (hence also normal), but not an isometry.} \]

\[\textbf{c) } (a-b)(a+b) = (a^2 + b^2)(a-b) = (-b)(a)(b) \\
\]
\[
\text{So } T \text{ is normal for all } a, b. \\
\text{ T is self-adjoint iff } b = 0. \\
\text{ T is an isometry iff } a^2 + b^2 = 1. \]
(2) We know \( \text{Null } T^* = (\text{Range } T)^\perp \), and
\[
\dim (\text{Range } T)^\perp = \dim W - \dim \text{Range } T
\]
so
\[
\dim \text{Null } T - \dim \text{Null } T^* = \dim \text{Null } T - (m - \dim \text{Range } T)
\]
\[
= \dim \text{Null } T - \dim \text{Range } T - m
\]
\[
= (\dim \text{Null } T + \dim \text{Range } T) - m
\]
\[
= \dim V - m \quad \text{(by rank-nullity)}
\]
\[
= n - m
\]

(3) (a) The operator \( -I \) is a square root of \( I \), on any space \( V \).

(b) Any matrix \( \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \) has
\[
\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix},
\]
and since there are infinitely many choices for \( a, b \in \mathbb{R} \), we get infinitely many square roots.

(c) Let \( T \) be normal. Pick \( n \) orthonormal e-vectors (possible by spectral thm), call \( gT = B \),
\[
M(T, B) = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}
\]
since \( \lambda_i \in \mathbb{C} \),

Then the matrix
\[
\begin{pmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix}
\]
is a square root of 
\[
M(T, B),
\]
so the operator \( S \) defined by \( S v_i = \sqrt{\lambda_i} v_i \) is a square root of \( T \).
Since $T$ is an isometry, $T^*T = I$.

Since $T$ is self-adjoint, $T^2 = T$.

So $T^2 = I$.

Therefore the only eigenvalues of $T$ are $\pm 1$.

Since $T$ has only positive eigenvalues, the only possible eigenvalue $\lambda = 1$.

By the real spectral theorem, since $T$ is self-adjoint, it is orthogonally diagonalizable, so there is a basis with respect to which the matrix of $T$ is

$$
\begin{pmatrix}
\lambda & 0 \\
0 & \ddots \\
0 & 0 & \lambda
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 \\
0 \\
\vdots
\end{pmatrix}
$$

So $T = I$.  \(\checkmark\)

By spectral theorem, since $T$ is normal, can find ONB

of eigenvectors $(v_1, \ldots, v_n)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Let $U_i = \text{Span}(v_i)$. Then $V = U_1 \oplus \cdots \oplus U_n$.

and $U_i \perp U_j$ (for $i \neq j$).

Let $P_i : V \to V$ be the orthogonal projection onto $U_i$.

and consider the operator $P = \lambda_1 P_1 + \cdots + \lambda_n P_n$.

Then $Pv_i = \lambda_1 P_1 v_i + \cdots + \lambda_n P_n v_i = \lambda_i v_i$ since $v_i \in U_i$ and $v_i \perp U_j \oplus U_j$ for $j \neq i$.

But $Tv_i = \lambda_i v_i$, so $T$ and $P$ do the same thing to each of the basis vectors, hence are the same map, by linearity. \(\checkmark\)