MATH 110 WORKSHEET, AUGUST 7TH

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(1) For each of the following operators, say whether it is normal, self-adjoint, or an isometry, or any combination of these. Where possible, try to understand the map geometrically.

  (a) T on $\mathbb{R}^3$ given by the matrix $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (in standard bases).

  (b) T on $\mathbb{R}^3$ given by the following rule: reflect a vector across the xy-plane, and then multiply it by 2.

  (c) T on $\mathbb{R}^2$ given by the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ (your answer will depend on a and b - say for which values of a and b it has the properties.

(2) Let $T\colon V \to W$ be a linear map and $\dim V = n$, $\dim W = m$. Prove that $\dim \operatorname{Null} T - \dim \operatorname{Null} T^* = n - m$.

(3) **Definition:** If $S, T \in \mathcal{L}(V)$, we say S is a *square root* of T if $S^2 = T$.

  (a) Find a square root of the identity operator on $\mathbb{R}^2$.

  (b) Prove that the 2×2 zero matrix has infinitely many square roots.

  (c) Prove that any normal operator on a complex space has a square root. [hint: use the spectral theorem]

(4) Let $T \in \mathcal{L}(V)$, with V a real vector space. Suppose T is unitary, self-adjoint, and has positive eigenvalues. Prove that T is the identity map on V.

(5) Challenge: Prove that every normal operator on a complex space is a linear combination of orthogonal projection operators.