MATH 110 WORKSHEET, AUGUST 5TH

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(1) Consider the functional ϕ on $P_2(\mathbb{R})$ which takes a polynomial $p(x)$ to the real number $\int_0^1 p(x)\,dx$. Find a "representing vector" for this functional, i.e., a vector $q(x)$ such that $\phi(p(x)) = \langle p(x), q(x)\rangle$ for all $p(x)$. Use the inner product $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$.

(2) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ be the map $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+2z \\ z+x \\ y \end{pmatrix}$. Find a formula for T^*. Both spaces are given the standard (Euclidean) inner product.

(3) Let $T \in \mathcal{L}(V)$. Prove that $\operatorname{Null} T^* = (\operatorname{Range} T)^\perp$. (it's in the book if you get stuck...)

(4) Let $R\colon V \to V$ be a linear map such that $R^2 = I$ (R is a "reflection").
   (a) Show that you can find subspaces U and W of V such that $V = U \oplus W$ and $R\big|_U = I$, $R\big|_W = -I$.
   (b) Show that U is orthogonal to W if and only if $R = R^*$.

① We want a $q(x)$ s.t. $\underbrace{\int_0^1 p(x)\,dx}_{\text{functional}} = \underbrace{\int_0^1 p(x)q(x)\,dx}_{\text{inner product.}}$

Just take $q(x) = 1$!

Alternatively, find an ONB for $P_2(\mathbb{R})$, call it v_1, v_2, v_3. Then from the proof of the representation thm, we know we should choose $q(x) = \phi(v_1)v_1 + \phi(v_2)v_2 + \phi(v_3)v_3$

(no complex conjugates required since we work over $\mathbb{R}$ here).

② We want $\left\langle \begin{pmatrix} x+y \\ y+2z \\ z+x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, T^*\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\rangle$

$= (x+y)a + (y+2z)b + (z+x)c + yd$

$= x(a+c) + y(a+b+d) + z(2b+c)$

$= \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a+c \\ a+b+d \\ 2b+c \end{pmatrix} \right\rangle$

$\Rightarrow T^*\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ a+b+d \\ 2b+c \end{pmatrix}$

(2) (cont)

Or, (alternate solution)

matrix of T (in standard bases)

is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

so matrix of T^* is its transpose (no conjugate required since over $\mathbb{R}$)

i.e., T^* given by $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$.

(3) Must show: everything in Null T^* $\perp$ to everything in Range T.

pick $u \in \text{Null } T^*$, $v \in \text{Range } T$, so $v = Tw$

$\langle u, v \rangle = \langle u, Tw \rangle = \langle T^* u, w \rangle = 0$ since $u \in \text{Null } T^*$

(4) (a) Just set $U = \{u \in V \mid Ru = u\}$

$W = \{w \in W \mid Rw = -w\}$

Then $R|_U = I_U$ and $R|_W = -I_W$ by def of U and W.

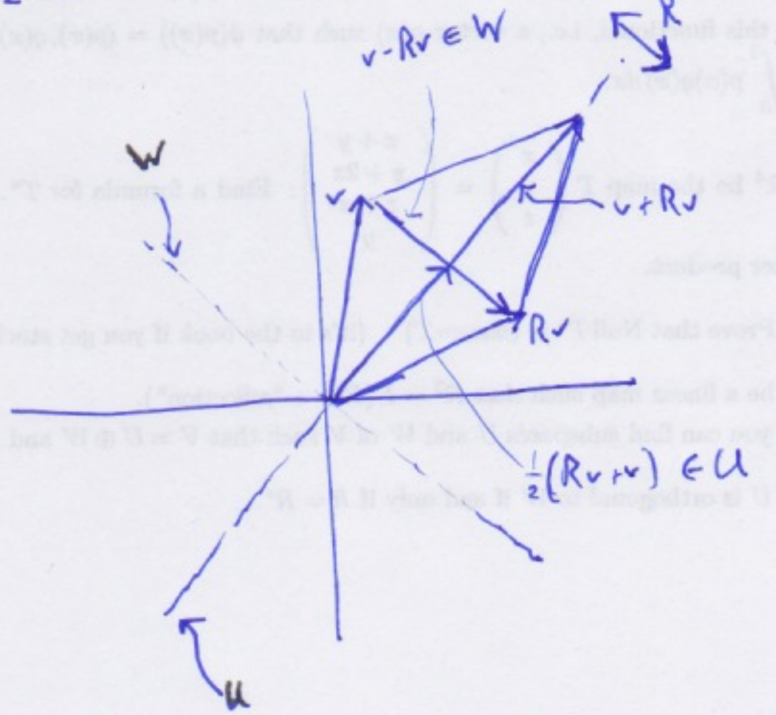
But why is $V = U \oplus W$?

(next page...)

Trick: pick $v \in V$.

Write $v = \frac{1}{2}(Rv + v) + \frac{1}{2}(v - Rv)$

Idea:



We must check:

$\frac{1}{2}(Rv + v) \in U$ and ~~$\frac{1}{2}(Rv + v)$~~ $\frac{1}{2}(v - Rv) \in W$

$$R\left(\tfrac{1}{2}(Rv + v)\right) = \tfrac{1}{2}(R^2 v + Rv) = \tfrac{1}{2}(v + Rv) \Rightarrow \tfrac{1}{2}Rv + v \in U \checkmark$$

$$R\left(\tfrac{1}{2}(v - Rv)\right) = \tfrac{1}{2}(Rv - R^2 v) = \tfrac{1}{2}(Rv - v) = -\tfrac{1}{2}(v - Rv) \Rightarrow \tfrac{1}{2}(v - Rv) \in W$$

This shows $V = U + W$.

To see $U \cap W = 0$, pick $v \in U \cap W$. Then $Rv = v$ since $v \in U$, and $Rv = -v$ since $v \in W$, so $v = -v$ hence $v = 0$ ✓

Thus $V = U \oplus W$

(4) (b) First, assume that $R = R^*$

To show $U \perp W$, pick $u \in U$, $w \in W$. We show $\langle u, w \rangle = 0$

note that $u = Ru$, $w = -Rw$

Then $\langle u, w \rangle = \langle Ru, w \rangle = \langle u, R^* w \rangle = \langle u, Rw \rangle$
$= \langle u, -w \rangle$
$= -\langle u, w \rangle$

so $2\langle u, w \rangle = 0$ hence $\langle u, w \rangle = 0$.

Next, assume $U \perp W$.

Weird way to show $R = R^*$: $R^2 = I$ implies that R is invertible and $R^{-1} = R$. Since inverses are unique, if we can show that R^* is also inverse to R, then R must be equal to R^*.

To show $R^* R = I$, pick $v_1, v_2 \in V$. We'll show $\langle v_1, v_2 \rangle = \langle v_1, R^* R v_2 \rangle$. Since v_1, v_2 are arbitrary, this means $R^* R = I$. So,.... write $v_1 = u_1 + w_1$, $v_2 = u_2 + w_2$, $u_i \in U$, $w_i \in W$

note: $Rv_i = u_i - w_i$

$\langle v_1, R^* R v_2 \rangle = \langle Rv_1, Rv_2 \rangle$
$= \langle u_1 - w_1, u_2 - w_2 \rangle$
$= \langle u_1, u_2 \rangle - \langle w_1, u_2 \rangle - \langle u_1, w_2 \rangle + \langle w_1, w_2 \rangle$
$= \langle u_1, u_2 \rangle + \langle w_1, u_2 \rangle + \langle u_1, w_2 \rangle + \langle w_1, w_2 \rangle$
$= \langle u_1 + w_1, u_2 + w_2 \rangle$
$= \langle v_1, v_2 \rangle$. Done!

Since $U \perp W$, so $\langle u_i, w_j \rangle = 0 = -0 = -\langle u_i, w_j \rangle$