Math 110, Summer 2013
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Homework 7 Solution

(1) For each of the following maps T, compute the adjoint map T^*, and say whether T is (i) normal, (ii) self-adjoint, and (ii) an isometry.

(a) The "horizontal shear" given by the matrix (in the standard basis) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(b) A rotation through $\pi/4$ around the y-axis in $\mathbb{R}^3$ (it doesn't matter the rotation is clockwise or counterclockwise - it won't affect your answers).

Solution:

(a) This matrix is not symmetric, so it's not normal. Therefore it cannot be either self-adjoint or an isometry.

(b) This is an isometry - it preserves lengths (and angles). Therefore it is normal as well. It is not self adjoint, as can be seen by looking at the rotation matrix - it's not symmetric.

(2) If T is a normal operator on a complex inner product space V, and $(T^*)^n$ (meaning the adjoint of T composed with itself n times) is invertible for some $n > 1$, prove that T is invertible also. [hint: it's fastest to use the spectral theorem] Is this result true for operators that are not normal? Explain why or give a counterexample.

Solution: Set $k = \dim V$ (n is taken already). Pick an ONB for V and write the matrices for T and T^* in this basis:

$$T = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix}, \quad T^* = \begin{pmatrix} \overline{\lambda}_1 & 0 & \cdots & 0 \\ 0 & \overline{\lambda}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\lambda}_k \end{pmatrix},$$

Now if $(T^*)^n$ is invertible, then for each $i = 1, \ldots, k$, $\overline{\lambda}_i^n \neq 0$, hence each $\overline{\lambda}_i \neq 0$, hence each $\lambda_i \neq 0$, so T is invertible.

The result is actually true for all operators! The eigenvalues of $(T^*)^n$ are $\{\lambda^n \mid \overline{\lambda} \text{ is an eigenvalue of } T\}$. By our assumption, since $(T^*)^n$ is invertible, none of these numbers are zero, therefore no $\overline{\lambda} = 0$, therefore T is also invertible.

(3) (Axler 7.7) If $T \in \mathcal{L}(V)$ is normal, prove that $\operatorname{Null} T^k = \operatorname{Null} T$ and $\operatorname{Range} T^k = \operatorname{Range} T$ for all $k > 0$.

Solution: Let T be normal. We can by the spectral theorem pick an ONB $(v_1, \ldots, v_n)$ with respect to which the matrix of T looks like

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Then $\operatorname{Null} T = \operatorname{Span}\{v_i \mid \lambda_i = 0\}$. Similarly, the matrix for T^k in this same basis is

$$\begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix},$$

so $\operatorname{Null} T^k = \operatorname{Span}\{v_i \mid \lambda_i^k = 0\}$. But $\lambda_i^k = 0$ iff $\lambda_i = 0$, so

$$\operatorname{Null} T = \operatorname{Span}\{v_i \mid \lambda_i = 0\} = \operatorname{Span}\{v_i \mid \lambda_i^k = 0\} = \operatorname{Null} T^k.$$

By looking at the same matrices,

$$\operatorname{Range} T^k = \operatorname{Span}\{v_i \mid \lambda_i^k \neq 0\} = \operatorname{Span}\{v_i \mid \lambda_i \neq 0\} = \operatorname{Null} T.$$

(4) Let T be an operator which is self-adjoint and nilpotent. Prove that T is the zero map. Give an example to show that this is not necessarily true if T is not self-adjoint.

Solution: Since T is self-adjoint, whether we work over $\mathbb{R}$ or $\mathbb{C}$, by the spectral theorem, T can be expressed as a diagonal matrix (using a suitable ONB), with its eigenvalues along the diagonal. But the only eigenvalue of a nilpotent operator is $\lambda = 0$ (proof: say $T^n = 0$, then $T^n v = \lambda^n v = 0$ for some nonzero v, so $\lambda^n = 0$, so $\lambda = 0$). Thus the matrix for T in this ONB is the zero matrix, hence T is the zero map.

For a counterexample in the non-self-adjoint case, let T be an operator on $\mathbb{R}^2$ whose matrix in the standard basis is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T^2 = 0$ but T is not the zero map.

(5) Find the Jordan normal form of the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -1/3 \\ 0 & 3/2 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

Solution: As a first step, we need to find the eigenvalues. It certainly seems like they're just 1 and 5, but we can't be sure without further computation. For now, let's just run with these two and if they don't produce enough generalized eigenvectors, then we'll know we were missing an eigenvalue somewhere and come back for it. If you compute the matrix $A - I$, you find it has a 1D nullspace, and that all successive powers of it do also (basically, you can't get rid of those powers of 4 on the diagonal). So there is only one-dimension worth of generalized eigenvectors for $\lambda = 1$, and they're actually eigenvectors. Choose e_1 as our basis. Now move on to $A - 5I$. Its null space is spanned by e_3. We want a basis for $\mathbb{R}^4$, so compute $(A - 5I)^2$ and look in its nullspace, which we find is spanned by e_2 and e_3 (e_3 was already in there from before. So we move on and look in the nullspace of $(A - 5I)^3$, which is spanned by e_2, e_3, e_4. To get the Jordan basis, we need a chain of vectors, starting with something in $\operatorname{Null}(A - 5I)^3$ that is not in $\operatorname{Null}(A - 5I)^2$ - take e_4. The next vector in the chain must be $(A - 5I)e_4$, and the third vector in the chain is $(A - 5I)^2 e_4$. This chain of three vectors (ending in an eigenvector) gives rise to a 3×3 Jordan block. We also have a 1×1 block from $\lambda = 1$. In particular, we are not missing any eigenvalues since these two blocks will give us a 4×4 matrix, namely

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

note that you could have reordered the blocks too, for a correct answer.

(6) Let T be an operator on an 8-dimensional space with three distinct eigenvalues $\lambda = 1, 2, 3$, and assume given the following data

$$\begin{aligned} \dim \operatorname{Null}(T - I)^2 &= 3 \\ \dim \operatorname{Null}(T - I)^3 &= 4 \\ \dim \operatorname{Null}(T - 2I)^2 &= 2 \\ \dim \operatorname{Null}(T - 3I) &= 2 \end{aligned}$$

What are the possible Jordan normal forms for T? (don't count different reordering of the blocks)

Solution: Let me abbreviate "independent generalized eigenvectors" by "GEs". From the given information, we know there are ≥ 2 GEs for $\lambda = 2$ and 3, and ≥ 4 GEs for $\lambda = 1$. Since these numbers add up to 8, they must be exact. For $\lambda = 3$, both are actually eigenvectors, so we get 2 1×1 blocks. For $\lambda = 2$, we cannot tell whether it is a 2D eigenspace, or a 1D eigenspace with one extra GE, so there could be two 1×1 blocks or one 2×2 block. For $\lambda = 1$, we cannot have a 3D eigenspace, since the chain would stop growing after $(T - I)^2$, which it doesn't. We cannot have a 1D eigenspace, either, since then the dimension would jump up by two when going from $\operatorname{Null} T - I$ to $\operatorname{Null}(T - I)^2$, and that can't happen coming from a 1D eigenspace. So we must have $\dim \operatorname{Null}(T - I) = 2$. Thus we have a 1×1 block, and a 3×3 block for $\lambda = 1$.

Summarizing: up to reordering of the blocks, there are two possible Jordan normal forms:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

(7) Let J be an $n \times n$ matrix which is in Jordan normal form. Prove that the maximum number of independent eigenvectors of A is $n-k$, where k is the number of 1s appearing just above the diagonal.

Solution: For each column a_i *without* a 1 above the diagonal, the vector e_i is an eigenvector, as we have seen in class. Since there are $n - k$ such columns, this shows that there are at least $n - k$ independent eigenvectors. The harder part is to show that there are at most $n-k$ independent eigenvectors. For this, we will prove that the Jordan normal form is essentially unique (up to reordering of the blocks). From our construction in class, the number and size of the blocks are determined by the dimensions of the various spaces $\operatorname{Null}(A-\lambda I)^j$. These spaces do not depend on a choice of basis. Consequently, if we have $J' = SJS^{-1}$, where J' is another matrix in Jordan normal form, then J and J' both represent the same map, but with respect to different bases. Specifically, J is the matrix with respect to the standard basis, and J' is the matrix for the same map with respect to the basis consisting of the columns of S. So the dimensions of the nullspaces mentioned above will be the same, and hence the number and sizes of the blocks, along with the eigenvalues along the diagonal, will be the same. So J and J' will be the same matrix, up to reordering of the blocks. Consequently, we cannot have more than $n-k$ eigenvectors, for then we would have another, distinct, Jordan form matrix for J which we just showed was impossible.

(8) Let A be a $n \times n$ matrix whose only eigenvalue is 4. If the matrix S is such that $A = SJS^{-1}$, where J is in Jordan normal form, prove that the columns of S must be generalized eigenvectors of A.

Solution: Let s_i be a column of S. To show S is a generalized eigenvector of A, we must show that $s_i \in \operatorname{Null}(A-4I)^k$, and in fact, we can just take $k = n$, by a result from class. So now compute

$$A - 4I = SJS^{-1} - 4I = SJS^{-1} - S(4I)S^{-1} = S(J - 4I)S^{-1},$$

so

$$(A - 4I)^n = S(J - 4I)^n S^{-1},$$

which is equivalent to

$$(A - 4I)^n S = S(J - 4I)^n$$

Now $(A - 4I)^n s_i$ is simply the ith column of the product on the left hand side of this equation. So all we have to do is show that the ith column of the matrix $S(J - 4I)^n$ is zero. But this follows since $(J - 4I)^n$ is the zero matrix! Reason: $J - 4I$ is an $n \times n$ upper triangular matrix with zeroes on the diagonal.