Math 110, Summer 2013
Instructor: James McIvor
Homework 7
Due Wednesday, August 14th

(1) For each of the following maps T, compute the adjoint map T^*, and say whether T is (i) normal, (ii) self-adjoint, and (ii) an isometry.

(a) The "horizontal shear" given by the matrix (in the standard basis) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(b) A rotation through $\pi/4$ around the y-axis in $\mathbb{R}^3$ (it doesn't matter the rotation is clockwise or counterclockwise - it won't affect your answers).

(2) If T is a normal operator on a complex inner product space V, and $(T^*)^n$ (meaning the adjoint of T composed with itself n times) is invertible for some $n > 1$, prove that T is invertible also. [hint: it's fastest to use the spectral theorem] Is this result true for operators that are not normal? Explain why or give a counterexample.

(3) (Axler 7.7) If $T \in \mathcal{L}(V)$ is normal, prove that $\operatorname{Null} T^k = \operatorname{Null} T$ and $\operatorname{Range} T^k = \operatorname{Range} T$ for all $k > 0$.

(4) Let T be an operator which is self-adjoint and nilpotent. Prove that T is the zero map. Give an example to show that this is not necessarily true if T is not self-adjoint.

(5) Find the Jordan normal form of the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -1/3 \\ 0 & 3/2 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

(6) Let T be an operator on an 8-dimensional space with three distinct eigenvalues $\lambda = 1, 2, 3$, and assume given the following data

$$\dim \operatorname{Null}(T - I)^2 = 3$$
$$\dim \operatorname{Null}(T - I)^3 = 4$$
$$\dim \operatorname{Null}(T - 2I)^2 = 2$$
$$\dim \operatorname{Null}(T - 3I) = 2$$

What are the possible Jordan normal forms for T? (don't count different reordering of the blocks)

(7) Let J be an $n \times n$ matrix which is in Jordan normal form. Prove that the maximum number of independent eigenvectors of A is $n-k$, where k is the number of 1s appearing just above the diagonal.

(8) Let A be a $n \times n$ matrix whose only eigenvalue is 4. If the matrix S is such that $A = SJS^{-1}$, where J is in Jordan normal form, prove that the columns of S must be generalized eigenvectors of A.