Math 110, Summer 2013
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Homework 6 Solution

(1) On your last midterm you proved that for a linear operator P on a finite-dimensional space V, the property $P^2 = P$ implies that $V = \operatorname{Null} P \oplus \operatorname{Range} P$. Now prove that if $P^2 = P$, then P is diagonalizable and its eigenvalues can only be 0 or 1.

Solution: Suppose λ is an eigenvalue of P, then $P^2 = P$ implies $\lambda^2 = \lambda$, so λ =0 or 1. We've seen that $E_0 = \operatorname{Null} P$. Also, $\operatorname{Range} P = E_1$, since if $v \in \operatorname{Range} P$ then $Pv = P^2 w$ for some w, but $P^2 w = Pw = v$, so $Pv = v$, hence $v \in E_1$. Conversely, if $v \in E_1$, then $Pv = v$, so $v \in \operatorname{Range} P$. Now, we are given that $V = \operatorname{Null} P \oplus \operatorname{Range} P = E_0 \oplus E_1$, so P is diagonalizable, by some theorem from a while ago.

(2) Let $V = \mathbb{R}^3$, and let U be the subspace $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x + y + z = 0 \right\}$.

(a) Find an orthonormal basis for $U^\perp$.

(b) Find a formula for the orthogonal projection P_U of V onto U. Your answer can either be a 3×3 matrix, or a formula of the form $P_U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$

Solution: U is 2D, so $U^\perp$ is 1D, so any nonzero vector in $Y^\perp$ will be a basis. The vector $(1, 1, 1)$ is in $U^\perp$. Now we just normalize it to get an orthonormal basis $u = \frac{1}{\sqrt{3}}(1, 1, 1)$ for $U^\perp$. For the second part, there are two ways to do it. One is to get an orthonormal basis for U and use that to project directly onto U. But since we already have an orthonormal basis for $U^\perp$, let's do the following. Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. We need a formula for $P_U v$. Well, the vector

$$w = \frac{\langle v, u \rangle}{\langle u, u \rangle} u = \langle v, u \rangle u = \frac{1}{3} \langle (x, y, z), (1, 1, 1) \rangle (1, 1, 1) = \frac{x + y + z}{3} (1, 1, 1)$$

is the orthogonal projection of v onto $U^\perp$, so $v - w$ is the orthogonal projection of v onto U. Thus

$$P_U v = v - w = (x, y, z) - \frac{x + y + z}{3} (1, 1, 1) = \frac{1}{3} \begin{pmatrix} 2x - y - z \\ 2y - x - z \\ 2z - x - y \end{pmatrix}.$$

(3) Let v_1, v_2, v_3 be vectors in $\mathbb{C}^3$, and A be the 3×3 matrix whose first column is v_1, second column is v_2, and third column is v_3. Prove that (v_1, v_2, v_3) is an orthonormal basis for $\mathbb{C}^3$ if and only if $A^{-1} = \overline{A}^T$, where the bar denotes complex conjugate, and the T denotes transpose, i.e., replacing the rows by the columns and vice versa.

Solution: The key observation is that

$$A\overline{A}^T = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle \\ \langle v_3, v_1 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_3 \rangle \end{pmatrix},$$

where here $\langle , \rangle$ denotes the standard hermitian inner product on $\mathbb{C}^3$. Now this matrix is equal to I if and only if $\langle v_i, v_j \rangle = 0$ if $i \neq j$ and 1 if $i = j$, which happens if and only if the v_i are orthonormal.

(4) (Axler 6.24) Find a polynomial q in $P_2(\mathbb{R})$ such that $p(1/2) = \int_0^1 p(x) q(x) \, dx$ for every $p \in P_2(\mathbb{R})$. [Hint: study the proof of 6.45, applying it to the functional $\phi(p(x)) = p(1/2)$.]

Solution: Use the orthonormal basis $e_1 = 1$, $e_2 = \sqrt{3}(2x - 1)$, $e_3 = \sqrt{5}(6x^2 - 6x + 1)$ found in problem 7 of HW5. Then the proof of the representation theorem tells us that the desired vector is (I have ignored the complex conjugate since we work over $\mathbb{R}$ in this problem):

$$e_1(1/2)e_1 + e_2(1/2)e_2 + e_3(1/2)e_3 = 1e_1 + 0e_2 + \sqrt{5}(6/4 - 6/2 + 1)e_3 = 1 - \frac{5}{2}(6x^2 - 6x + 1) = -15x^2 + 15x - 3/2$$

(5) (Axler 6.27) If $T \in \mathcal{L}(\mathbb{F}^n)$ is given by

$$T \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{pmatrix},$$

find a formula for the adjoint T^*.

Solution: We can work in the standard basis here - it's orthonormal, so easy to compute adjoints! The matrix for this map then becomes

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

so its adjoint is given by the matrix

$$A^T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

so the formula for T^* is

$$T \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ 0 \end{pmatrix}.$$