Math 110, Summer 2013
Instructor: James McIvor
Homework 6
Due Thursday, August 8th

(1) On your last midterm you proved that for a linear operator P on a finite-dimensional space V, the property $P^2 = P$ implies that $V = \operatorname{Null} P \oplus \operatorname{Range} P$. Now prove that if $P^2 = P$, then P is diagonalizable and its eigenvalues can only be 0 or 1.

(2) Let $V = \mathbb{R}^3$, and let U be the subspace $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x + y + z = 0 \right\}$.

(a) Find an orthonormal basis for $U^\perp$.

(b) Find a formula for the orthogonal projection P_U of V onto U. Your answer can either be a 3×3 matrix, or a formula of the form $P_U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$

(3) Let v_1, v_2, v_3 be vectors in $\mathbb{C}^3$, and A be the 3×3 matrix whose first column is v_1, second column is v_2, and third column is v_3. Prove that (v_1, v_2, v_3) is an orthonormal basis for $\mathbb{C}^3$ if and only if $A^{-1} = \overline{A}^T$, where the bar denotes complex conjugate, and the T denotes transpose, i.e., replacing the rows by the columns and vice versa.

Example: If $A = \begin{pmatrix} 1 & i & 2-i \\ 3+i & 0 & 2i \\ 1 & 1 & 2+2i \end{pmatrix}$, $\overline{A}^T = \begin{pmatrix} 1 & 3-i & 1 \\ -i & 0 & 1 \\ 2+i & -2i & 2-2i \end{pmatrix}$

(4) (Axler 6.24) Find a polynomial q in $P_2(\mathbb{R})$ such that $p(1/2) = \int_0^1 p(x)q(x)\,dx$ for every $p \in P_2(\mathbb{R})$. [Hint: study the proof of 6.45, applying it to the functional $\phi(p(x)) = p(1/2)$.]

(5) (Axler 6.27) If $T \in \mathcal{L}(\mathbb{F}^n)$ is given by

$$T \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{pmatrix},$$

find a formula for the adjoint T^*.