Math 110, Summer 2013
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Homework 5 Solution

(1) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Find a basis (v_1, v_2, v_3) for $\mathbb{R}^3$ with respect to which the matrix for T is upper triangular.

Solution: Note that e_2 is an eigenvector for T, so we can take this as the first vector in our basis, so $v_1 = e_2$. Then we can choose, for instance $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, which happens also to be an eigenvector: $Tv_2 = 2v_2$. In particular, $v_2 \in \text{Span}(v_1, v_2)$, which is all we need for upper-triangularity. Finally, v_3 can be any vector not in the span of (v_1, v_2) (it can't be in the span or else they wouldn't be independent). Take $v_3 = e_1$. Then $Tv_3 = v_2$. Thus the matrix of T with respect to this basis is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is upper-triangluar. You may have also just diagonalized T, which is an acceptable solution.

(2) In class we proved that any operator on a nonzero finite-dimensional vector space over $\mathbb{C}$ has at least one eigenvalue. Show that the finite-dimensionality is a necessary hypothesis by verifying that the operator $T \in \mathcal{L}(P(\mathbb{C}))$ given by $Tp(x) = xp(x)$ has no eigenvalues.

Proof: Suppose λ is a scalar, and $p(x) = a_n x^n + \cdots + a_1 x + a_0$. We will show that If $Tp(x) = \lambda p(x)$, then $p(x) = 0$, so that λ cannot be an eigenvalue. Since λ is arbitrary here, this will show there can be no eigenvalues. So suppose $Tp(x) = \lambda p(x)$ holds. Then

$$a_n x^{n+1} + a_{n-1}x^n + \cdots + a_1 x^2 + a_0 x = \lambda a_n x^n + \lambda a_{n-1} x^{n-1} + \cdots + \lambda a_1 x + \lambda a_0$$

Equating the coefficients of like powers gives the following equations:

$$\begin{aligned} a_n &= 0 \\ a_{n-1} &= \lambda a_n \\ &\vdots \\ a_0 &= \lambda a_1 \\ 0 &= \lambda a_0 \end{aligned}$$

These equations imply that each $a_i = 0$, so $p(x)$ is the zero polynomial, which is what we wanted to show.

(3) Let $T \in \mathcal{L}(V)$. Suppose that *every* nonzero vector in V is an eigenvector for T. Prove that T is a scalar multiple of the identity operator, i.e., $T = cI$ for some $c \in \mathbb{F}$.

Proof: We claim that there is only one eigenvalue for T. For if there are more than one, pick nonzero eigenvectors $u \in E_{\lambda_1}$ and $v \in E_{\lambda_2}$. Then since $u + v$ is itself an eigenvector with some eigenvalue λ, we have

$$\lambda u + \lambda v = \lambda(u + v) = T(u + v) = Tu + Tv = \lambda_1 u + \lambda_2 v,$$

and since u, v are independent (they come from distinct eigenspaces), it implies that $\lambda = \lambda_1 = \lambda_2$, a contradiction, since we assumed the λ_i distinct. So there is only one eigenvalue, just call it c. But then for any $v \in V$, since v is an eigenvector with eigenvalue c, we have $Tv = cv$, which means that T is the same map as cI.

(4) (This was pretty tough - sorry!) Let $v_1, \ldots, v_n$ be eigenvectors corresponding to *distinct* eigenvalues $\lambda_1, \ldots, \lambda_n$ of $T \in \mathcal{L}(V)$. Let W be a T-invariant subspace. Prove that if $v_1 + \cdots + v_n \in W$, then each $v_i \in W$.

Hints:

(a) Show W is also invariant under each $T - \lambda_i I$, for each i.
(b) Show that the operators $T - \lambda_i I$ and $T - \lambda_j I$ commute, i.e., you can switch their order.
(c) Finally, let $w = v_1 + \cdots + v_n$. If you want to show, for instance, that $v_1 \in W$, look at $(T - \lambda_2 I)(T - \lambda_3 I) \cdots (T - \lambda_n I)w$. It's in W by the invariance you proved in (a). But now actually calculate what vector it is - a multiple of v_1!

Proof: Following the hint, we first establish that W is also invariant under $T - \lambda I$, where λ is any scalar. Pick $w \in W$. Then $(T - \lambda I)w = Tw - \lambda w$, which is in W because Tw is (W is T-invariant), and λw is (W is closed under scaling.

Next we prove that $(T - \lambda_i I)(T - \lambda_j I) = (T - \lambda_j I)(T - \lambda_i I)$. This is because scalar multiples of the identity commute with all operators, so

$$(T-\lambda_i I)(T-\lambda_j I) = T\circ T - \lambda_i I \circ T - T \circ \lambda_j I + \lambda_i I \circ \lambda_j I = T \circ T - \lambda_j I \circ T - T \circ \lambda_i I + \lambda_j I \circ \lambda_i I = (T-\lambda_j I)(T-\lambda_i I)$$

which shows they commute.

Now we prove that

$$v_1 = \frac{1}{(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_n)}(T - \lambda_2 I)\cdots(T - \lambda_n I)w,$$

which shows that $v_1 \in W$. The other indices are similar.

Apply the operators $T - \lambda_n I, T - \lambda_{n-1} I, \ldots, T - \lambda_2 I$ to w. Since we can switch the order of the $T - \lambda_i I$'s, and each $v_i \in \operatorname{Null} T - \lambda_i I$, all terms except the one involving v_1 drop out, leaving

$$(T - \lambda_2 I)(T - \lambda_3 I)\cdots(T - \lambda_n I)w = (T - \lambda_2 I)(T - \lambda_3 I)\cdots(T - \lambda_n I)v_1 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_n)v_1$$

Now divide both sides by $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_n)$ (possible since all λ_i are distinct) and you get the equation claimed above, which finishes the proof.

(5) Let $T \colon V \to V$ be an operator with the property that for every orthonormal list $(v_1, \ldots, v_n)$ in V, the list $(Tv_1, \ldots, Tv_n)$ is orthonormal.

(a) Prove that $\|Tv\| = \|v\|$ for all $v \in V$.
Proof: If $v = 0$, we're done. If not, the vector $u = \frac{1}{\|v\|}v$ has $\|u\| = 1$. Now extend (u) to an orthonormal basis for V. Then the list $(Tu, \ldots)$ is orthonormal, so in particular $\|Tu\| = 1$, hence $\|Tv\| = \|(\|v\|)u\| = \|v\|\|u\| = \|v\|$. Done.

(b) Prove that if λ is an eigenvalue of T, then $|\lambda| = 1$.
Proof: For this it's probably easiest to use part (a). Let λ be an eigenvalue, so there is $v \neq 0$ such that $Tv = \lambda v$. Then since $\|Tv\| = \|v\|$, we get $\|v\| = \|Tv\| = \|\lambda v\| = |\lambda|\|v\|$, and dividing by $\|v\|$ (possible since $v \neq 0$) gives $|\lambda| = 1$.

(c) Perhaps using (a) for insight, give an example of such a map when $V = \mathbb{R}^2$ (do not use the identity map as your example).
Solution: In English, part (a) says that T is an operator which preserves lengths. Examples in $\mathbb{R}^2$ include rotations and reflections.

(6) (a) Prove that if $\langle u, v\rangle = 0$ for all $u \in V$, then v must be the zero vector.

Proof: Take $u = v$, then $\|v\|^2 = \langle v, v\rangle = 0$, so $\|v\| = 0$ hence $v = 0$.

(b) Prove that if $v, w \in V$ are such that $\langle u, v\rangle = \langle u, w\rangle$ for all $u \in V$, then $v = w$.

Proof: It is given that $\langle u, v\rangle - \langle u, w\rangle = \langle u, v - w\rangle = 0$. By (a), this means $v - w = 0$, so $v = w$.

(c) Prove the formula

$$\langle u, v\rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4},$$

when V is a real vector space.

Proof:

$$\begin{aligned}\|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle - (\langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle) \\ &= \langle u,v\rangle + \langle u,v\rangle + \langle u,v\rangle + \langle u,v\rangle = 4\langle u,v\rangle,\end{aligned}$$

which proves the formula after dividing by 4. Note that we used the symmetry property (as opposed to just conjugate symmetry) in going to the last line, so the additional assumption that V is over $\mathbb{R}$ was necessary.

(7) (Axler 6.10) Let

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$$

be our inner product on $P_2(\mathbb{R})$. Apply Gram-Schmidt to the basis $(1, x, x^2)$ to obtain an orthonormal basis for $P_2(\mathbb{R})$.

Solution: The first vector, 1, already has length 1, so we choose $e_1 = 1$.

Then set

$$e_2 = \frac{x - \langle x, 1\rangle 1}{\|x - \langle x, 1\rangle 1\|} = \frac{x - \frac{1}{2}}{\frac{1}{2\sqrt{3}}} = \sqrt{3}(2x-1)$$

and

$$e_3 = \frac{x^2 - \langle x^2, 1\rangle 1 - \langle x^2, \sqrt{3}(2x-1)\rangle\sqrt{3}(2x-1)}{\|x^2 - \langle x^2, 1\rangle 1 - \langle x^2, \sqrt{3}(2x-1)\rangle\sqrt{3}(2x-1)\|} = \frac{x^2 - \frac{1}{3} - \frac{\sqrt{3}}{6}\sqrt{3}(2x-1)}{\|x^2 - \frac{1}{3} - \frac{\sqrt{3}}{6}\sqrt{3}(2x-1)\|} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\frac{1}{180}}} = \sqrt{5}(6x^2 - 6x + 1)$$

This should convince you that orthonormal bases can be ugly in polynomial spaces!