Math 110, Summer 2013
Instructor: James McIvor
Homework 5
Due Wednesday, July 31st

(1) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Find a basis (v_1, v_2, v_3) for $\mathbb{R}^3$ with respect to which the matrix for T is upper triangular.

(2) In class we proved that any operator on a nonzero finite-dimensional vector space over $\mathbb{C}$ has at least one eigenvalue. Show that the finite-dimensionality is a necessary hypothesis by verifying that the operator $T \in \mathcal{L}(P(\mathbb{C}))$ given by $Tp(x) = xp(x)$ has no eigenvalues.

(3) Let $T \in \mathcal{L}(V)$. Suppose that *every* nonzero vector in V is an eigenvector for T. Prove that T is a scalar multiple of the identity operator, i.e., $T = cI$ for some $c \in \mathbb{F}$.

(4) Let $v_1, \ldots, v_n$ be eigenvectors corresponding to *distinct* eigenvalues $\lambda_1, \ldots, \lambda_n$ of $T \in \mathcal{L}(V)$. Let W be a T-invariant subspace. Prove that if $v_1 + \cdots + v_n \in W$, then each $v_i \in W$.

Hints:
(a) Show W is also invariant under each $T - \lambda_i I$, for each i.
(b) Show that the operators $T - \lambda_i I$ and $T - \lambda_j I$ commute, i.e., you can switch their order.
(c) Finally, let $w = v_1 + \cdots + v_n$. If you want to show, for instance, that $v_1 \in W$, look at $(T - \lambda_2 I)(T - \lambda_3 I) \cdots (T - \lambda_n I)w$. It's in W by the invariance you proved in (a). But now actually calculate what vector it is - a multiple of v_1!

(5) Let $T\colon V \to V$ be an operator with the property that for every orthonormal list $(v_1, \ldots, v_n)$ in V, the list $(Tv_1, \ldots, Tv_n)$ is orthonormal.
(a) Prove that $\|Tv\| = \|v\|$ for all $v \in V$.
(b) Prove that if λ is an eigenvalue of T, then $|\lambda| = 1$.
(c) Perhaps using (a) for insight, give an example of such a map when $V = \mathbb{R}^2$ (do not use the identity map as your example).

(6) (a) Prove that if $\langle u, v\rangle = 0$ for all $u \in V$, then v must be the zero vector.
(b) Prove that if $v, w \in V$ are such that $\langle u, v\rangle = \langle u, w\rangle$ for all $u \in V$, then $v = w$.
(c) Prove the formula

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4},$$

when V is a *real* vector space.

(7) (Axler 6.10) Let

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$$

be our inner product on $P_2(\mathbb{R})$. Apply Gram-Schmidt to the basis $(1, x, x^2)$ to obtain an orthonormal basis for $P_2(\mathbb{R})$.