Math 110, Summer 2013
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Homework 4 Solution

(1) Which of the following maps are isomorphisms? Explain why or why not.

(a) $T\colon P_3(\mathbb{F}) \to \mathbb{F}^3$ given by $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix}$.

Solution: This is not an isomorphism - there are NO isomorphisms between $P_3(\mathbb{F})$ and $\mathbb{F}^3$ since they do not have the same dimension.

(b) $T\colon \mathbb{F}^3 \to \mathbb{F}^3$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x-z \end{pmatrix}$.

Solution: This is not an isomorphism, since it's not injective. It has the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ in its null space.

(c) $T\colon P_n(\mathbb{F}) \to P_n(\mathbb{F})$ given by $T(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n) = (a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n)$. (note: there was a mistake here - it originally said $P(\mathbb{F})$ instead of $P_n(\mathbb{F})$, and the formula actually doesn't make sense if it's left as $P(\mathbb{F})$. Sorry for any confusion.)
Solution: This is an isomorphism. We saw in class that a map is an isomorphism if it sends a basis to a basis. This map sends the standard basis $(1, x, \ldots, x^n)$ to the list $(x^n, x^{n-1}, \ldots, 1)$, which is also a basis.

(d) $T\colon \mathcal{L}(\mathbb{F}, V) \to V$ given by, for $S \in \mathcal{L}(\mathbb{F}, V)$, $T(S) = S(1)$.

Solution: We saw in class that $\dim \mathcal{L}(\mathbb{F}, V) = \dim \mathbb{F} \cdot \dim V = 1 \cdot \dim V = \dim V$, so to check the map is an isomorphism, it's enough to check it's injective (then we get surjectivity for free). So we prove it's injective. Pick $S \in \operatorname{Null} T$. We show that S is the zero map $\mathbb{F} \to V$. This means we have to show that $S(c) = 0$ for all $c \in \mathbb{F}$. Since $T(S) = 0$, we know $S(1) = 0$, and then by linearity, for any $c \in \mathbb{F}$, $S(c) = cS(1) = c \cdot 0 = 0$, so S is the zero map, hence T is injective, hence an isomorphism.

(2) Prove that isomorphism (denoted $\cong$) is an *equivalence relation* on the set of vector spaces. That is, prove

(a) $V \cong V$ for every vector space V.

Proof: The identity map is an isomorphism $V \to V$, so V is isomorphic to itself.

(b) If V, W are two vector spaces with $V \cong W$, then $W \cong V$.
Proof: If $V \cong W$, then there is a map $T\colon V \to W$ which has an inverse. Call the inverse S. But then since $S\colon W \to V$ is a map with an inverse, we see that $W \cong V$.

(c) If U, V, W are three vector spaces such that $U \cong V$ and $V \cong W$, then $U \cong W$.

Proof: First off, since $U \cong V$, $\dim U = \dim V$ by some theorem from class ("isomorphic spaces have the same dimension, and vice versa"). Similarly $\dim V = \dim W$. Thus all three dimensions are the same and in particular $\dim U = \dim W$. In fact, by the same theorem, this already implies that they're isomorphic. But if you prefer to use the definition, we need to produce an isomorphism $U \to W$, and since the dimensions are the same it's enough to check surjectivity (or injectivity) alone. By our assumptions, we have $T\colon U \to V$ and $S\colon V \to W$ - the given isomorphisms. We claim $S \circ T$ is an isomorphism $U \to W$. To see the subjectivity, pick $w \in W$. Since S is an isomorphism, there is $v \in V$ with $Sv = w$, and since T is an isomorphism, there is a $u \in U$ with $Tu = v$. Putting this together shows $w = Sv = S(Tu) = (S \circ T)u$, so $S \circ T$ is surjective, hence an isomorphism. Thus $U \cong W$.

(3) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -y \end{pmatrix}$. Find a subspace U of $\mathbb{R}^2$ such that $T\big|_U$ is the identity map on U. Find a subspace W of $\mathbb{R}^2$ such that $T\big|_W$ is the zero map on W.

Solution: Let U be the zero subspace. Then for all $v \in U$ (the only choice is $v = 0$) $Tv = 0 = v$, so T is the identity map on U. This silly choice of U turns out to be the only one! For W, we could actually take the same example, since on the zero subspace, the identity map and the zero map are the SAME MAP! There's only one input, after all. But for a more interesting example, we could take W to be the x-axis. Then $T\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $T\big|_W = 0$.

(4) (True or false? If true, prove it. If false, find a counterexample.) If $T \in \mathcal{L}(V)$ and U is a subspace of V that is T-invariant, then U contains a non-zero eigenvector for T.

Solution: FALSE! We always have two obvious invariant subspaces: the zero subspace, and the whole space. If we take U to be the zero subspace, then it's invariant, but doesn't contain any nonzeroo eigenvectors, because it doesn't contain any nonzero vectors at all! Alternatively, take for example $V = \mathbb{R}^2$. We've seen that a rotation through $\pi/4$ radians has no nonzero eigenvectors. So if we let $U = \mathbb{R}^2$ also, then U is invariant, but doesn't contain any nonzero eigenvectors - another counterexample.

(5) Consider the operator $T\colon P_2(\mathbb{F}) \to P_2(\mathbb{F})$ given by $Tp(x) = xp'(x)$. Find a basis for $P_2(\mathbb{F})$ with respect to which the matrix for T is diagonal (in other words, diagonalize T).

Solution: We need to produce a basis of eigenvectors. So we solve $Tp(x) = \lambda p(x)$. Write $p(x) = ax^2 + bx + c$. Then our equation looks like

$$xp'(x) = 2ax^2 + bx = \lambda ax^2 + \lambda bx + \lambda c,$$

and equating the coefficients of like powers of x gives us the system

$$\begin{aligned} 2a &= \lambda a \\ b &= \lambda b \\ 0 &= \lambda c. \end{aligned}$$

From the first equation, we see that either $\lambda = 0$ or $c = 0$. If $\lambda = 0$, the the other two equations tell us that $a = b = 0$, so $\lambda = 0$ is an eigenvalue, whose eigenvectors are the constant polynomials, spanned by the polynomial 1. So now assume that $\lambda \neq 0$. Then c must be zero. From the second equation, either $\lambda = 1$ or $b = 0$. If $\lambda = 1$, b could be anything, but the first equation forces $a = 0$. So $\lambda = 1$ is an eigenvalue, with eigenvectors bx, i.e., spanned by x. Finally assume instead that $b = 0$. Then we look at the final equation and see that $\lambda = 2$ (of course $a = 0$ gives a solution, but then our polynomial is the zero polynomial, and this doesn't tell us about eigenvalues). So $\lambda = 2$ is an eigenvalue, with eigenvectors ax^2.

Summarizing, we have found three eigenvalues with eigenspaces

$$\begin{aligned} E_0 &= \text{Span}(1) \\ E_1 &= \text{Span}(x) \\ E_2 &= \text{Span}(x^2) \end{aligned}$$

Since each of these are 1-dimensional, we can produce a basis for $P_2(\mathbb{F})$ by choosing one vector from each, say the basis $B = (1, x, x^2)$, the standard basis. Then

$$M(T, B) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

which is diagonal.

(6) Consider the operator $T\colon \mathbb{F}^3 \to \mathbb{F}^3$ given by $Tx = Ax$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ -2 & 0 & 3 \end{pmatrix}$. Find a basis for $\mathbb{F}^3$ with respect to which the matrix for T is diagonal (in other words, diagonalize T).

Solution: As in 5, we find eigenvectors/values. The equation

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives us the system of equations

$$\begin{aligned} x &= \lambda x \\ -x + 2y + z &= \lambda y \\ -2x + 3z &= \lambda z \end{aligned}$$

Look at the first equation. Either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then plugging this in to the other two equations gives

$$\begin{aligned} -x + y + z &= 0 \\ -2x + 2z &= 0, \end{aligned}$$

so $x = z$ and $y = 0$. Thus we have a one-dimensional eigenspace $E_1 = \text{Span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Now consider the case when $x = 0$. Plugging this in to the other equations gives the system

$$\begin{aligned} 2y + z &= \lambda y \\ 3z &= \lambda z \end{aligned}$$

From the second equation, either $z = 0$ or $\lambda = 3$. If $z = 0$, then $\lambda = 2$ and y could be anything, so we have another eigenspace $E_2 = \text{Span} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Finally, if (still using $x = 0$) $\lambda = 3$, then the second equation reads $2y + z = 3y$, so $z = y$. This gives us the third eigenspace, $E_3 = \text{Span} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Thus $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors for T, and in fact

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$$

is the "diagonal decomposition" of A.

(7) If $P \in \mathcal{L}(V)$ satisfies $P^2 = P$,

(a) prove that the only possible eigenvalues of P are 0 and 1.

Proof: Suppose λ is an eigenvalue of P. Then for some $v \neq 0$, $Pv = \lambda v$, and applying P gives $P^2 v = \lambda P v$, so $Pv = \lambda P v$. Thus $(1 - \lambda) P v = 0$. Therefore $\lambda = 1$ or $Pv = 0$. So $\lambda = 1$ is a possible eigenvalue, and otherwise we must have $Pv = 0$, which means v is in the nullspace, which is the same as the 0-eigenspace, so P has zero as an eigenvalue.

(b) prove that the set of eigenvectors with eigenvalue 1 is equal to the range of P.

Proof:

$$\begin{aligned} \text{Range}\, P &= \{w \mid w = Pv \text{ for some } v \in V\} \\ &= \{w \mid w = P^2 v \text{ for some } v \in V\} \\ &= \{w \mid w = P(Pv) \text{ for some } v \in V\} \\ &= \{w \mid w = Pw\} \\ &= E_1. \end{aligned}$$

(going from the third line to the fourth is not completely obvious - owing to the disappearance of the "for some $v \in V$" in the set definition. Can you see why this is true?)

(8) Let $S, T \in \mathcal{L}(V)$ be such that $ST = TS$ (we say they "commute")

(a) Prove that T^n and S commute, for any $n \geq 0$.

Proof: $T^n S = T^{n-1}ST = T^{n-2}ST^2 = \cdots = TST^{n-1} = ST^n$, where at each step we switched the order of S and one of the T's, using our assumption.

(b) Let $p(x)$ be any polynomial, and let $p(T) \in \mathcal{L}(V)$ be the operator obtained by replacing x by T, as defined in class. Prove that $\operatorname{Null} p(T)$ is invariant under S.

Proof: (note: this will be heavily used if we ever make it to the Jordan form section of the end of the course) To show $\operatorname{Null} p(T)$ is S-invariant, we pick $v \in \operatorname{Null} p(T)$, and show that Sv is also in $\operatorname{Null} p(T)$. So let $v \in \operatorname{Null} p(T)$. To see that Sv in $\operatorname{Null} p(T)$, we apply the operator $p(T)$ to the vector Sv, and show that the answer is zero. First write $p(T) = a_0 I + a_1 T + \cdots + a_n T^n$. Then:

$$p(T)(Sv) = (a_0 I + a_1 T + \cdots + a_n T^n)(Sv) \tag{1}$$
$$= a_0 I(Sv) + a_1 T(Sv) + \cdots a_n T^n(Sv) \tag{2}$$
$$= a_0 S(Iv) + a_1 S(Tv) + \cdots + a_n S(T^n v) \tag{3}$$
$$= S(a_0 Iv + a_1 Tv + \cdots + a_n T^n v) \tag{4}$$
$$= S(p(T)v) \tag{5}$$
$$= S(0) = 0, \tag{6}$$

where going from line (2)-(3) we used the fact from part (a) that S commutes with all powers of T, and going from line (5)-(6) we used that $v \in \operatorname{Null} p(T)$. All the other steps just used the definitions of polynomial operators and linearity.

(9) Let $T \in \mathcal{L}(V)$. Prove that if v is a non-zero eigenvector of T which is not in $\operatorname{Range} T$, then $v \in \operatorname{Null} T$.

Proof: Let $Tv = \lambda v$, with $v \neq 0$, and assume that $v \notin \operatorname{Range} T$. There are two cases: $\lambda = 0$ or $\lambda \neq 0$. If $\lambda \neq 0$, then $v = \frac{1}{\lambda} Tv = T(\frac{1}{\lambda} v) \in \operatorname{Range} T$, and we assumed this was not the case. So it must be that $\lambda = 0$. But then $Tv = \lambda v = 0v = 0$, so $v \in \operatorname{Null} T$. Done. (note: you should be familiar by now with the idea that the 0-eigenspace is the same as the nullspace, which is essentially the content of the final sentence)