(1) Which of the following maps are isomorphisms? Explain why or why not.

(a) \( T : P_3(F) \to \mathbb{F}^3 \) given by \( T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} \).

Solution: This is not an isomorphism - there are NO isomorphisms between \( P_3(F) \) and \( \mathbb{F}^3 \) since they do not have the same dimension.

(b) \( T : \mathbb{F}^3 \to \mathbb{F}^3 \) given by \( T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \\ x - z \end{pmatrix} \).

Solution: This is not an isomorphism, since it’s not injective. It has the vector \( \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \) in its null space.

(c) \( T : P_n(F) \to P_n(F) \) given by \( T(a_0 + a_1 x + \cdots + a_n x^n) = (a_n + a_{n-1} x + \cdots + a_1 x^{n-1} + a_0 x^n) \).

(note: there was a mistake here - it originally said \( P(F) \) instead of \( P_n(F) \), and the formula actually doesn’t make sense if it’s left as \( P(F) \). Sorry for any confusion.)

Solution: This is an isomorphism. We saw in class that \( \text{dim } P_3(F) = 3 \) and since the dimensions are the same it’s enough to check surjectivity (or injectivity) alone. So we prove it’s injective. Pick \( T : \mathbb{F}^3 \to \mathbb{F}^3 \) the given isomorphisms. We claim \( T \) is an isomorphism, it’s enough to check it’s injective (then we get surjectivity for free). So we prove it’s injective. Pick \( S \) as \( S(c) = cS(1) = c \cdot 0 = 0 \), so \( S \) is the zero map, hence \( T \) is injective, hence an isomorphism.

(d) \( T : \mathcal{L}(F,V) \to V \) given by, for \( S \in \mathcal{L}(F,V) \), \( T(S) = S(1) \).

Solution: We saw in class that \( \text{dim } \mathcal{L}(F,V) = \text{dim } F \cdot \text{dim } V = 1 \cdot \text{dim } V = \text{dim } V \), so to check the map is an isomorphism, it’s enough to check it’s injective (then we get surjectivity for free). So we prove it’s injective. Pick \( S \) as \( S \in \text{Null } T \). We show that \( S \) is the zero map \( F \to V \). This means we have to show that \( S(c) = 0 \) for all \( c \in F \). Since \( T(S) = 0 \), we know \( S(1) = 0 \), and then by linearity, for any \( c \in F \), \( S(c) = cS(1) = c \cdot 0 = 0 \), so \( S \) is the zero map, hence \( T \) is injective, hence an isomorphism.

(2) Prove that isomorphism (denoted \( \cong \)) is an equivalence relation on the set of vector spaces. That is, prove

(a) \( V \cong V \) for every vector space \( V \).

Proof: The identity map is an isomorphism \( V \to V \), so \( V \) is isomorphic to itself.

(b) If \( V, W \) are two vector spaces with \( V \cong W \), then \( W \cong V \).

Proof: If \( V \cong W \), then there is a map \( T : V \to W \) which has an inverse. Call the inverse \( S \). But then since \( S : W \to V \) is a map with an inverse, we see that \( W \cong V \).

(c) If \( U, V, W \) are three vector spaces such that \( U \cong V \) and \( V \cong W \), then \( U \cong W \).

Proof: First off, since \( U \cong V \), \( \text{dim } U = \text{dim } V \) by some theorem from class (“isomorphic spaces have the same dimension, and vice versa”). Similarly \( \text{dim } V = \text{dim } W \). Thus all three dimensions are the same and in particular \( \text{dim } U = \text{dim } W \). In fact, by the same theorem, this already implies that they’re isomorphic. But if you prefer to use the definition, we need to produce an isomorphism \( U \to W \), and since the dimensions are the same it’s enough to check surjectivity (or injectivity) alone. By our assumptions, we have \( T : U \to V \) and \( S : V \to W \) - the given isomorphisms. We claim \( S \circ T \) is an isomorphism \( U \to W \). To see the subjectivity, pick \( w \in W \). Since \( S \) is an isomorphism, there is \( u \in U \) with \( Su = w \), and since \( T \) is an isomorphism, there is a \( v \in U \) with \( Tu = v \). Putting this together shows \( w = Sv = S(Tu) = (S \circ T)u \), so \( S \circ T \) is surjective, hence an isomorphism. Thus \( U \cong W \).
(3) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -y \end{pmatrix}$. Find a subspace $U$ of $\mathbb{R}^2$ such that $T|_U$ is the identity map on $U$. Find a subspace $W$ of $\mathbb{R}^2$ such that $T|_W$ is the zero map on $W$.

**Solution:** Let $U$ be the zero subspace. Then for all $v \in U$ (the only choice is $v = 0$) $Tv = 0 = v$, so $T$ is the identity map on $U$. This silly choice of $U$ turns out to be the only one! For $W$, we could actually take the same example, since on the zero subspace, the identity map and the zero map are the SAME MAP! There’s only one input, after all. But for a more interesting example, we could take $W$ to be the $x$-axis. Then $T\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $T|_W = 0$.

(4) (True or false? If true, prove it. If false, find a counterexample.) If $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$ that is $T$-invariant, then $U$ contains a non-zero eigenvector for $T$.

**Solution:** FALSE! We always have two obvious invariant subspaces: the zero subspace, and the whole space. If we take $U$ to be the zero subspace, then it’s invariant, but doesn’t contain any non-zero eigenvectors, because it doesn’t contain any non-zero vectors at all! Alternatively, take for example $V = \mathbb{R}^2$. We’ve seen that a rotation through $\pi/4$ radians has no non-zero eigenvectors. So if we let $U = \mathbb{R}^2$ also, then $U$ is invariant, but doesn’t contain any non-zero eigenvectors - another counterexample.

(5) Consider the operator $T: P_2(\mathbb{F}) \to P_2(\mathbb{F})$ given by $Tp(x) = xp'(x)$. Find a basis for $P_2(\mathbb{F})$ with respect to which the matrix for $T$ is diagonal (in other words, diagonalize $T$).

**Solution:** We need to produce a basis of eigenvectors. So we solve $Tp(x) = \lambda p(x)$. Write $p(x) = ax^2 + bx + c$. Then our equation looks like

$$xp'(x) = 2ax^2 + bx = \lambda ax^2 + \lambda bx + \lambda c,$$

and equating the coefficients of like powers of $x$ gives us the system

$$\begin{align*}
2a &= \lambda a \\
b &= \lambda b \\
0 &= \lambda c.
\end{align*}$$

From the first equation, we see that either $\lambda = 0$ or $c = 0$. If $\lambda = 0$, the the other two equations tell us that $a = b = 0$, so $\lambda = 0$ is an eigenvalue, whose eigenvectors are the constant polynomials, spanned by the polynomial 1. So now assume that $\lambda \neq 0$. Then $c$ must be zero. From the second equation, either $\lambda = 1$ or $b = 0$. If $\lambda = 1$, $b$ could be anything, but the first equation forces $a = 0$. So $\lambda = 1$ is an eigenvalue, with eigenvectors $bx$, i.e., spanned by $x$. Finally assume instead that $b = 0$. Then we look at the final equation and see that $\lambda = 2$ (of course $a = 0$ gives a solution, but then our polynomial is the zero polynomial, and this doesn’t tell us about eigenvalues). So $\lambda = 2$ is an eigenvalue, with eigenvectors $ax^2$.

Summarizing, we have found three eigenvalues with eigenspaces

$$
E_0 = \text{Span}(1) \\
E_1 = \text{Span}(x) \\
E_2 = \text{Span}(x^2)
$$

Since each of these are 1-dimensional, we can produce a basis for $P_2(\mathbb{F})$ by choosing one vector from each, say the basis $B = (1, x, x^2)$, the standard basis. Then

$$M(T, B) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix},$$

which is diagonal.

(6) Consider the operator $T: \mathbb{F}^3 \to \mathbb{F}^3$ given by $Tx = Ax$, where $A = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 2 & 1 \\
-2 & 0 & 3
\end{pmatrix}$. Find a basis for $\mathbb{F}^3$ with respect to which the matrix for $T$ is diagonal (in other words, diagonalize $T$).
Solution: As in 5, we find eigenvectors/values. The equation
\[
\begin{pmatrix}
1 & 0 & 0 \\
-1 & 2 & 1 \\
-2 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \lambda
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]
gives us the system of equations
\[
\begin{align*}
x &= \lambda x \\
-x + 2y + z &= \lambda y \\
-2x + 3z &= \lambda z
\end{align*}
\]
Look at the first equation. Either \( \lambda = 1 \) or \( x = 0 \). If \( \lambda = 1 \), then plugging this in to the other two equations gives
\[
\begin{align*}
-x + y + z &= 0 \\
-2x + 2z &= 0,
\end{align*}
\]
so \( x = z \) and \( y = 0 \). Thus we have a one-dimensional eigenspace \( E_1 = \text{Span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \).

Now consider the case when \( x = 0 \). Plugging this in to the other equations gives the system
\[
\begin{align*}
2y + z &= \lambda y \\
3z &= \lambda z
\end{align*}
\]
From the second equation, either \( z = 0 \) or \( \lambda = 3 \). If \( z = 0 \), then \( \lambda = 2 \) and \( y \) could be anything, so we have another eigenspace \( E_2 = \text{Span} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \). Finally, if (still using \( x = 0 \)) \( \lambda = 3 \), then the second equation reads
\[
2y + z = 3y, \text{ so } z = y.
\]
This gives us the third eigenspace, \( E_3 = \text{Span} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \).

Thus \( \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} \) is a basis for \( \mathbb{R}^3 \) consisting of eigenvectors for \( T \), and in fact
\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}^{-1}
\]
is the “diagonal decomposition” of \( A \).

(7) If \( P \in \mathcal{L}(V) \) satisfies \( P^2 = P \),

(a) prove that the only possible eigenvalues of \( P \) are 0 and 1.

\textbf{Proof:} Suppose \( \lambda \) is an eigenvalue of \( P \). Then for some \( v \neq 0 \), \( Pv = \lambda v \), and applying \( P \) gives \( P^2 v = \lambda P v \), so \( P v = \lambda P v \). Thus \((1-\lambda)P v = 0 \). Therefore \( \lambda = 1 \) or \( P v = 0 \). So \( \lambda = 1 \) is a possible eigenvalue, and otherwise we must have \( P v = 0 \), which means \( v \) is in the nullspace, which is the same as the 0-eigenspace, so \( P \) has zero as an eigenvalue.

(b) prove that the set of eigenvectors with eigenvalue 1 is equal to the range of \( P \).

\textbf{Proof:}
\[
\text{Range } P = \{ w | w = P v \text{ for some } v \in V \}
\]
\[
= \{ w | w = P^2 v \text{ for some } v \in V \}
\]
\[
= \{ w | w = P(P v) \text{ for some } v \in V \}
\]
\[
= \{ w | w = Pw \}
\]
\[
= E_1.
\]
(going from the third line to the fourth is not completely obvious - owing to the disappearance of the “for some \( v \in V \) in the set definition. Can you see why this is true?)

(8) Let \( S, T \in \mathcal{L}(V) \) be such that \( ST = TS \) (we say they “commute”)

(a) Prove that \( T^n \) and \( S \) commute, for any \( n \geq 0 \).

**Proof:** \( T^n S = T^{n-1} ST = T^{n-2} ST^2 = \cdots = TST^{n-1} = ST^n \), where at each step we switched the order of \( S \) and one of the \( T \)'s, using our assumption.

(b) Let \( p(x) \) be any polynomial, and let \( p(T) \in \mathcal{L}(V) \) be the operator obtained by replacing \( x \) by \( T \), as defined in class. Prove that \( \text{Null} \, p(T) \) is invariant under \( S \).

**Proof:** (note: this will be heavily used if we ever make it to the Jordan form section of the end of the course) To show \( \text{Null} \, p(T) \) is \( S \)-invariant, we pick \( v \in \text{Null} \, p(T) \), and show that \( Sv \) is also in \( \text{Null} \, p(T) \). So let \( v \in \text{Null} \, p(T) \). To see that \( Sv \) in \( \text{Null} \, p(T) \), we apply the operator \( p(T) \) to the vector \( Sv \), and show that the answer is zero. First write \( p(T) = a_0I + a_1T + \cdots + a_nT^n \). Then:

\[
\begin{align*}
(1) & \quad p(T)(Sv) = (a_0I + a_1T + \cdots + a_nT^n)(Sv) \\
(2) & \quad = a_0I(Sv) + a_1T(Sv) + \cdots + a_nT^n(Sv) \\
(3) & \quad = a_0S(Iv) + a_1S(Tv) + \cdots + a_nS(T^n v) \\
(4) & \quad = S(a_0Iv + a_1Tv + \cdots + a_nT^n v) \\
(5) & \quad = S(p(T)v) \\
(6) & \quad = S(0) = 0,
\end{align*}
\]

where going from line (2)-(3) we used the fact from part (a) that \( S \) commutes with all powers of \( T \), and going from line (5)-(6) we used that \( v \in \text{Null} \, p(T) \). All the other steps just used the definitions of polynomial operators and linearity.

(9) Let \( T \in \mathcal{L}(V) \). Prove that if \( v \) is a non-zero eigenvector of \( T \) which is not in \( \text{Range} \, T \), then \( v \in \text{Null} \, T \).

**Proof:** Let \( Tv = \lambda v \), with \( v \neq 0 \), and assume that \( v \notin \text{Range} \, T \). There are two cases: \( \lambda = 0 \) or \( \lambda \neq 0 \). If \( \lambda \neq 0 \), then \( v = \frac{1}{\lambda} Tv = T(\frac{1}{\lambda} v) \in \text{Range} \, T \), and we assumed this was not the case. So it must be that \( \lambda = 0 \). But then \( Tv = \lambda v = 0v = 0 \), so \( v \in \text{Null} \, T \). Done. (note: you should be familiar by now with the idea that the 0-eigenspace is the same as the nullspace, which is essentially the content of the final sentence)