Math 110, Summer 2013
Instructor: James McIvor
Homework 4
Due THURSDAY, July 25th

(1) Which of the following maps are isomorphisms? Explain why or why not.

(a) $T\colon P_3(\mathbb{F}) \to \mathbb{F}^3$ given by $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix}$.

(b) $T\colon \mathbb{F}^3 \to \mathbb{F}^3$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x-z \end{pmatrix}$.

(c) $T\colon P(\mathbb{F}) \to P(\mathbb{F})$ given by $T(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n) = (a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n)$.

(d) $T\colon \mathcal{L}(\mathbb{F}, V) \to V$ given by, for $S \in \mathcal{L}(\mathbb{F}, V)$, $T(S) = S(1)$.

(2) Prove that isomorphism (denoted $\cong$) is an *equivalence relation* on the set of vector spaces. That is, prove
(a) $V \cong V$ for every vector space V.
(b) If V, W are two vector spaces with $V \cong W$, then $W \cong V$.
(c) If U, V, W are three vector spaces such that $U \cong V$ and $V \cong W$, then $U \cong W$.

(3) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -y \end{pmatrix}$. Find a subspace U of $\mathbb{R}^2$ such that $T\big|_U$ is the identity map on U. Find a subspace W of $\mathbb{R}^2$ such that $T\big|_W$ is the zero map on W.

(4) (True or false? If true, prove it. If false, find a counterexample.) If $T \in \mathcal{L}(V)$ and U is a subspace of V that is T-invariant, then U contains a non-zero eigenvector for T.

(5) Consider the operator $T\colon P_2(\mathbb{F}) \to P_2(\mathbb{F})$ given by $Tp(x) = xp'(x)$. Find a basis for $P_2(\mathbb{F})$ with respect to which the matrix for T is diagonal (in other words, diagonalize T).

(6) Consider the operator $T\colon \mathbb{F}^3 \to \mathbb{F}^3$ given by $Tx = Ax$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ -2 & 0 & 3 \end{pmatrix}$. Find a basis for $\mathbb{F}^3$ with respect to which the matrix for T is diagonal (in other words, diagonalize T).

(7) If $P \in \mathcal{L}(V)$ satisfies $P^2 = P$,
(a) prove that the only eigenvalues of P are 0 and 1, and
(b) prove that the set of eigenvectors with eigenvalue 1 is equal to the range of P.

(8) Let $S, T \in \mathcal{L}(V)$ be such that $ST = TS$ (we say they "commute")
(a) Prove that T^n and S commute, for any $n \geq 0$.
(b) Let $p(x)$ be any polynomial, and let $p(T) \in \mathcal{L}(V)$ be the operator obtained by replacing x by T, as defined in class. Prove that $\operatorname{Null} p(T)$ is invariant under S.

(9) Let $T \in \mathcal{L}(V)$. Prove that if v is a non-zero eigenvector of T which is not in $\operatorname{Range} T$, then $v \in \operatorname{Null} T$.