(1) Which of the following maps are isomorphisms? Explain why or why not.

(a) $T: P_3(F) \rightarrow F^3$ given by $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix}$.

(b) $T: F^3 \rightarrow F^3$ given by $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \\ x - z \end{pmatrix}$.

(c) $T: P(F) \rightarrow P(F)$ given by $T(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n) = (a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n)$.

(d) $T: \mathcal{L}(F, V) \rightarrow V$ given by, for $S \in \mathcal{L}(F, V)$, $T(S) = S(1)$.

(2) Prove that isomorphism (denoted $\cong$) is an equivalence relation on the set of vector spaces. That is, prove

(a) $V \cong V$ for every vector space $V$.

(b) If $V$, $W$ are two vector spaces with $V \cong W$, then $W \cong V$.

(c) If $U, V, W$ are three vector spaces such that $U \cong V$ and $V \cong W$, then $U \cong W$.

(3) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -y \end{pmatrix}$. Find a subspace $U$ of $\mathbb{R}^2$ such that $T|_U$ is the identity map on $U$. Find a subspace $W$ of $\mathbb{R}^2$ such that $T|_W$ is the zero map on $W$.

(4) (True or false? If true, prove it. If false, find a counterexample.) If $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$ that is $T$-invariant, then $U$ contains a non-zero eigenvector for $T$.

(5) Consider the operator $T: P_2(F) \rightarrow P_2(F)$ given by $T(p(x)) = xp'(x)$. Find a basis for $P_2(F)$ with respect to which the matrix for $T$ is diagonal (in other words, diagonalize $T$).

(6) Consider the operator $T: F^3 \rightarrow F^3$ given by $Tx = Ax$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ -2 & 0 & 3 \end{pmatrix}$. Find a basis for $F^3$ with respect to which the matrix for $T$ is diagonal (in other words, diagonalize $T$).

(7) If $P \in \mathcal{L}(V)$ satisfies $P^2 = P$,

(a) prove that the only eigenvalues of $P$ are 0 and 1, and

(b) prove that the set of eigenvectors with eigenvalue 1 is equal to the range of $P$.

(8) Let $S, T \in \mathcal{L}(V)$ be such that $ST = TS$ (we say they “commute”)

(a) Prove that $T^n$ and $S$ commute, for any $n \geq 0$.

(b) Let $p(x)$ be any polynomial, and let $p(T) \in \mathcal{L}(V)$ be the operator obtained by replacing $x$ by $T$, as defined in class. Prove that $\text{Null } p(T)$ is invariant under $S$.

(9) Let $T \in \mathcal{L}(V)$. Prove that if $v$ is a non-zero eigenvector of $T$ which is not in $\text{Range } T$, then $v \in \text{Null } T$. 