Math 110, Summer 2013
Instructor: James McIvor
Homework 3
Due Wednesday, July 17th
+2 bonus points for submitting it on Monday, July 15th

(1) (Axler 3.10) Prove that there does not exist a linear map $\mathbb{F}^5 \to \mathbb{F}^2$ whose null space is $\{(x_1, \dots, x_5) \,|\, x_1 = 3x_2, x_3 = x_4 = x_5\}$.

Solution: Using the trick learned in class, we see that the given subspace has dimension 2 (it is a subspace of $\mathbb{F}^5$ defined by three independent equations, and 5-3=2). Moreover, since the range is a subspace of $\mathbb{F}^2$, its dimension is at most two. So the rank-nullity equation cannot be satisfied by such a map.

(2) (Axler 3.23) Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$. Here ST and TS are shorthand for $S \circ T$ and $T \circ S$. (this is useful - it means when you're checking that two maps are inverses, you only need to check one of the two equations)

Solution: First, as we proved in class, the condition $TS = I$ implies that T is surjective and S is injective. But by the rank-nullity law, as soon as a map $V \to V$ is surjective, it is also injective. So both maps are surjective and injective, hence are isomorphisms. Applying the inverse of T to $TS = I$ shows that in fact S is the inverse to T. Thus $ST = I$, by the definition of inverse. The other direction is similar.

(3) Let $T\colon V \to W$ and $S\colon W \to U$ be linear maps. Prove that ST is the zero map if and only if $\operatorname{Range} T \subseteq \operatorname{Null} S$ (recall that ST is the zero map means $(ST)v = 0$ for all v in V).

Solution: Suppose first that $ST = 0$. Then pick $v \in \operatorname{Range} T$, so $v = Tw$ for some $w \in V$. Then $Sv = STw = 0$, so $v \in \operatorname{Null} T$. Conversely, suppose $\operatorname{Range} T \subseteq \operatorname{Null} S$. Then for any $v \in V$, $STv = S(Tv) = 0$ since $Tv \in \operatorname{Range} T$. Since v was arbitrary, ST is the zero map.

(4) Find a linear map $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ whose null space is $U = \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x = w, 2y = z\}$ and whose range is $W = \{(x, y, z) \in \mathbb{R}^3 \,|\, y = z\}$.

Solution: First we choose a basis, say $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ for U, and extend it to a basis

$$u_1, u_2, v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

for $\mathbb{R}^4$. Now we define the map T by

$$\begin{aligned} Tu_1 &= 0 \\ Tu_2 &= 0 \\ Tv_1 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ Tv_2 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Then $\operatorname{Null} T \subseteq U$ since u_1, u_2 go to zero, and is no larger than U since Tv_1 and Tv_2 are independent. Moreover, since Tv_1 and Tv_2 span W, the range of the map is W.

(5) (a) Prove that the map $T\colon P_2(\mathbb{F}) \to P_3(\mathbb{F})$ given by $Tp(x) = p'(x) - xp(x)$ is injective.

Proof: Suppose $p(x) \in \operatorname{Null} T$. Then $Tp(x) = p'(x) - xp(x) = 0$, so writing $p(x) = ax^2 + bx + c$ we have

$$2ax + b = ax^3 + bx^2 + cx,$$

and equating the coefficients of the like powers of x gives the following equations:

$$\begin{aligned} 0 &= a \\ 0 &= b \\ 2a &= c \end{aligned}$$

which imply that $a = b = c = 0$, so $p(x) = 0$, hence $\operatorname{Null} T = 0$, so T is injective.

(b) Prove that for every $c \in \mathbb{F}$ (including $c = 0$), the evaluation map $T_c \colon P(\mathbb{F}) \to \mathbb{F}$ as defined in the previous HW is surjective.

Proof: Let $c \in \mathbb{F}$, and let $a \in \mathbb{F}$. We will find a pre-image under T_c for this number a. The constant polynomial $p(x) = a$ for all x works, because $T_c p(x) = p(c) = a$.

(6) Let T be the map of problem 5(a). Using the bases $B_1 = (x^2, x, 1)$ and $B_2 = (1, 1 - x^2, x, x^3)$ for $P_2(\mathbb{F})$ and $P_3(\mathbb{F})$, respectively, compute $M(T, B_1, B_2)$.

Solution: We compute

$$\begin{aligned} Tx^2 &= 2x - x^3 = 0 \cdot 1 + 0 \cdot (1 - x^2) + 2 \cdot x + (-1) \cdot x^3 \\ Tx &= 1 - x^2 = 0 \cdot 1 + 1 \cdot (1 - x^2) + 0 \cdot x + 0 \cdot x^3 \\ T1 &= -x = 0 \cdot 1 + 0 \cdot (1 - x^2) + (-1) \cdot x + 0 \cdot x^3, \end{aligned}$$

so

$$M(T, B_1, B_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

(7) If $T \colon V \to V$ is a linear map whose matrix with respect to the basis $(v_1, \ldots, v_5)$ is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{pmatrix},$$

find Tv, where $v = v_1 + v_2 + v_3 + v_4 + v_5$.

Solution: The given matrix tells us that $Tv_1 = v_1 + 2v_2 + 3v_3 + 4v_4 + 5v_5$, $Tv_2 = 2v_1 + 3v_2 + 4v_3 + 5v_4 + 6v_5$, etc, so

$$Tv = (1 + 2 + 3 + 4 + 5)v_1 + (2 + 3 + 4 + 5 + 6)v_2 + \cdots = 15v_1 + 20v_2 + 25v_3 + 30v_4 + 35v_5$$