Math 110, Summer 2013
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Homework 2 Solution

(1) (Axler 2.11) If V is a finite-dimensional vector space and U a subspace of V with $\dim U = \dim V$, prove that $U = V$.

Proof: Pick a basis for U. It has $\dim V$ vectors, since U and V have the same dimension. But this list is also an independent list of vectors in V, so by the extension lemma it can be extended to a basis of V. But since the list already has the right length, we do not need to add in any vectors, i.e., it is already a basis for V.

(2) (Axler 2.17) Prove that if $U_1, \dots, U_m$ are subspaces of a finite-dimensional space V such that $V = U_1 \oplus \cdots \oplus U_m$, then

$$\dim V = \dim U_1 + \cdots + \dim U_m$$

Proof: Pick a basis for each U_i, and put them together to form a long list of vectors in V. The number of vectors in this list is $\dim U_1 + \cdots + \dim U_m$, so we will be done if we can show this list is a basis for V, since then the number of vectors in this list will also be $\dim V$. The list spans V, since any vector in V can be broken up into pieces from each U_i, and each of the piece pieces can be written using the vectors in our list. The list is independent, because if 0 is a linear combination of these vectors, then grouping the terms from each U_i together and calling them u_i, we have $0 = u_1 + \cdots + u_m$ and by one of our characterizations of direct sum, this forces each $u_i = 0$. But for each i, u_i is written in terms of a basis for U_i, so the coefficients are all zero. Therefore all the coefficients of our original representation for 0 are zero, proving independence.

(3) Suppose V is a vector space of dimension n, and U is a subspace of V of dimension m, and that W is another subspace of V such that $V = U + W$. What are the possible values for $\dim W$? What are the possible values for $\dim W$ if we assume further that $V = U \oplus W$? Justify your answers.

Solution: We know $\dim W = n - m + \dim U \cap W$. The smallest $\dim U \cap W$ could be is 0, so the smallest $\dim W$ could be is $n - m$. On the other hand, since $U \cap W$ is a subspace of both U and W, $\dim U \cap W$ is no bigger than either of $\dim U$ and $\dim W$. $\dim U \cap W \leq \dim W$ implies $\dim W \leq n - m + \dim W$, which tells us nothing since $n \geq m$ anyway (U is a subsace of V). On the other hand, the inequality $\dim U \cap W \leq m$ tells us that $\dim W \leq n$. So altogether, we've found that $n - m \leq \dim W \leq n$.

On the assumption that we have a direct sum, the intersection is trivial, so $\dim W = n - m$.

(4) Prove that the following functions are linear maps:

(a) "Evaluation map": Let $c \in \mathbb{F}$. The map $T_c \colon P(\mathbb{F}) \to \mathbb{F}$ is given by $T_c p(x) = p(c)$.
Proof: Pick $p, q \in P(\mathbb{F})$. Then $T_c(p + q) = (p + q)(c) = p(c) + q(c) = T_c(p) + T_c(q)$, so T_c is additive. Pick also $a \in \mathbb{F}$. Then $T_c(ap) = (ap)(c) = a(p(c)) = aT_c p$, so T_c is homogeneous.

(b) "Multiplication by x": $T \colon P(\mathbb{F}) \to P(\mathbb{F})$ is given by $Tp(x) = xp(x)$.
Proof: Pick $p, q \in P(\mathbb{F})$. Then $T(p + q)(x) = x(p + q)(x) = xp(x) + xq(x) = Tp(x) + Tq(x)$, so T is additive. Pick also $a \in \mathbb{F}$. Then $T(ap)(x) = x(ap)(x) = axp(x) = aTp(x)$, so T is homogeneous.

(5) Prove what I call the "Construction Theorem": Let $\dim V = n$, and $(v_1, \dots, v_n)$ be a basis for V, and let $w_1, \dots, w_n$ be any n vectors in W. Then there exists a unique linear map $T \colon V \to W$ such that $Tv_i = w_i$ for each $i = 1, \dots, n$.

Proof: First we must define what the map T should be. We do this as follows: for any input vector $v \in V$, we first write it as $a_1 v_1 + \cdots + a_n v_n$. Then we define Tv to be:

$$Tv = a_1 w_1 + \cdots + a_n w_n$$

This is our definition of T. It satisfies $Tv_i = w_i$, because if we pick the input vector v to be $v = v_i$, then we write $v_i = 0v_1 + \cdots + 1v_i + \cdots + 0v_n$, so according to the definition above,

$$Tv_i = 0w_1 + \cdots + 1w_i + \cdots + 0v_n = w_i$$

Now we check it's linear. If we have two vectors $u, v \in V$, we first write them each in terms of the basis: $v = a_1v_1 + \cdots + a_nv_n$ and $u = b_1v_1 + \cdots + b_nv_n$. Now we compute

$$\begin{aligned} T(v+u) &= T((a_1v_1 + \cdots + a_nv_n) + (b_1v_1 + \cdots + b_nv_n)) \\ &= T((a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n) \\ &= (a_1+b_1)w_1 + \cdots + (a_n+b_n)w_n \\ &= (a_1w_1 + \cdots + a_nw_n) + (b_1w_1 + \cdots + b_nw_n) \\ &= Tv + Tu \end{aligned}$$

Similarly, if $c \in \mathbb{F}$ is any scalar, we have

$$\begin{aligned} T(cv) &= T(c(a_1v_1 + \cdots + a_nv_n)) \\ &= T(ca_1v_1 + \cdots + ca_nv_n) \\ &= ca_1w_1 + \cdots + ca_nw_n \\ &= c(a_1w_1 + \cdots + a_nw_n) \\ &= cTv \end{aligned}$$

Now we show such a map T is unique. So let S be another map which is linear and satisfies $Tv_i = w_i$. Then pick any $v \in V$. We will show $Sv = Tv$, which shows that T and S are the same map, because v was arbitrary. Write $v = a_1v_1 + \cdots + a_nv_n$. Using the linearity of S we have $Sv = a_1Sv_1 + \cdots + a_nSv_n$. Using the fact that $Sv_i = w_i$ we have that $Sv = a_1w_1 = \cdots + a_nw_n = Tv$. Done!

(6) Let V be a vector space, and U, W two subspaces such that $V = U \oplus W$. We define a map $P_U : V \to V$ (the "projection onto U") as follows. Pick any v in V. Write it as $v = u + w$, for some $u \in U$ and $w \in W$. Then set $P_U(v) = u$.

(a) Prove that P_U is a linear map.

Proof: I will write P instead of P_U, for short. Pick two vectors $v_1, v_2 \in V$, and write them first as $v_1 = u_1 + w_1$, $v_2 = u_2 + w_2$ (where $u_i \in U$, $w_i \in W$). This is possible because $V = U + W$. Then $v_1 + v_2 = u_1 + w_1 + u_2 + w_2 = (u_1 + u_2) + (w_1 + w_2)$, and this allows us to calculate that $P(v_1 + v_2) = u_1 + u_2 = Pv_1 + Pv_2$, so P is additive. Now pick a scalar $a \in \mathbb{F}$ and a vector $v = u + w$ in V. Then $av = au + aw$ so $P(av) = au = aPv$ so P is homogeneous.

(b) Prove that $P_U^2 = P_U$ (here P_U^2 means $P_U \circ P_U$).

Proof: Note first that if $u \in U \subseteq V$, then $Pu = u$. So for arbitrary $v = u+w \in V$, $P^2v = P(P(u+w)) = P(u) = u = Pv$, so P^2 and P are the same map.

(7) Consider the one-dimensional complex vector space $\mathbb{C}^1$. Let $T : \mathbb{C}^1 \to \mathbb{C}^1$ be given by $T(a + bi) = a$. Is T linear? Explain why or why not.

Solution: T is not linear - it is additive but not homogeneous. For example, $T(i) = 0$, so $iT(i) = 0$. But $T(i \cdot i) = T(-1) = -1$. Since $iT(i) \neq T(i \cdot i)$, the map is not homogeneous.

(8) (Axler 3.1) Prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there is a scalar $a \in \mathbb{F}$ such that $Tv = av$ for every v in V.

Proof: Pick a basis (u) for V. Now since Tu is some vector in V as well, it must be a multiple of u, call it au. Now pick an arbitrary vector $v \in V$. We can write it as cu for some $c \in \mathbb{F}$. Then $Tv = T(cu) = cT(u) = c(au) = a(cu) = av$.

(9) Consider the following two functions: $S_1 : \mathbb{F} \to \mathcal{L}(P(\mathbb{F}), \mathbb{F})$ given by, for $c \in \mathbb{F}$, $S_1c = T_c$ (where T_c is the evaluation map defined in problem 4a), and $S_2 : \mathcal{L}(P(\mathbb{F}), \mathbb{F}) \to \mathbb{F}$, where $S_2(T) = T(x^n)$ (here n is some fixed natural number).

(a) Verify that S_1 is not linear.

Solution: Pick two numbers in $\mathbb{F}$, for example 0 and 1. Then $S_1(0) = T_0$ and $S_1(1) = T_1$, while $S_1(1 + 0) = S_1(1) = T_1$, so the question is whether T_1 and $T_0 + T_1$ are the same map. They're not. Reason - plug in a polynomial, for example $x + 1$. Then $T_1(x + 1) = 2$, whereas $(T_0 + T_1)(x + 1) =$

$T_0(x+1)+T_1(x+1)=1+2=3$, so they can't be the same function. Thus S_1 is not additive. (In fact, it's not homogeneous, either. What a silly function!)

(b) For which natural number(s) n is the composite function $S_2 \circ S_1 : \mathbb{F} \to \mathbb{F}$ nevertheless still linear?
Solution: Let us fix an arbitrary n and see what this composite does. It is a map from $\mathbb{F}$ to $\mathbb{F}$, that takes a number c, turns it into the evaluation map T_c, and then plugs in the polynomial x^n to this evaluation map. More precisely, $(S_2 \circ S_1)(c) = S_2(S_1(c)) = S_2(T_c) = T_c(x^n) = c^n$. So for a fixed n, the function $S_2 \circ S_1$ takes c to c^n - it's the "nth power map". This is not linear unless we choose n to be equal to 1. You can see this by looking at the graphs - only for $n = 1$ is it a line!