Math 110, Summer 2013
Instructor: James McIvor
Homework 2
Due Wednesday, July 10th

(1) (Axler 2.11) If V is a finite-dimensional vector space and U a subspace of V with $\dim U = \dim V$, prove that $U = V$.

(2) (Axler 2.17) Prove that if $U_1, \dots, U_m$ are subspaces of a finite-dimensional space V such that $V = U_1 \oplus \dots \oplus U_m$, then

$$\dim V = \dim U_1 + \dots + \dim U_m$$

(3) Suppose V is a vector space of dimension n, and U is a subspace of V of dimension m, and that W is another subspace of V such that $V = U + W$. What are the possible values for $\dim W$? What are the possible values for $\dim W$ if we assume further that $V = U \oplus W$? Justify your answers.

(4) Prove that the following functions are linear maps:
 (a) "Evaluation map": Let $c \in \mathbb{F}$. The map $T_c \colon P(\mathbb{F}) \to \mathbb{F}$ is given by $T_c p(x) = p(c)$.
 (b) "Multiplication by x": $T \colon P(\mathbb{F}) \to P(\mathbb{F})$ is given by $Tp(x) = xp(x)$.

(5) Prove what I call the "Construction Theorem": Let $\dim V = n$, and $(v_1, \dots, v_n)$ be a basis for V, and let $w_1, \dots, w_n$ be any n vectors in W. Then there exists a unique linear map $T \colon V \to W$ such that $Tv_i = w_i$ for each $i = 1, \dots, n$.

(6) Let V be a vector space, and U, W two subspaces such that $V = U \oplus W$. We define a map $P_U \colon V \to V$ (the "projection onto U") as follows. Pick any v in V. Write it as $v = u + w$, for some $u \in U$ and $w \in W$. Then set $P_U(v) = u$.
 (a) Prove that P_U is a linear map.
 (b) Prove that $P_U^2 = P_U$ (here P_U^2 means $P_U \circ P_U$).

(7) Consider the one-dimensional complex vector space $\mathbb{C}^1$. Let $T \colon \mathbb{C}^1 \to \mathbb{C}^1$ be given by $T(a + bi) = a$. Is T linear? Explain why or why not.

(8) (Axler 3.1) Prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there is a scalar $a \in \mathbb{F}$ such that $Tv = av$ for every v in V.

(9) Consider the following two functions: $S_1 \colon \mathbb{F} \to \mathcal{L}(P(\mathbb{F}), \mathbb{F})$ given by, for $c \in \mathbb{F}$, $S_1 c = T_c$ (where T_c is the evaluation map defined in problem 4a), and $S_2 \colon \mathcal{L}(P(\mathbb{F}), \mathbb{F}) \to \mathbb{F}$, where $S_2(T) = T(x^n)$ (here n is some fixed natural number).
 (a) Verify that S_1 is not linear.
 (b) For which natural number(s) n is the composite function $S_2 \circ S_1 \colon \mathbb{F} \to \mathbb{F}$ nevertheless still linear?