(1) (Axler 2.11) If $V$ is a finite-dimensional vector space and $U$ a subspace of $V$ with $\dim U = \dim V$, prove that $U = V$.

(2) (Axler 2.17) Prove that if $U_1, \ldots, U_m$ are subspaces of a finite-dimensional space $V$ such that $V = U_1 \oplus \cdots \oplus U_m$, then
$$\dim V = \dim U_1 + \cdots + \dim U_m$$

(3) Suppose $V$ is a vector space of dimension $n$, and $U$ is a subspace of $V$ of dimension $m$, and that $W$ is another subspace of $V$ such that $V = U + W$. What are the possible values for $\dim W$? What are the possible values for $\dim W$ if we assume further that $V = U \oplus W$? Justify your answers.

(4) Prove that the following functions are linear maps:
(a) “Evaluation map”: Let $c \in \mathbb{F}$. The map $T_c: P(\mathbb{F}) \rightarrow \mathbb{F}$ is given by $T_c(p(x)) = p(c)$.
(b) “Multiplication by $x$”: $T: P(\mathbb{F}) \rightarrow P(\mathbb{F})$ is given by $Tp(x) = xp(x)$.

(5) Prove what I call the “Construction Theorem”: Let $\dim V = n$, and $(v_1, \ldots, v_n)$ be a basis for $V$, and let $w_1, \ldots, w_n$ be any $n$ vectors in $W$. Then there exists a unique linear map $T: V \rightarrow W$ such that $Tv_i = w_i$ for each $i = 1, \ldots, n$.

(6) Let $V$ be a vector space, and $U, W$ two subspaces such that $V = U \oplus W$. We define a map $P_U: V \rightarrow V$ (the “projection onto $U$”) as follows. Pick any $v$ in $V$. Write it as $v = u + w$, for some $u \in U$ and $w \in W$. Then set $P_U(v) = u$.
(a) Prove that $P_U$ is a linear map.
(b) Prove that $P_U^2 = P_U$ (here $P_U^2$ means $P_U \circ P_U$).

(7) Consider the one-dimensional complex vector space $\mathbb{C}^1$. Let $T: \mathbb{C}^1 \rightarrow \mathbb{C}^1$ be given by $T(a + bi) = a$. Is $T$ linear? Explain why or why not.

(8) (Axler 3.1) Prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there is a scalar $a \in \mathbb{F}$ such that $Tv = av$ for every $v$ in $V$.

(9) Consider the following two functions: $S_1: \mathbb{F} \rightarrow \mathcal{L}(P(\mathbb{F}), \mathbb{F})$ given by, for $c \in \mathbb{F}$, $S_1c = T_c$ (where $T_c$ is the evaluation map defined in problem 4a), and $S_2: \mathcal{L}(P(\mathbb{F}), \mathbb{F}) \rightarrow \mathbb{F}$, where $S_2(T) = T(x^n)$ (here $n$ is some fixed natural number).
(a) Verify that $S_1$ is not linear.
(b) For which natural number(s) $n$ is the composite function $S_2 \circ S_1: \mathbb{F} \rightarrow \mathbb{F}$ nevertheless still linear?