Math 110, Summer 2013
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Homework 1 Solution

(1) (a) Write $\frac{1+i}{1-i}$ in the form $a+bi$, for some $a,b \in \mathbb{R}$.
Solution: $\dfrac{1+i}{1-i} = \dfrac{1+i}{1-i}\dfrac{1+i}{1+i} = \dfrac{1+2i-1}{2} = i$

(b) Find all complex numbers z which satisfy $z^2 = -4i$.
Solution: Since $|-4i| = 4$, any such z must have length 2, so can be written as $z = 2e^{i\theta}$. Then $z^2 = 4e^{2i\theta} = 4e^{\frac{3\pi i}{2}}$, since $-i = e^{\frac{3\pi i}{2}}$, so θ must be $3\pi/4$ or $7\pi/4$. Note that $4e^{\frac{7\pi i}{2}} = -4e^{\frac{3\pi i}{2}}$, so we could also write the answer as $z = \pm e^{\frac{3\pi i}{2}}$

(2) Axler, Chapter 1 problem 3: Prove that for every vector v in V, $-(-v) = v$ (in other words, prove that v is the additive inverse of $-v$.)
Proof: By definition of $-v$, we have $v + (-v) = 0$, and this equation also says that v is the additive inverse of $-v$, since adding it to $-v$ gives the zero vector.

(3) Axler, Chapter 1 problem 4: Prove that if $a \in \mathbb{F}$, $v \in V$ and $av = 0$, then either $a = 0$ or $v = 0$.
Proof: We will show that if $av = 0$ and $a \neq 0$, then $v = 0$. Multiply the equation $av = 0$ by the scalar $\frac{1}{a}$ (this makes sense since $a \neq 0$), and you get $v = 0$. Done!

(4) Axler, Chapter 1 problem 8: Prove that the intersection of any collection of subspaces of V is itself a subspace of V.
Proof: Note - in class I said it was okay to prove it for a *finite* collection of subspaces (this just makes the notation a little nicer). So let $U_1, \ldots, U_n$ be subspaces of V, and let $U = U_1 \cap \cdots \cap U_n$. Note that by definition of intersection, to be in U, a vector must be in each of the U_i. We show 3 things:
(a) U contains the zero vector. This is because $0 \in U_i$ for each $i = 1, \ldots, n$ (they're all subspaces). Hence 0 is in their intersection U.
(b) U is closed under addition: Pick $v, w \in U$. Then $v, w \in U_i$ for each $i = 1, \ldots, n$ and since these are subspaces, they're closed under addition so $v + w \in U_i$ for each $i = 1, \ldots, n$. Hence $v + w \in U$.
(c) U is closed under scalar multiplication. Pick $v \in U$ and a scalar $c \in \mathbb{F}$. Since $v \in U$, v is in each U_i. Since each U_i is closed under scaling, cv is in each U_i. So $cv \in U$.
Thus U satisfies the conditions of the subspace test.

(5) Prove that $\{p(x) \in P(\mathbb{F}) \,|\, p'(x) = 0\}$ is a subspace of $P(\mathbb{F})$.

Proof: Set $W = \{p(x) \in P(\mathbb{F}) \,|\, p'(x) = 0\}$. As in 4, we check the three conditions of the subspace test.
(a) $0 \in W$ because $0' = 0$.
(b) If $p(x), q(x)$ are two polynomials in W, then $p'(x) = q'(x) = 0$, so

$$(p+q)'(x) = p'(x) + q'(x) = 0 + 0 = 0$$

so $(p+q)(x)$ is in W.
(c) If $p(x) \in W$ and $c \in \mathbb{F}$, then $(cp)'(x) = c(p'(x)) = c \cdot 0 = 0$, so $(cp)(x) \in W$.

(6) Let $V = \mathbb{R}^3$, and let $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x + z = 0 \right\}$.
(a) Find a subspace W_1 of $\mathbb{R}^3$ such that $V \neq U + W_1$.
Solution: There are many choices for W_1, but any correct answer must be a subspace of U. Some examples are: $W_1 = U$, $W_1 = \{0\}$, W_1 is the y-axis, etc.

(b) Find a subspace W_2 of $\mathbb{R}^3$ such that $V = U + W_2$ but $V \neq U \oplus W_2$.
Solution: The correct choices of W_2 are $W_2 = \mathbb{R}^3$, or $W_2 =$ any plane other than U.

(7) Let $V = P_2(\mathbb{F})$, the space of polynomials of degree at most two, with coefficients in $\mathbb{F}$.
(a) Find examples of subspaces U and W of V such that $V \neq U + W$

Solution: For example, let $U = \{c \,|\, c \in \mathbb{F}\}$ (the constant polynomials), and let $W = \{cx \,|\, c \in \mathbb{F}\}$ (the multiples of x^2).

(b) Find examples of subspaces U and W of V such that $V = U + W$ but $V \neq U \oplus W$.
Solution: For example, take $U = \{a + bx \,|\, a, b \in \mathbb{F}\}$ and $W = \{ax + bx^2 \,|\, a, b \in \mathbb{F}\}$ you can certainly build any quadratic polynomial using these subspaces, but they intersect in the space $\{ax \,|\, a \in \mathbb{F}\}$, so the sum is not direct.

(8) Find a polynomial $p(x)$ such that $(1 + x + x^2, 1 - x + x^2, p(x))$ spans $P_2(\mathbb{F})$.
Solution: Pick any polynomial at random and you're almost certain to get a correct answer. For instance, the polynomial $p(x) = x^2$ works. Or $p(x) = 1$, or $p(x) = 1 + x$, or $p(x) = 1 + 2x + 3x^2$, or ...

(9) Consider the subspace $W = \{(x, y, z, w) \in \mathbb{R}^4 \,|\, 2x = z, y = 2w\}$ of $\mathbb{R}^4$.
(a) Find a list of vectors in W which spans W but is not linearly independent.
Solution: Such a list must have more than 2 vectors in it, since W is 2-dimensional, so any spanning list with 2 vectors would be a basis, hence independent. So for example,take

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

(b) Find a list of vectors in W which is linearly independent but does not span W.
Solution: Such a list must have exactly one vector in it. Just write down any nonzero vector which is in W.
(c) Find a basis for W.
Solution: Take the list from (a) but remove the redundant third vector. There are many other choices as well.

(10) Axler, Chapter 2 problem 2: Prove that if $(v_1, \ldots, v_n)$ is linearly independent in V, then so is the list $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$.
Proof: Suppose there are scalars $c_1, \ldots, c_n$ such that

$$c_1(v_1 - v_2) + \cdots + c_{n-1}(v_{n-1} - v_n) + c_n v_n = 0.$$

We must show that these c_is have to be zero. Rearrange the above equation to get

$$c_1 v_1 + (c_2 - c_1)v_2 + (c_3 - c_2)v_3 + \cdots + (c_{n-1} - c_n)v_n = 0.$$

Since the v_is form an independent list, we must have

$$\begin{aligned} c_1 &= 0 \\ c_2 - c_1 &= 0 \\ c_3 - c_2 &= 0 \\ &\vdots \\ c_n - c_{n-1} &= 0. \end{aligned}$$

These equations imply that each c_i is zero. Hence the list is independent.

(11) Axler Chapter 2 problem 3: Suppose $(v_1, \ldots, v_n)$ is a linearly independent list in V and w is some vector in V. Prove that if the list $(v_1 + w, v_2 + w, \ldots, v_n + w)$ is linearly dependent, then w must be in the span of $(v_1, \ldots, v_n)$.
Proof: Since $(v_1 + w, v_2 + w, \ldots, v_n + w)$ is dependent, there are scalars c_i, not all zero, such that

$$c_1(v_1 + w) + c_2(v_2 + w) + \cdots + c_n(v_n + w) = 0.$$

Rearranging this we get

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n + (c_1 + \cdots + c_n)w = 0.$$

The scalar $(c_1 + \cdots + c_n)$ cannot be zero, for if it were, we would have $c_1 v_1 + \cdots + c_n v_n = 0$, and independence of the v_is would force all the c_is to be zero. But we said above that the c_is are not all zero, so that's impossible.

Thus, since the scalar $(c_1 + \cdots c_n)$ is nonzero, we can divide by it and solve for w:

$$w = -\frac{1}{c_1 + \cdots + c_n}\left(c_1 v_1 + \cdots c_n v_n\right),$$

and this shows that w is in the span of the v_is.

(12) Let E be the subset of $P_5(\mathbb{F})$ consisting of *even* polynomials (this means they must satisfy $p(-x) = p(x)$). Prove that E is actually a subspace of $P_5(\mathbb{F})$, find a basis for E, and prove that your answer is actually a basis.

Solution: To check E is a subspace, do the usual 3 things:

(a) Let $z(x)$ be the zero polynomial (so $z(x) = 0$, no matter what x is). Then certainly $z(-x) = z(x)$, since they're both always equal to zero. Thus $z(x) \in E$.

(b) If $p, q \in E$, then $p(x) = p(-x)$ and $q(x) = q(-x)$ for all x. But then

$$(p+q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p+q)(x),$$

which shows that $p + q \in E$, so E is closed under addition.

(c) If c is a scalar and $p \in E$, the $(cp)(-x) = c(p(-x)) = c(p(x)) = (cp)(x)$, so $cp \in E$ and therefore E is closed under scaling.

So E is a subspace. The simplest basis for E is the list $(1, x^2, x^4)$. First off, notice that each of these three polynomials satisfies the condition to be in E. This list is a subset of the list $(1, x, x^2, x^3, x^4, x^5)$ in $P_5(\mathbb{F})$, which we saw in class to be independent. A subset of an independent list is still independent, so $(1, x^2, x^4)$ is an independent list in E.

To see that it spans, take any polynomial $\sum_{k=0}^{5} a_k x^k$ which is in E. Then we know that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_5 x^5 = a_0 + a_1(-x) + a_2(-x)^2 + \cdots + a_5(-x)^5$$

We can cancel all the terms with odd index, and add the even index terms to the left side to get

$$2a_1 x + 2a_3 x^3 + 2a_5 x^5 = 0,$$

which implies that $a_1 = a_3 = a_5 = 0$. Thus our arbitrary polynomial in E can be written as

$$a_0 + a_2 x^2 + a_4 x^4,$$

which show that it's in the span of $(1, x^2, x^4)$. So this list is a basis for E.