Announce Pizza again.

Office Hours: Thu 11:30-12:30

Change

3 Useful/Important Points About Proof-Writing

1. Really important! Do NOT state them
2. Do NOT state them
3. Do NOT state them

Theorem: All humans are from Mars.

Example of a bad proof:

Prove: All humans are female.

Proof: (Yet another bad proof)

Counter: All humans have X-chromosomes.
For a more mathematical, but still logical proof:

**Theorem:** \( 0 = 1 \)

**Proof:**
\[
0 \cdot 0 = 0.0 \tag{what we want to show}
\]
\[
0 = 0 \tag{by Prop last week}
\]

Since \( 0 \cdot 0 \) is true, so is \( 0 = 1 \).

It doesn't work if we go backwards, we have to divide by zero!

(2) "If-and-only-if" proofs.

"\( P \iff Q \) means "if \( P \) then \( Q \) and if \( Q \) then \( P \)"

Usually, you have to prove both directions separately. Often one way is much easier.

**Example**

**Prop:** Let \((V_1, \ldots, V_n)\) be an independent list of vectors in \( V \), all distinct, and let
\[
U_i = \text{Span } \{ V_i \}, \ldots, U_n = \text{Span } \{ V_n \}.
\]

Then
\[
V = U_1 + \cdots + U_n \iff V = U_1 \oplus \cdots \oplus U_n
\]

**Proof:**

\[
\text{(The direction } \implies \text{)} \quad \text{if } V = U_1 \oplus \cdots \oplus U_n,
\]

then \( V = U_1 + \cdots + U_n \)

We have to show that \( V = U_1 + \cdots + U_n \).

But this is part of the definition of a direct sum. [Notice that this makes no part of direction is very easy. We didn't even use the linear independence!]

\[
\implies \quad \text{We use Prop 1.6, which says}
\]

we have to check:

(i) \( V = U_1 + \cdots + U_n \)

(ii) The only \( v \in V \) to write \( 0 \) as a sum \( u_1 + \cdots + u_n \), \( u_i \in U_i, \quad i \) with \( u_i \neq 0 \) for all \( i \)

(i) \( U_1 + \cdots + U_n \neq V \), by our assumption

(ii) Let \( \mathbf{0} = U_1 + \cdots + U_n \), where each \( u_i \in U_i \)

we have to check \( u_i = 0 \) for all \( i \).

Since each \( u_i \in U_i \), and \( U_i = \text{Span } V_i \),

we have \( u_i = c_i V_i \) for some \( c_i \in F \).

So \( \mathbf{0} = c_1 V_1 + \cdots + c_n V_n \).

Since \((V_1, \ldots, V_n)\) is independent, \( c_i = 0 \forall i \).

Thus \( \mathbf{0} = U \), \( U = 0 \forall i \).

Therefore by Prop 1.8, \( V = U_1 \oplus \cdots \oplus U_n \).

This proves the \( \implies \) direction.
To prove two sets are equal, say \( A = B \), you must show \( A \subseteq B \) and \( B \subseteq A \).

Again, it is often the case that one inclusion is much easier than the other.

Example:

**Prop.:** \( R = \{ z \in \mathbb{C} \mid z = \overline{z} \} \) (\( \overline{z} \) is the complex conjugate: \( \overline{a+bi} = a-bi \))

**Proof:**
1) \( R \subseteq \{ z \in \mathbb{C} \mid z = \overline{z} \} \)
   \[ \text{Proof: Let } a = a + 0i; \in R \]
   \[ \text{then } \overline{a} = a + 0i = a \]
   \[ \Rightarrow a \in \{ z \mid z = \overline{z} \} \]
2) \( \{ z \mid z = \overline{z} \} \subseteq R \)
   \[ \text{Proof: Let } z \in \mathbb{C} \text{ be such that } z = \overline{z} \]
   \[ \text{then if } z = a + bi, \text{ we have} \]
   \[ z = \overline{z} \Rightarrow a + bi = a - bi \]
   \[ \Rightarrow 2bi = 0 \]
   \[ \Rightarrow b = 0 \]
   \[ \Rightarrow z = a + 0i \Rightarrow z \in R / / \]

Rest of class: WS.

T/F Answers: 0 T 2 F 0 F 0 T 0 F 0 T 0 F