Math 256B Notes

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April 16, 2012

Abstract

These are my notes for Martin Olsson's Math 256B: Algebraic Geometry course, taught in the Spring of 2012, to be updated continually throughout the semester. In a few places, I have added an example or fleshed out a few details myself. I apologize for any inaccuracies this may have introduced. Please email any corrections, questions, or suggestions to mcivor at math.berkeley.edu.

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Professor Olsson's office hours are Wednesdays 9-11AM in 879 Evans Hall.

1 Wednesday, January 18: Differentials

1.1 An Algebraic Description of the Cotangent Space

We want to construct the *relative cotangent sheaf* associated to a morphism $f\colon X \to Y$. The motivation is as follows. A differential, or dually, a tangent vector, should be something like the data of a point in a scheme, together with an "infinitesimal direction vector" at that point. Algebraically, this can be described by taking the morphism $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ and mapping it into our scheme. These two schemes have the same topological space, and the nilpotency in $k[\epsilon]/(\epsilon^2)$ is meant to capture our intuition of the "infinitesimal". More generally, we can take any "infinitesimal thickening", i.e., a closed immersion $T_0 \hookrightarrow T$ defined by a square-zero ideal sheaf $J \subset \mathcal{O}_T$ and map that into X. Of course, we want this notion to be relative, so we should really map this thickening into the morphism f, which amounts to filling in the dotted arrow in following diagram

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, hook] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

which should be thought of as representing the following picture in the case when $Y = \operatorname{Spec} k$

picture omitted/pending

Example 1.1. Take $Y = \operatorname{Spec} k$, $T = \operatorname{Spec} k$ and $T_0 = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$. Consider the set of morphisms filling in following diagram:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x \in X"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k[\epsilon]/(\epsilon^2) \arrow[r] \arrow[ur, dotted] & \operatorname{Spec} k \end{tikzcd} \quad \text{or, equivalently,} \quad \begin{tikzcd} k & \mathcal{O}_{X,x} \arrow[l] \\ k[\epsilon]/(\epsilon^2) \arrow[u, "\epsilon \mapsto 0"] \arrow[ur, dotted] & k \arrow[l] \arrow[u] \end{tikzcd}$$

Commutativity of the square forces the composite $k \to k$ to be the identity, so the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{X,x}$ maps to zero in k, hence along the dotted arrow, $\mathfrak{m}$ gets mapped into (ϵ), giving a new diagram

$$\begin{tikzcd} k & \mathcal{O}_{X,x}/\mathfrak{m}^2 \arrow[l, "a"] \\ k[\epsilon]/(\epsilon^2) \arrow[u, "\epsilon \mapsto 0"] \arrow[ur, dotted] & k \arrow[l] \arrow[u, "b"'] \end{tikzcd}$$

But $k \oplus \mathfrak{m}/\mathfrak{m}^2 \cong \mathcal{O}_{X,x}/\mathfrak{m}^2$ via the map $(\alpha, \beta) \mapsto b(\alpha) + \beta$. Moreover, $k \oplus \mathfrak{m}/\mathfrak{m}^2$ can be given the structure of a k-algebra by defining the multiplication $(\alpha, \beta) \cdot (\alpha', \beta') = (\alpha\alpha', \alpha\beta' + \alpha'\beta)$. Also, as k-algebras, $k[\epsilon]/(\epsilon^2) \cong k \oplus k \cdot \epsilon$, so in fact we are looking for dotted arrows filling in the following diagram of k-algebras:

$$\begin{tikzcd} k & k \oplus \mathfrak{m}/\mathfrak{m}^2 \arrow[l, "a"] \\ k[\epsilon]/(\epsilon^2) \arrow[u, "\epsilon \mapsto 0"] \arrow[ur, dotted] & k \arrow[l] \arrow[u, "b"'] \end{tikzcd}$$

Such arrows are determined by the map $\mathfrak{m}/\mathfrak{m}^2 \to k$. Thus the set of arrows filling in diagram 1 is in bijection with $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. This justifies the definition of the *tangent space of X over k at x* as $(\mathfrak{m}/\mathfrak{m}^2)^\vee$.

Greater generality can be achieved by working with X as a scheme over some arbitrary scheme Y, and also by replacing the *ring of dual numbers* $k[\epsilon]/(\epsilon^2)$ by a more general construction. Before doing so, we need the notion of relative Spec.

1.2 Relative Spec

Given a scheme X and a quasicoherent sheaf of algebras $\mathcal{A}$ on X, we show how to construct a scheme $\underline{\operatorname{Spec}}_X(\mathcal{A})$. Loosely speaking, the given sheaf $\mathcal{A}$ includes both the data of a ring over each affine open of X, and the gluing data for these rings. We take Spec of each ring $\mathcal{A}(U)$ for opens U of X, and glue them according to the specifications of the sheaf $\mathcal{A}$.

Alternatively, $\underline{\operatorname{Spec}}_X(\mathcal{A})$ is the representing object for the functor

$$F\colon (\text{Schemes}/X)^{op} \to \text{Set}$$

which sends $T \xrightarrow{f} X$ to the set of $\mathcal{O}_T$-algebra morphisms $f^*\mathcal{A} \to \mathcal{O}_T$. Two things need to be checked, namely that this functor is representable at all,

and that the scheme obtained by gluing described above actually is the representing object. Here's why it's locally representable:

If X and T are affine, say $X = \operatorname{Spec} B$, and $T = \operatorname{Spec} R$, with $\mathcal{A} = \tilde{A}$ for some B-algebra A, then $f^*\mathcal{A} = (A \otimes_B R)^\sim$, by definition of the pullback sheaf. By the equivalence of categories between quasicoherent sheaves of $\mathcal{O}_T$-modules and R-modules, the set of $\mathcal{O}_T$-algebra maps $f^*\mathcal{A} \to \mathcal{O}_T$ is in natural bijection with the set of R-algebra maps $A \otimes_B R \to R$, so we are looking for dotted maps in the following diagram

$$\begin{tikzcd} A \otimes_B R \arrow[r, dotted] & R \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$

But the vertical map on the left fits into a fibered square as follows

$$\begin{tikzcd} A \arrow[r] & A \otimes_B R \arrow[r, dotted] & R \\ B \arrow[u] \arrow[r] & R \arrow[u] \arrow[ur] & \end{tikzcd}$$

Moreover, to give a map of R-algebras $A \otimes_B R \to R$ is equivalent, by the universal property of the fibered square, to giving a map $A \to R$ along the top:

$$\begin{tikzcd} A \arrow[r] \arrow[rr, dotted, bend left] & A \otimes_B R \arrow[r, dotted] & R \\ B \arrow[u] \arrow[r] & R \arrow[u] \arrow[ur] & \end{tikzcd}$$

This shows that in this case, the set of maps of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$ is in bijection with the maps of R-algebras $A \to R$, and hence that $\operatorname{Spec} A$ represents the functor F. The remaining details of representability are omitted.

In fact, the relative Spec functor defines an equivalence of categories

$$\begin{tikzcd} \{\text{affine morphisms } Y \to X\} \arrow[rr, bend right] & & \{\text{qcoh } \mathcal{O}_X\text{-algebras}\} \arrow[ll, bend right, "\underline{\operatorname{Spec}}_X"'] \end{tikzcd}$$

where the map in the other direction sends the affine morphism $Y \xrightarrow{f} X$ to $f_*\mathcal{O}_Y$.

1.3 Dual Numbers

In the example of 1.1 we took a scheme X over k and mapped the "infinitesimal thickening" $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ into $X \to \operatorname{Spec} k$. In fact this thickening can be globalized as follows. If X is a scheme and $\mathcal{F}$ a quasicoherent sheaf of $\mathcal{O}_X$-algebras, define a quasicoherent sheaf of $\mathcal{O}_X$-algebras by

$$\mathcal{O}_X[\mathcal{F}] = \mathcal{O}_X \oplus \mathcal{F}$$

with the multiplicative structure given open-by-open by the rule

$$(a, b) \cdot (a', b') = (aa', ab' + a'b)$$

just like before. We then have a diagram of $\mathcal{O}_X$-algebras

$$\begin{tikzcd} \mathcal{O}_X & \mathcal{O}_X \oplus \mathcal{F} \arrow[l, "\mathrm{pr}_1"'] \\ \mathcal{O}_X \arrow[u, "\mathrm{Id}"] \arrow[ur] & \end{tikzcd}$$

where pr_1 denotes projection onto the first factor and Id is the identity map. We apply the relative Spec functor to this diagram, and define the scheme $X[\mathcal{F}]$ to be $\underline{\mathrm{Spec}}_X\, \mathcal{O}_X[\mathcal{F}]$, which by the construction comes with a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{Id}"'] & X[\mathcal{F}] \arrow[dl] \\ X & \end{tikzcd}$$

Example 1.2. Consider the affine case, where $B \xrightarrow{f} A$ is a ring map, and M an A-module. Here we are thinking of X in the above discussion as the scheme $\operatorname{Spec} A$ over the scheme $\operatorname{Spec} B$, and the dual numbers are just $A[M] \cong A \oplus M$, with multiplication defined as above. Thus the diagram of $\mathcal{O}_X$-algebras above, when working over B, looks like

$$\begin{tikzcd} A & A[M] \arrow[l] \\ A \arrow[u, "\mathrm{Id}"] \arrow[ur] & B \arrow[l] \arrow[u] \end{tikzcd}$$

A natural question to ask is: what maps along the diagonal, $A \to A[M]$, make this diagram commute? We take up this question in the next lecture.

2 Friday, January 20th: More on Differentials

Given a ring map $B \to A$ and A-module M, we form $A[M]$ as last time, with the "dual multiplication rule". In the case $A = M = k$, we recover the initial example: $A[M] \cong k[\epsilon]/(\epsilon^2)$.

2.1 Relationship with Leibniz Rule

The module $A[M]$ comes with maps *of A-algebras*

$$A \xrightarrow{\sigma_0} A[M] \xrightarrow{\pi} A$$

whose composition is the identity. We would like to study sections of the map $A[M] \to A$. First we need a definition.

Definition 2.1. If $B \to A$ is a ring map and M is an A-module, a *B-linear derivation of A into M* is a map $\partial\colon A \to M$ satisfying $\partial(b) = 0$ for all b in B, and $\partial(aa') = a\partial(a') + a'\partial(a)$ for al a, a' in A.

The second property is referred to as the Leibniz rule. Note also that by $\partial(b)$, we really mean map b into A using the given ring map, and then apply ∂. The set of B-linear derivations of A into M is denoted $\mathrm{Der}_B(A, M)$, or often just $\mathrm{Der}(M)$ if the ring map $B \to A$ is clear from context. Note also that $\mathrm{Der}(M)$ itself has the structure of an A-module (even though its elements are *not* A-linear maps!): if a is in A and ∂ is in $\mathrm{Der}(M)$, define $(a \cdot \partial)(x) = a \cdot \partial(x)$, the dot on the right being the action of A on M.

For an example, let $B = \mathbb{R}$, $A = C^\infty(\mathbb{R})$, and $M = \Omega^1(\mathbb{R})$, the A-module of smooth 1-forms on $\mathbb{R}$. Then take $\partial = d$, the usual differential operator. The conditions in this case say simply that d kills the constants and satisfies the usual product rule for differentiation.

Proposition 2.1. *The set of A-algebra maps $\phi\colon A \to A[M]$ such that $\pi \circ \phi = \mathrm{Id}$ is in bijection with the set $\mathrm{Der}_B(A, M)$.*

Proof. The bijection is defined by sending a derivation ∂ to the map $A \to A[M]$ given by $a \mapsto a + \partial(a)$. The resulting map is a section of π since it is the identity on the first component; it is A-linear since it is additive in both factors A and M, and for a, a' in A,

$$aa' \mapsto aa' + \partial(aa') = aa' + a\partial(a') + a'\partial(a) = (a + \partial(a))(a' + \partial(a')),$$

the last equality as a result of the definition of the multiplication in $A[M]$.

The map in the other direction sends an A-linear map $\phi\colon A \to A[M]$ to $\partial(a) = \phi(a) - a$. Note that the image of ∂ actually lies in M since $\pi \circ \phi = \mathrm{Id}$.

Moreover, ∂ is a derivation since, writing $\phi(a) = a + m$ for some m in M, we have

$$\begin{aligned}\partial(aa') = \phi(aa') - aa' &= \phi(a)\phi(a') - aa' = (a + m)(a' + m') - aa' \\ &= am' + a'm = a(\phi(a') - a') + a'(\phi(a) - a) \\ &= a\partial(a') + a'\partial(a)\end{aligned}$$

□

2.2 Relation with the Diagonal

Now, to study these maps ϕ from a slightly different point of view, we think of M above as varying over all A-modules, and construct a universal derivation. To begin with, take Spec of the maps above, giving a diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\pi"] \arrow[dr, "\mathrm{Id}"'] & \operatorname{Spec} A[M] \arrow[d, "\sigma_0"] \\ & \operatorname{Spec} A \end{tikzcd}$$

Now, putting the diagram over B, we are looking for ϕ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "\pi"] \arrow[rr, bend left, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & \operatorname{Spec} A[M] \arrow[r, "\phi"] \arrow[d, "\sigma_0"] & \operatorname{Spec} A \arrow[d] \\
& \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B
\end{tikzcd}$$

Let $\Delta\colon \operatorname{Spec} A \to \operatorname{Spec} A \otimes_B A$ denote the diagonal morphism, corresponding to multiplication $A \otimes_B A \to A$. Then to give a morphism ϕ above is equivalent to giving a map ϕ such that

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "\pi"] \arrow[dr] \arrow[ddr, bend right, "\sigma_0"'] & \operatorname{Spec} A[M] \arrow[d, "\sigma_0 \otimes \phi"] \arrow[dr, "\phi"] \arrow[ddl, bend left, "\Delta"] & \\
& \operatorname{Spec} A \otimes_B A \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \operatorname{Spec} A \arrow[d] \\
& \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B
\end{tikzcd}$$

commutes, by the universal property of the fiber product.

If the appearance of the diagonal is surprising, you should think of it in the following way. The composite

$$X \to X \times_Y X \to X$$

can be interpretted as expressing X as a retract of the diagonal, sort of like a tubular neighborhood of X in $X \times_Y X$. We think of the image of a point x in X in $X \times_Y X$ as "two nearby points", which then map back down along the second map to the original point in X.

In any case, the above situation leads to the following algebraic problem. We are interested in $\phi\colon A \otimes_B A \to A[M]$ (note that this is not exactly the same ϕ as above) such that the following commutes:

$$\begin{tikzcd}
& A[M] \arrow[dl, "\pi"'] & \\
A & & A \otimes_B A \arrow[ul, dotted, "\phi"'] \arrow[ll, "\sigma_0"] \\
& & A \arrow[u, "a \mapsto a \otimes 1"'] \arrow[uul, bend left]
\end{tikzcd}$$

Let I be the kernel of the multiplication $A \otimes_B A \to A$. Then the composite

$$I \to A \otimes_B A \xrightarrow{\phi} A[M] \xrightarrow{\pi} A$$

is zero, by commutativity of the triangle above; hence the image of I lies in $M \subset A[M]$, and there is an induced dotted map

$$
\begin{tikzcd}
 & A[M] \arrow[dl, "\pi"'] & M \arrow[l] \arrow[dl, dotted] \\
A & A \otimes_B A \arrow[l] \arrow[u, "\phi"'] & I \arrow[l] \\
 & A \arrow[u, "a \mapsto a \otimes 1"'] \arrow[uul, bend left, "\sigma_0"] &
\end{tikzcd}
$$

Now since $M^2 = 0$ in $A[M]$, $I^2 \subset A \otimes_B A$ maps to 0 in $A[M]$ under any such ϕ, hence the desired ϕ must factor through $A \otimes_B A/I^2$. Thus $A \otimes_B A/I^2$ is the universal object of the form $A[M]$, as M varies. In fact, $A \otimes_B A \cong A[I/I^2]$, since the map $A[I/I^2] \to A \otimes_B A/I^2$ sending $a + \gamma$ to $(a \otimes 1) + \gamma$ fits into the following map of exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, "\cong"'] & A[I/I^2] \arrow[r, "\pi"] \arrow[d] & A \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & I/I^2 \arrow[r] & A \otimes A/I^2 \arrow[r, "\text{mult}"] & A \arrow[r] & 0
\end{tikzcd}
$$

so it is an isomorphism by the five lemma.

Summarizing our results, we have established

Theorem 2.1. *The following four sets are in natural bijection:*

1. $\mathrm{Der}_B(A, M)$
2. $\{A$*-algebra maps* $\phi\colon A \to A[M] \mid \pi \circ \phi = \mathrm{Id}\}$
3. $\{A$*-algebra maps* $\mid$ $\begin{tikzcd} A[I/I^2] \arrow[rr] \arrow[dr] & & A[M] \arrow[dl] \\ & A & \end{tikzcd}$ $\}$
4. $\mathrm{Hom}_A(I/I^2, M)$

The bijection between 3 and 4 is given by sending ϕ to its restriction to the second factor, I/I^2. We can express the equivalence of 1 and 4 as saying that the functor

$$\mathrm{Der}\colon \mathrm{Mod}_A \to \mathrm{Mod}_A$$

as described above is represented by I/I^2.

2.3 Kähler Differentials

We have seen that every A-linear map $A \to A[M]$ corresponds to a derivation $\partial\colon A \to M$; the map $A \to A[I/I^2]$ is associated to the *universal derivation* $\mathrm{d}\colon A \to I/I^2$, defined by

$$\mathrm{d}(a) = a \otimes 1 - 1 \otimes a$$

The fact that this lies in I is implied by the following:

Lemma 2.1. *The kernel I of the multiplication map $A \otimes_B A \to A$ is generated by elements of the form $1 \otimes a - a \otimes 1$.*

Proof. Clearly these elements are in I. Conversely, let $\Sigma x \otimes y$ be any element of I. Then $\Sigma xy = 0$, so $\Sigma(xy \otimes 1) = 0$, giving $\Sigma x \otimes y = \Sigma x(1 \otimes y - y \otimes 1)$. $\square$

Theorem 2.2. *The map* d *defined above is a derivation, and is universal in the following sense: for any B-linear derivation $\partial\colon A \to M$, there is a unique A-linear map $\phi\colon I/I^2 \to M$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\partial"'] & I/I^2 \arrow[d, "\phi"] \\ & M \end{tikzcd}$$

commutes.

Proof. The universal property follows from the preceding discussion of I/I^2. It remains only to check that d is actually a derivation. For b in B, we have

$$\mathrm{d}(b) = 1 \otimes b - b \otimes 1 = b \otimes 1 - b \otimes 1 = 0$$

using the fact that the tensor product is taken over B. For the Leibniz rule, first note that I is generated by elements of the form $1 \otimes a - a \otimes 1$. Then let a and a' be elements of A, and compute

$$\begin{aligned} \mathrm{d}(aa') - a\,\mathrm{d}(a') - a'\,\mathrm{d}(a) &= aa' \otimes 1 - 1 \otimes aa' - a(a' \otimes 1 - 1 \otimes a') - a'(a \otimes 1 - 1 \otimes a) \\ &= aa' \otimes 1 - 1 \otimes aa' - aa' \otimes 1 + a \otimes a' - a'a \otimes 1 + a' \otimes a \\ &= -1 \otimes aa' - aa' \otimes 1 + a \otimes a' + a' \otimes a \\ &= (a \otimes 1 - 1 \otimes a)(a' \otimes 1 - 1 \otimes a'), \end{aligned}$$

and this is zero in I/I^2.

$\square$

Remark. The above computation used the A-module structure on $A \otimes_B A$ given by $a \cdot (x \otimes y) = ax \otimes y$. But there is also the action given by $a \cdot (x \otimes y) = x \otimes ay$. Both of these actions, of course, restrict to I, but in fact when working modulo I^2, these two restrictions agree. This follows from Lemma 1: if a is an element of A, then the difference of the two actions of a on I can be described as multiplication by $(1 \otimes a - a \otimes 1)$ (using the usual multiplication for the tensor product of rings). But the lemma shows that this is in I, and hence its action on I is zero modulo I^2. Professor Olsson used this in his computation that d satisfies the Leibniz rule.

Definition 2.2. The A-module I/I^2 is called the *module of (relative) Kähler differentials* of $B \to A$, and denoted $\Omega_{A/B}$. Note that it comes together with the derivation $\mathrm{d}\colon A \to \Omega_{A/B}$.

In light of the previous discussion, one thinks of $\Omega_{A/B}$ as being the algebraic analogue of the cotangent bundle. Here is an example to further support this terminology.

Example 2.1. With notation as above, let $B = k$, and $A = k[x, y]/(y2 - x^3 - 5)$. Then

$$\Omega_{A/B} \cong \frac{A\,\mathrm{d}x \oplus A\,\mathrm{d}y}{2y\,\mathrm{d}y - 3x^2\,\mathrm{d}x},$$

as you might expect from calculus. The universal derivaton $\mathrm{d}\colon A \to \Omega_{A/B}$ is given by the usual differentiation formula: $\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$; the quotient ensures that this is well defined when we work with polynomials modulo $y^2 - x^3 - 5$. The universal property is expressed by the following diagram:

$$\begin{tikzcd} A \arrow[rr, "\mathrm{d}"] \arrow[dr, "\partial"'] & & \Omega_{A/B} \arrow[dl, dotted, "\phi"] \\ & M & \end{tikzcd}$$

The map $\phi\colon \Omega_{A/B} \to M$ guaranteed by the theorem sends $\mathrm{d}x$ and $\mathrm{d}y$ to $\partial(x)$ and $\partial(y)$, respectively.

3 Monday, Jan. 23rd: Globalizing Differentials

Today we will globalize the construuctions of last week, obtaining the sheaf $\Omega^1_{X/Y}$ of relative differentials associated to a morphism of schemes $X \to Y$. We could do this by gluing together the modules $\Omega_{A/BB}$ of last time, but instead we give a global definition, and then check that it reduces to the original definition in the affine case.

Recall that for a ring map $B \to I$, letting I be the kernel of $A \otimes_B A \to A$, we defined $\Omega_{A/B} = I/I^2$. To relate this to today's construction, it will be helpful to bear in mind the isomorphism

$$I/I^2 \cong I \otimes_{A \otimes_B A} A$$

which emphasizes the fact that this object is a pullback along the diagonal.

3.1 Definition of the Sheaf of Differentials

Let $f\colon X \to Y$ be a morphism of schemes. It induces a map $\Delta_{X/Y}\colon X \to X \times_Y X$, from the following diagram:

$$\begin{tikzcd} X \arrow[dr, dotted, "\Delta_{X/Y}"] \arrow[drr, bend left, "\mathrm{Id}"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\ & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

The map $\Delta_{X/Y}$ comes with a map of sheaves $\Delta^{\#}_{X/Y}\colon \mathcal{O}_{X \times_Y X} \to (\Delta_{X/Y})_* \mathcal{O}_X$ on $X \times_Y X$, which we will just abbreviate by Δ. Although really a map of sheaves of rings, we will regard it as a map of sheaves of $\mathcal{O}_{X \times X}$-modules. Let $\mathcal{J}$ be its kernel, again regarded as an $\mathcal{O}_{X \times X}$-module.

Definition 3.1. The *sheaf of relative differentials* of $X \to Y$ is the $\mathcal{O}_X$-module $\Omega^1_{X/Y} = \Delta^* \mathcal{J}$.

Remark. Note that $\Delta_{X/Y}$ is not necessarily a closed immersion, so a priori $\mathcal{J}$ is just a sheaf of $\mathcal{O}_{X\times_Y X}$-modules.

Now from the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/Y}"] \arrow[dr, "\mathrm{Id}"'] & X \times_Y X \arrow[d, bend right, "\mathrm{pr}_1"'] \arrow[d, bend left, "\mathrm{pr}_2"] \\ & X \end{tikzcd}$$

we get maps of sheaves $\delta_i\colon \mathrm{pr}_i^{-1}\,\mathcal{O}_X \to \mathcal{O}_{X\times_Y X}$ for $i = 1, 2$. Applying the functor $\Delta_{X/Y}^{-1}$ to these morphisms gives maps

$$\Delta^{-1}\,\mathrm{pr}_i^{-1}\,\mathcal{O}_X \xrightarrow{\Delta^{-1}\delta_i} \Delta^{-1}\mathcal{O}_{X\times_Y X},$$

but by the commutativity of the diagram of schemes, we have $\Delta^{-1}\,\mathrm{pr}_i^{-1}\,\mathcal{O}_X \cong \mathcal{O}_X$. Thus we have maps of sheaves on X, $\Delta^{-1}\delta_i\colon \mathcal{O}_X \to \Delta^{-1}\mathcal{O}_{X\times_Y X}$. Both maps (for $i = 1, 2$) agree when followed by the map $\Delta^{-1}\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X$, so their difference maps into the kernel of $\Delta^{-1}\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X$, giving

$$\Delta^{-1}\delta_1 - \Delta^{-1}\delta_2\colon \mathcal{O}_X \to \ker(\Delta^{-1}\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X)$$

Recall that $\mathcal{J}$ was defined as the kernel of $\mathcal{O}_{X\times_Y X} \to \Delta_*\mathcal{O}_X$. Applying Δ^{-1}, we have

$$\Delta^{-1}\mathcal{J} = \Delta^{-1}\ker(\mathcal{O}_{X\times_Y X} \to \Delta_*\mathcal{O}_X)$$

Using the left-exactness of Δ^{-1}, we can push it across the kernel and then apply the adjunction between Δ^{-1} and Δ_* to get

$$\Delta^{-1}\mathcal{J} = \ker(\Delta^{-1}\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X)$$

Putting this together we now have a map

$$\Delta^{-1}\delta_1 - \Delta^{-1}\delta_2\colon \mathcal{O}_X \to \Delta^{-1}\mathcal{J}$$

Now follow this with the map $\Delta^{-1}\mathcal{J} \to \Delta^*\mathcal{J} = \Omega^1_{X/Y}$. We have now, at great length, constructed the *universal derivation* $\mathrm{d} = \Delta^{-1}\delta_1 - \Delta^{-1}\delta_2\colon \mathcal{O}_X \to \Omega^1_{X/Y}$, which is the global version of the map $\mathrm{d}A \to \Omega^1_{A/B}$ in the affine case.

3.2 Relating the Global and Affine Constructions

We now verify that our new definition of the cotangent sheaf agrees with the affine definition locally, in the process showing further that $\Omega^1_{X/Y}$ is quasicoherent.

Proposition 3.1. *1. $\Omega^1_{X/Y}$ is quasicoherent.*

2. If $f\colon X \to Y$ is locally of finite type and Y is locally Noetherian, then $\Omega^1_{X/Y}$ is coherent.

Proof. Pick x in X, let $y = f(x)$, take an open affine neighborhood $\operatorname{Spec} B$ of y, and choose an open $\operatorname{Spec} A$ of X which lies in the preimage of $\operatorname{Spec} B$ and contains x. This gives a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\text{open}"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} B \arrow[r, "\text{open}"] & Y \end{tikzcd}$$

mapping the inclusion $\operatorname{Spec} A \to X$ into the fiber product gives a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\Delta_{A/B}"] \arrow[d, "\text{open}"'] & \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A \arrow[d] \\ X \arrow[r, "\Delta_{X/Y}"] & X \times_Y X \end{tikzcd}$$

Let $J = \ker(A \otimes_B A \to A)$. Our goal is to show that

$$\Omega^1_{X/Y}\Big|_{\operatorname{Spec} A} \cong (J/J^2)^\sim,$$

which will prove simultaneously that $\Omega^1_{X/Y}$ is quasicoherent, and that it agrees with our old definition in the affine case. To do so we make the following computation

$$\begin{aligned}
\Omega^1_{X/Y}\Big|_{\operatorname{Spec} A} &\cong \left(\Delta^{-1}_{X/Y}\mathcal{J} \otimes_{\Delta^{-1}\mathcal{O}_{X\times X}} \mathcal{O}_X\right)\Big|_{\operatorname{Spec} A} && (1)\\
&\cong \left(\Delta^{-1}\mathcal{J}\right)\Big|_{\operatorname{Spec} A} \otimes_{(\Delta^{-1}\mathcal{O}_{X\times X})|_{\operatorname{Spec} A}} \mathcal{O}_X\Big|_{\operatorname{Spec} A} && (2)\\
&\cong \ker(\Delta^{-1}\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X)\Big|_{\operatorname{Spec} A} \otimes_{\Delta^{-1}(A\otimes_B A)^\sim} A^\sim && (3)\\
&\cong \Delta^{-1}_{A/B}\big(\ker\left((A\otimes_B A)^\sim \to A^\sim\right)\big) \otimes_{\Delta^{-1}(A\otimes_B A)^\sim} A^\sim && (4)\\
&\cong \Delta^{-1}J^\sim \otimes_{\Delta^{-1}(A\otimes_B A)^\sim} A^\sim && (5)\\
&\cong \Delta^*(J^\sim) && (6)\\
&\cong \left(J \otimes_{A\otimes_B A} A\right)^\sim && (7)\\
&\cong \left(J/J^2\right)^\sim && (8)
\end{aligned}$$

The justifications for the steps are as follows:

(1) Definition of $\Omega^1_{X/Y}$ (and definition of pullback sheaf)

(2) Tensor product commutes with restriction

(3) Definition of $\mathcal{J}$

(4) Left-exactness of Δ^{-1} and commutativity of the square immediately above this computation, and for the subscript on the $\otimes$, the fact that Δ^{-1} commutes with restriction

(5) Most importantly, the fact that $\ker\big((A\otimes_B A)^\sim \to A^\sim\big)$ is quasicoherent, which follows from the fact that in the affine case, the diagonal is a closed immersion (note that $\mathcal{J}$, on the other hand, may not be quasicoherent, precisely because in general, the diagonal map need not be a closed immersion) *Thanks to Chang-Yeon for pointing this out to me*

(6) Definition of pullback

(7) Pullback corresponds to tensor product over affines

(8) An observation made at the very beginning of this lecture

This proves part 1 of the theorm. Part 2 was an exercise on HW 1. □

Note: At this point Prof. Olsson made some remark about "the key fact here is a certain property of the diagonal map ..." I didn't write this down, but if anyone remembers what he said here, please let me know.

Corollary 3.1. *If $B \to A$ is a ring map and f is an element of A, then the unique map*

$$\Omega^1_{A/B} \otimes_A A_f \to \Omega^1_{A'/B}$$

is an isomorphism.

Proof. This follows immediately from quasicoherence, but here's an alternate proof. Check that the localization of $\mathrm{d}\colon A \to \Omega_{A/B}$ gives an A_f-linear derivation $\mathrm{d}_f\colon A_f \to \Omega_{A_f/B}$. Precomposing d_f with the localization map $A \to A_f$ gives a derivation $A \to \Omega_{A_f/B}$ so by the universal property of $\Omega_{A_f/B}$ there is a unique A-linear map $\Omega_{A/B} \to \Omega_{A_f/B}$ which commutes with the localiztion map, d, and d_f. Now one checks that this map $\Omega_{A/B} \to \Omega_{A_f/B}$ satifies the universal property of the localization $\Omega_{A/B} \to \Omega_{A/B} \otimes A_f$. This also was assigned on HW 1, so I will not fill in the details. □

3.3 Relation with Infinitesimal Thickenings

We motivated our discussion of differentials with talk of mapping infinitesimal thickenings into a morphism $X \to Y$. Having now made our explicit definition of the cotangent sheaf, it's time we related it back to that initial discussion. The following theorem says loosely that if there is a way to do this at all, the number of ways to do so is controlled by the cotangent sheaf.

First fix some notation. Let $T_0 \to T$ be a closed immersion defined by a square-zero quasicoherent (follows from closed immersion) ideal sheaf $\mathcal{J}$. Although $\mathcal{J}$ is a sheaf on T, the fact that $\mathcal{J}^2 = 0$ means the action of $\mathcal{O}_T$ on $\mathcal{J}$ factors through that of $\mathcal{O}_{T_0}$, so in fact we may regard $\mathcal{J}$ as a sheaf on T_0 (recall that their topologial spaces are the same). Let x_0 be a point of X, and let S be the set of morphisms filling in the following diagram

$$\begin{tikzcd} T_0 \arrow[r, "x_0"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r, "y"] \arrow[ur, dotted] & Y \end{tikzcd}$$

Theorem 3.1. *Either $S = \emptyset$ or else there is a simply transitive action of* $\mathrm{Hom}_{\mathcal{O}_{T_0}}(x_0^*\Omega^1_{X/Y}, \mathcal{J})$ *on S.*

Intuitively, this says that if we look at a point x in our scheme X, although there is no preferred choice of tangent direction at X, once one is chosen, we can obtain all the other by a rotation. In other words, the difference of two tangent directions is given by a vector in the cotangent space.

We illustrate the theorem with a concrete example.

Example 3.1. Let k be an algebraically closed field and X/k a finite-type k-scheme, with X regular, connected, and of dimension 1. Thus if x is a closed point, the local ring $\mathcal{O}_{X,x}$ is a DVR.

Proposition 3.2. *In the situation above, $\Omega^1_{X/k}$ is locally free of rank 1.*

Proof. We know that $\Omega^1_{X/Y}$ is coherent by part 2 of Proposition 2. We first show that for any closed point x of X, the k-vector space $\Omega^1_{X/k}(x) = \Omega^1_{X/k,x}/\mathfrak{m}_x\Omega^1_{X/k,x}$ (equality since k is algebraically closed) is one-dimensional. For this we calculate the dimension of the dual vector space $\mathrm{Hom}_k(\Omega^1_{X/Y}(x), k)$, which by the pevious theorem is equal to the set of dotted arrows

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k[\epsilon]/\epsilon^2 \arrow[r] \arrow[ur, dotted] & \operatorname{Spec} k \end{tikzcd}$$

and we calculated in Example 1 of Lecture 1 that this is equal to $\mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$, which is one-dimensional since $\mathcal{O}_{X,x}$ is a DVR. Now, applying "geometric Nakayama" (see Vakil 14.7D), we can lift a generator of $\Omega^1_{X/k}$ to some affine neighborhood of x. Thus $\Omega^1_{X,k}$ is locally free of rank 1.

□

4 Wednesday, Jan. 25th: Exact Sequences Involving the Cotangent Sheaf

Before proving two sequences which give a further geometric perspective on all we've done so far, it would be nice if the construction of $\Omega^1_{X/Y}$ were functorial in some appropriate sense. Since this sheaf is associated to a *morphism* $X \to Y$, we should consider a "morphism of morphisms", i.e., a commutative diagram.

4.1 Functoriality of $\Omega^1_{X/Y}$

Theorem 4.1. *Given a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

there is an associated morphism $g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ of sheaves on X'.

Proof. In the following diagram, the two outer "trapezoids" are the commutative square given to us, and the upper left square is the usual fiber square

$$\begin{tikzcd}
X' \times_{Y'} X' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[dd] \\
X' \arrow[r] \arrow[d] & Y' \arrow[dr] & \\
X \arrow[rr] & & Y
\end{tikzcd}$$

We get a map $g \times g \colon X' \times_{Y'} X' \to X \times_Y X$ by the universal property of $X \times_Y X$, which also commutes with the relevant diagonal maps, giving us the square

$$\begin{tikzcd}
X' \arrow[r, "\Delta_{X'/Y'}"] \arrow[d, "g"'] & X' \times_{Y'} X' \arrow[d, "g \times g"] \\
X \arrow[r, "\Delta_{X/Y}"'] & X \times_Y X
\end{tikzcd}$$

From the definition of $\mathcal{J}$ we have a sequence

$$\mathcal{J}_{X/Y} \to \mathcal{O}_{X \times_Y X} \to (\Delta_{X/Y})_* \mathcal{O}_X$$

of sheaves on $X \times_Y X$, and an analogous sequence of sheaves on $X' \times_{Y'} X'$. Applying $(g \times g)^*$ to the first sequence gives

$$(g \times g)^* \mathcal{J}_{X/Y} \to (g \times g)^* \mathcal{O}_{X' \times_{Y'} X'} \to (g \times g)^* (\Delta_{X/Y})_* \mathcal{O}_X$$

on $X' \times_{Y'} X'$. Now commutativity of the square above shows that $(g \times g) \circ \Delta_{X'/Y'} = \Delta_{X/Y} \circ g$, so $(g \times g)^* (\Delta_{X/Y})_* \mathcal{O}_X = (\Delta_{X'/Y'})_* \, g^* \mathcal{O}_X$. But the pullback of the structure sheaf along any morphism is always the structure sheaf, so $g^* \mathcal{O}_X \cong \mathcal{O}_{X'}$, and also $(g \times g)^* \mathcal{O}_{X \times_Y X} \cong \mathcal{O}_{X' \times_{Y'} X'}$. This gives us a diagram of sheaves on $X' \times_{Y'} X'$:

$$\begin{tikzcd}
(g \times g)^* \mathcal{J}_{X/Y} \arrow[r] & \mathcal{O}_{X' \times_{Y'} X'} \arrow[r] \arrow[d, "\cong"] & (g \times g)^* (\Delta_{X/Y})_* \mathcal{O}_X \arrow[d] \\
\mathcal{J}_{X'/Y'} \arrow[r] & \mathcal{O}_{X' \times_{Y'} X'} \arrow[r] & (\Delta_{X'/Y'})_* \mathcal{O}_{X'}
\end{tikzcd}$$

Even after applying $(g \times g)^*$, the composite along the top is still zero, hence the map $(g \times g)^* \mathcal{J}_{X/Y} \to \mathcal{O}_{X' \times_{Y'} X'} \overset{\cong}{\to} \mathcal{O}_{X' \times_{Y'} X'} \to (\Delta_{X'/Y'})_* \mathcal{O}_{X'}$ is zero, so it factors through $\mathcal{J}_{X'/Y'}$, giving us a map $(g \times g)^* \mathcal{J}_{X/Y} \to \mathcal{J}_{X'/Y'}$. Now if we apply $\Delta^*_{X'/Y'}$ to this morphism we get a map

$$g^* \Omega^1_{X/Y} = g^* \Delta^*_{X/Y} \mathcal{J}_{X/Y} = \Delta^*_{X'/Y'} (g \times g)^* \mathcal{J}_{X/Y} \to \Delta^*_{X'/Y'} \mathcal{J}_{X'/Y'} = \Omega^1_{X'/Y'}$$

which is what we were after.

□

In the affine case, this construction goes as follows. We are given a diagram

$$\begin{tikzcd} \operatorname{Spec} A' \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} B' \arrow[r] & \operatorname{Spec} B \end{tikzcd}$$

which yields maps of rings

$$\begin{tikzcd} I' \arrow[r] & A' \otimes_{B'} A' \arrow[r] & A' \\ I \arrow[r] \arrow[u, "\alpha"] & A \otimes_B A \arrow[r] \arrow[u, "\beta"] & A \arrow[u] \end{tikzcd}$$

Here α is just the restriction of β to I; commutativity of the square on the right ensures that it factors through I'.

First we form the $A' \otimes_{B'} A'$-module $I \otimes_{A \otimes_B A} \left(A' \otimes_{B'} A'\right)$, which is the analogue of the pullback $(g \times g)^* \mathcal{J}$. Now we have an $A \otimes_B A$-linear map $\alpha \colon I \to I'$, and by the change of base adjunction

$$\operatorname{Hom}_{A' \otimes_{B'} A'} \left(I \otimes_{A \otimes_B A} \left(A' \otimes_{B'} A'\right), I'\right) \cong \operatorname{Hom}_{A \otimes_B A}(I, I')$$

this gives a map $I \otimes_{A \otimes_B A} \left(A' \otimes_{B'} A'\right) \to I'$ of $A' \otimes_{B'} A'$-modules, which corresponds to the map $(g \times g)^* \mathcal{J}_{X/Y} \to \mathcal{J}_{X'/Y'}$ in the above. Finally we apply the functor $- \otimes_{A' \otimes_{B'} A'} A'$ to this map, which gives a map

$$\left(I \otimes_{A \otimes_B A} \left(A' \otimes_{B'} A'\right)\right) \otimes_{A' \otimes_{B'} A'} A' \to I' \otimes_{A' \otimes_{B'} A'} A'$$

But $I' \otimes_{A' \otimes_{B'} A'} A' = \Omega^1_{A'/B'}$ by definition, and by "changing base all over the place,"

$$\left(I \otimes_{A \otimes_B A} \left(A' \otimes_{B'} A'\right)\right) \otimes_{A' \otimes_{B'} A'} A' = I \otimes_{A \otimes_B A} A' = \left(I \otimes_{A \otimes_B A} A\right) \otimes_A A'$$

and this is $\Omega^1_{A/B} \otimes_A A'$, so we have produced the desired map. If you work through the construction, you find that this map is just $\mathrm{d}a \otimes a' \mapsto a' \cdot \mathrm{d}(g(a))$, where g is the given map $A \to A'$. In particular, this map is surjective if g is.

4.2 First Exact Sequence

The following construction is what Vakil calls the "relative cotangent exact sequence" (Thm 23.2.24).

Theorem 4.2. *Given morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is an exact sequence of sheaves on X:*

$$f^* \Omega^1_{X/Y} \to \Omega^1_{X/Z} \to \Omega^1_{Y/Z}$$

Proof. The result essentially follows from the following diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta_{X/Y}"] \arrow[rr, bend left, "\Delta_{X/Z}"] \arrow[dr, "f"'] & X \times_Y X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\
& Y \arrow[r, "\Delta_{Y/Z}"] & Y \times_Z Y
\end{tikzcd}$$

in which the square is cartesian - it is a *magic square* (see Vakil 2.3R and take $X_1 = X_2 = X$). Let's see what this says in the affine case. Then the given morphisms correspond to ring maps $A \leftarrow B \leftarrow C$, and the diagram above looks like

$$\begin{tikzcd}
A & A \otimes_B A \arrow[l] & A \otimes_C A \arrow[l, two heads] \\
& B \arrow[ul] \arrow[u] & B \otimes_C B \arrow[l] \arrow[u]
\end{tikzcd}$$

We have a right exact sequence

$$\ker(B \otimes_C B \to B) \to B \otimes_C B \to B \to 0$$

to which we apply the right exact functor $- \otimes_{B \otimes_C B} A \otimes_C A$, giving us

$$\big(A \otimes_C A\big) \otimes_{B \otimes_C B} \big(\ker(B \otimes_C B \to B)\big) \to \big(A \otimes_C A\big) \to \big(A \otimes_C A\big) \otimes_{B \otimes_C B} B \to 0$$

Here we have changed base at the middle term: $\big(A \otimes_C A\big) \otimes_{B \otimes_C B} \big(B \otimes_C B\big) \cong A \otimes_C A$. This is where the magic square comes in: it says that $A \otimes_B A$, which sits at the top left of that square, satisfies the universal property of the tensor product of B and $A \otimes_C A$ over the base $B \otimes_C B$, so our sequence becomes

$$\big(A \otimes_C A\big) \otimes_{B \otimes_C B} \big(\ker(B \otimes_C B \to B)\big) \to \big(A \otimes_C A\big) \to A \otimes_B A \to 0$$

Now the term on the left is already what we want; the other two terms sit in their own sequences

$$\begin{tikzcd}
& A & A & \\
\big(A \otimes_C A\big) \otimes \big(\ker(B \otimes B \to B)\big) \arrow[r] & A \otimes_C A \arrow[r] \arrow[u] & A \otimes_B A \arrow[r] \arrow[u] & 0 \\
& \ker\big(A \otimes_C A \to A\big) \arrow[u] & \ker\big(A \otimes_B A \to A\big) \arrow[u] &
\end{tikzcd}$$

There is an induced map $\ker\big(A \otimes_C A \to A\big) \to \ker\big(A \otimes_B A \to A\big)$. We would like to also find a map $\big(A \otimes_C A\big) \otimes \big(\ker(B \otimes B \to B)\big) \to \ker\big(A \otimes_C A \to A\big)$. For this we have to show that the composite

$$\big(A \otimes_C A\big) \otimes \big(\ker(B \otimes B \to B)\big) \to A \otimes_C A \to A$$

is zero. But this is clear if you remember that we just changed the base on the middle term. So the map on the left is just $\mathrm{Id} \otimes i$, where $i\colon \big(\ker(B \otimes B \to B)\big) \to B \otimes B$ is the inclusion. Then we follow this with multiplcation, mapping us into A. But the magic square above show that anything in the kernel of $B \otimes B \to B$ maps to 0 in A. So the leftmost map factors through the kernel of the center column, and we obtain our sequence

$$\big(A \otimes_C A\big) \otimes_{B \otimes_C B} \big(\ker(B \otimes_C B \to B)\big) \to \ker\big(A \otimes_C A \to A\big) \to \ker\big(A \otimes_B A \to A\big) \to 0$$

which by definition is

$$f^*\Omega^1_{C/B} \to \Omega^1_{C/A} \to \Omega^1_{B/A} \to 0$$

□

4.3 Second Exact Sequence

The next Theorem describes how differentials behave under a closed immersion. Of course, since the construction is relative, we should look at a closed immersion $X \to Y$ over some base scheme Z.

Theorem 4.3. *Let*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

be a diagram of schemes, where i is a closed immersion defined by the ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$. Then the map $\mathcal{I} \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_{Y/Z}$ induces an $\mathcal{O}_X$-linear map $\bar{i}\colon i^*\mathcal{I} \to i^*\Omega^1_{Y/Z}$ which fits into an exact sequence

$$i^*\mathcal{I} \xrightarrow{\bar{d}} i^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to 0$$

Proof. (This is the corrected version, actually given in the following lecture) We'll prove it only in the affine case. In this setting, we have a diagram

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r, "i"] \arrow[dr] & \operatorname{Spec} A \arrow[d, "f"] \\ & \operatorname{Spec} B \end{tikzcd}$$

and we wish to exhibit an exact sequence

$$I/I^2 \xrightarrow{\bar{\mathrm{d}}} \Omega^1_{A/B} \otimes_A A/I \to \Omega^1_{A/I/B} \to 0$$

We claim that the sequence can be found buried in the belly of this beast:

$$\begin{tikzcd}
I \otimes_{A\otimes_B A} A \arrow[d, "a\otimes f\mapsto a\otimes f - f\otimes a"'] & \ker(A\otimes_B A \to A) \arrow[d] \arrow[dr] & \\
I\otimes A \oplus A\otimes I \arrow[r] \arrow[dr, "\Sigma"'] & \ker(A\otimes_B A \to A/I) \arrow[r, two heads] \arrow[d] & \ker(A/I\otimes_B A/I \to A/I) \\
& I \arrow[d] & \\
& 0 &
\end{tikzcd}$$

Here Σ is the map defined by $a\otimes f + a'\otimes f' \mapsto af + a'f'$, extended by linearity.

I haven't worked through this yet. I'll come back to it. In case I forget, the argument in Matsumura is nice and short. Read that instead.

$\square$

Example 4.1 (Jacobian Matrix Calculation)**.** Let $B \to A$ be a finite type ring map, with A and B Noetherian. We compute $\Omega^1_{A/B}$ as follows. First write

$$A = B[x_1, \dots, x_n]/(f_1, \dots, f_s)$$

Then we have morphisms

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] & \mathbb{A}^n_B \arrow[d] \\
& \operatorname{Spec} B
\end{tikzcd}$$

We have computed already that $\Omega^1_{B[x_i]/B} =\cong B\,\mathrm{d}x_1 \oplus \cdots \oplus B\,\mathrm{d}x_n$, and pulling this back to $\operatorname{Spec} A$ is the same as tensoring up with A, giving

$$\Omega^1_{B[x_i]/B}\Big|_A \cong A\,\mathrm{d}x_1 \oplus \cdots \oplus A\,\mathrm{d}x_n$$

Now by the previous proposition, this surjects onto $\Omega^1_{A/B}$ with kernel I/I^2, where $I = (f_1, \dots, f_s)$. Furthermore, there is an obvious surjection $A^s \to I/I^2$, sending the i-th basis element of A^s to f_i. Since precomposing with a surjection does not change the cokernel, the sequence

$$A^s \to A\,\mathrm{d}x_1 \oplus \cdots \oplus A\,\mathrm{d}x_n \to \Omega^1_{A/B}$$

expresses $\Omega^1_{A/B}$ as the cokernel of the map $A^s \to A\,\mathrm{d}x_1 \oplus \cdots \oplus A\,\mathrm{d}x_n$. But consider the image of a basis element e_i under this map. It maps first to $f_i \bmod I^2$. Recall that the map I/I^2 sends the class of f to $\mathrm{d}f = \sum \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$. Thus in terms of the bases $\{e_i\}$ for A^s and $\{\,\mathrm{d}x_j\}$ for $A\,\mathrm{d}x_1 \oplus \cdots \oplus A\,\mathrm{d}x_n$, this map is represented by the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$, the familiar *Jacobian matrix*. This is the key to computing $\Omega^1_{A/B}$ in many applications.

5 Friday, Jan. 27th: Differentials on $\mathbb{P}^n$

Remark. The first problem on HW 1 should have convinced you that at least in algebraic geometry, inseparable field extensions are not pathologies, despite what your Galois Theory course may have led you to believe. One common situation in which they arise is the following. Say we have a finite type k-scheme X, where k is algebraically closed, and a morphism $X \to Y$ whose fibers are curves. Then one is led to consider the map of fields $k(x) \to k(y)$, where $k(y)$ is typically not a perfect field. It may be, for example, of the form $k(t_1, \ldots, t_n)$, such as appeared on the assignment.

5.1 Preliminaries

Anyway, today will will compute the cotangent sheaf of $\mathbb{P}^n_S$. There are many more concrete proofs (e.g., in Hartshorne), but we will take the functorial approach. Recall that $\mathbb{P}^n_S$ represents the functor

$$\Big(\text{Schemes}/S\Big)^{\text{op}} \to \text{Set}$$

sending $\big(T \to S\big)$ to the set of all surjections $\mathcal{O}_T^{n+1} \to \mathcal{L}$, where $\mathcal{L}$ is a locally free sheaf of rank one on T, up to isomorphism. In particular, plugging $\mathbb{P}^n_S$ itself in to this functor, we get an isomorphism

$$h_{\mathbb{P}^n_S}(\mathbb{P}^n_S) \cong F(\mathbb{P}^n_S),$$

and the image of $\mathrm{Id}_{\mathbb{P}^n_S}$ under this isomorphism is the surjection

$$\mathcal{O}_{\mathbb{P}^n_S}^{n+1} \to \mathcal{O}_{\mathbb{P}^n_S}(1)$$

where $\mathcal{O}_{\mathbb{P}^n_S}(-1)$ is the so-called "twisting sheaf". A HW problem from last term computed the abelian group structure of $\mathrm{Pic}(\mathbb{P}^n_S)$ under tensor product. Applying $- \otimes_{\mathcal{O}_{\mathbb{P}^n_S}} \mathcal{O}_{\mathbb{P}^n_S}(-1)$ (which is right exact) to this surjection gives another surjection

$$\mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_S}$$

The following theorem asserts that the cotangent sheaf of $\mathbb{P}^n_S$ is the kernel of this surjection.

Theorem 5.1. *$\Omega^1_{\mathbb{P}^n_S/S}$ is locally free of rank n, and there is an exact sequence*

$$0 \to \Omega^1_{\mathbb{P}^n_S/S} \to \mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_S} \to 0$$

5.2 A Concrete Example

To warm up, let's take $S = \operatorname{Spec} k$, with k algebraically closed. We have seen that cotangent vectors can be regarded as dotted maps filling in the following diagram

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] \arrow[d] & \mathbb{P}^n_k \arrow[d] \\
\operatorname{Spec} k[\epsilon]/\epsilon^2 \arrow[r] \arrow[ur, dotted] & \operatorname{Spec} k
\end{tikzcd}$$

Since k is algebraically closed, the map along the top is given by choosing a (closed) point $[a_0 : \ldots : a_n]$ of $\mathbb{P}^n_k$.

To give a map from $k[\epsilon]/\epsilon^2$ into $\mathbb{P}^n_k$ is to give a point $[a_0 + \epsilon b_0 : \ldots : a_n + \epsilon b_n]$ which agrees with the first point modulo ϵ (since the vertical map on the left is reduction mod ϵ. So since the a_is have already been chosen, filling in the diagram entail choosing the b_is. But we could rescale our coordinates by an element of $k[\epsilon]/\epsilon^2$ and possibly still get the same map. Since $(a_i + \epsilon b_i)(x + \epsilon y) = a_i x + \epsilon(b_i x + a_i y)$, we are forced to take $x = 1$ in order to get the diagram to commute. So the map is determined by choosing the b_is, and the ambiguity in doing so can be expressed as the corkernel of the map sending $y \mapsto (a_0 y, \ldots, a_n y)$. Thus we conclude, loosely speaking, that the (co)tangent space is the cokernel of a map $k \to k^{n+1}$.

To relate this to the theorem, take the dual of the sequence which appears there:

$$0 \to \mathcal{O}_{\mathbb{P}^n_S} \to \mathcal{O}_{\mathbb{P}^n_S}(1)^{n+1} \to \left(\Omega^1_{\mathbb{P}^n_S}\right)^\vee \to 0$$

where the final term is by definition the tangent sheaf of $\mathbb{P}^n_S$ over S. This is the coordinate-free expression of our computation above.

Compare this with the topological desription of complex projective space. There we have a $\mathbb{C}^*$-bundle

$$\begin{tikzcd} \mathbb{C}^{n+1} \setminus \{0\} \arrow[d, "f"] \\ \mathbb{CP}^n \end{tikzcd}$$

and if p is a nonzero vector in $\mathbb{C}^{n+1}$, the tangent space to $\mathbb{C}^{n+1}$ at p is just $\mathbb{C}^{n+1}$, whereas the tangent space of $\mathbb{CP}^n$ at $f(p)$ is isomorphic to $\mathbb{C}^n$, and we think of this as being the tangent space of $\mathbb{C}^{n+1}$ module the tangent space of the fiber over $f(p)$, which is just a line:

$$0 \to T_{f^{-1}f(p)}(p) \to T_{\mathbb{C}^{n+1}\setminus\{0\}}(p) \to T_{\mathbb{CP}^n}(f(p)) \to 0$$

$$0 \to \mathbb{C} \to \mathbb{C}^{n+1} \to \mathbb{C}^n \to 0$$

5.3 Functorial Characterization of $\Omega^1_{X/Y}$

Before we tackle the proof, let's ask what functor $\Omega^1_{X/Y}$ represents. We've basically already done this. Given $X \to Y$, a morphism of schemes, define

$$\mathbb{T}_{X/Y} \colon \mathrm{QCoh}(X) \to \mathrm{Set}$$

by sending $\mathcal{F}$ to $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{F})$. Given a quasicoherent sheaf $\mathcal{F}$, we saw the other day how to construct the following diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}"] \arrow[d] & X \arrow[d] \\ X[\mathcal{F}] \arrow[r] \arrow[ur, "s_0"] & Y \end{tikzcd}$$

Recall that, by analogy with the usual dual numbers, the scheme $X[\mathcal{F}]$ comes equipped with morphisms of sheaves $\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{F} \to \mathcal{O}_X$. In particular, there *is* a distinguished arrow s_0 filling in the diagram, though of course there may be others. Since we have a distinguished choice of arrow, Theorem 3.1 says that the set of all such arrows is isomorphic to the set of morphisms $\Omega^1_{X/Y}, \mathcal{F})$. In other words, the content of that theorem is really that $\Omega^1_{X/Y}$ represents this functor $\mathbb{T}_{X/Y}$.

6 Monday, January 30th: Proof of the Computation of $\Omega^1_{\mathbb{P}^n_S/S}$

Today we aim to prove Theorem 5.1. We will do this in three steps, the second of which is the most involved. Remember that we have to show that $\Omega^1_{\mathbb{P}^n_S/S}$ fits into an exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n_S/S} \to \mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_S/S} \to 0.$$

Following the discussion of last time, all that remains is to check that $\Omega^1_{\mathbb{P}^n_S/S}$ is the kernel of the map on the right. Let $\mathcal{K}$ be this kernel. The following is an outline of the argument:

1. For any quasicoherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_S$,
$$\operatorname{coker}\left(\mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S/S}, \mathcal{F}) \to \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S/S}(-1)^{n+1}, \mathcal{F})\right) \cong \mathcal{H}om(\mathcal{K}, \mathcal{F})$$
2. For any quasicoherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_S$,
$$\operatorname{coker}\left(\mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S/S}, \mathcal{F}) \to \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S/S}(-1)^{n+1}, \mathcal{F})\right) \cong \mathcal{H}om(\Omega^1_{\mathbb{P}^n_S/S}, \mathcal{F})$$
3. By 1 and 2, $\mathcal{H}om(\mathcal{K}, \mathcal{F}) \cong \mathcal{H}om(\Omega^1_{\mathbb{P}^n_S/S}, \mathcal{F})$, and therefore $\mathcal{K} \cong \Omega^1_{\mathbb{P}^n_S/S}$

Step 3 is basically just an application of Yoneda: once we've shown that $\mathcal{H}om(\mathcal{K}, \mathcal{F}) \cong \mathcal{H}om(\Omega^1_{\mathbb{P}^n_S/S}, \mathcal{F})$, then taking global sections gives $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}\left(\mathcal{K}, \mathcal{F}\right) \cong \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}\left(\Omega^1_{\mathbb{P}^n_S/S}, \mathcal{F}\right)$, which implies $\Omega^1_{\mathbb{P}^n_S/S} \cong \mathcal{K}$, finishing the proof.

Now we prove the first two steps.

Proof of Step 1. We already have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_S/S} \to 0$$

We claim that application of the functor $\mathcal{H}om$ gives another exact sequence of sheaves on $\mathbb{P}^n_S$:

$$0 \to \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S}, \mathcal{F}) \to \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1}, \mathcal{F}) \to \mathcal{H}om(\mathcal{K}, \mathcal{F}) \to 0$$

The exactness follows from the following lemma

Lemma 6.1. *Let*

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{E}' \to 0$$

be an exact sequence of quasicoherent sheaves on X, where $\mathcal{E}'$ is locally free of finite rank. Then for all $\mathcal{F}$ in $QCoh(X)$, the sequence of sheaves on X

$$0 \to \mathcal{H}om(\mathcal{E}', \mathcal{F}) \to \mathcal{H}om(\mathcal{E}, \mathcal{F}) \to \mathcal{H}om(\mathcal{K}, \mathcal{F}) \to 0$$

is exact in the category of $\mathcal{O}_X$-modules.

Proof of Lemma. Exactness can be checked on stalks. Since $\mathcal{E}'$ is locally free of rank r for some r, every point of X has a neighborhood U on which $\mathcal{E}'\big|_U \cong \mathcal{O}_U^r$. Therefore replacing X by U we may assume $\mathcal{E}' \cong \mathcal{O}_X^r$, generated by basis elements e_i in $\Gamma(X, \mathcal{E}')$, $i = 1, \ldots, r$. Then define a section by lifting each e_i to one of its preimages in $\Gamma(X, \mathcal{E})$, giving a splitting

$$\mathcal{E} \cong \mathcal{K} \oplus \mathcal{E}'$$

Then

$$\begin{aligned} \mathcal{H}om(\mathcal{E}, \mathcal{F}) &\cong \mathcal{H}om(\mathcal{K} \oplus \mathcal{E}', \mathcal{F}) \\ &\cong \mathcal{H}om(\mathcal{K}, \mathcal{F}) \oplus \mathcal{H}om(\mathcal{E}', \mathcal{F}) \end{aligned}$$

which implies the exactness of the sequence in the statement of the lemma. □

Now, taking $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1}$ and $\mathcal{E}' = \mathcal{O}_{\mathbb{P}^n_S}$ (which is locally free of rank one), we have proved step 1. □

Step 2 is the heart of the proof, and we will not finish it today. Before we begin, here are a few facts which we will be useful to bear in mind.

Our approach involves the fact that $\mathbb{P}^n_S$ represents the functor

$$F\colon \Big(\text{Schemes}/S\Big)^{\text{op}} \to \text{Set}$$

which sends an S-scheme $T \to S$ to the set of surjections of $\mathcal{O}_T$-modules $\mathcal{O}_T^{n+1} \to \mathcal{L}$ where $\mathcal{L}$ is a line bundle on T, up to some equivalence relation that we won't worry about here. But recall the way in which this F is actually a (contravariant) functor: given a morphism

$$\begin{tikzcd} T' \arrow[rr, "g"] \arrow[dr] & & T \arrow[dl] \\ & S & \end{tikzcd}$$

of S-schemes, the induced morphism $F(f)$ sends the surjection $\mathcal{O}_T^{n+1} \xrightarrow{\pi} \mathcal{L}$ in $F(T)$ to the surjection $\mathcal{O}_{T'}^{n+1} \xrightarrow{g^*\pi} g^*\mathcal{L}$ in $F(T')$.

Why is $g^*\pi$ still a surjection? Because g^* is right exact, being a left adjoint to the pushforward g_*. Why is $g^*\mathcal{L}$ a line bundle? Because pullback sends vector bundles to vector bundles: this can be checked locally, and if $T' = \operatorname{Spec} A$, $T = \operatorname{Spec} B$, and $\mathcal{L} \cong (B^\sim)^n$, we have $g^*\mathcal{L} \cong (B^\sim)^n \otimes_{(B^\sim)} A^\sim \cong (B^n \otimes_B A)^\sim \cong (A^\sim)^n$ as $B^\sim$-modules. Now we beign the proof.

Proof of Step 2. Let U be any open of $\mathbb{P}^n_S$, and $\mathcal{F}$ any quasicoherent sheaf on $\mathbb{P}^n_S$. Then $\mathcal{H}om\left(\Omega^1_{\mathbb{P}^n_S}, \mathcal{F}\right)(U) = \operatorname{Hom}_{\mathcal{O}_U}\left(\Omega^1_{\mathbb{P}^n_S}\big|_U, \mathcal{F}\big|_U\right)$, and by Theorem 3.1, this set is equal to the set of dotted maps filling in the following diagram:

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d] & \mathbb{P}^n_S \arrow[d] \\ U[\mathcal{F}] \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

Notice that in applying the theorem, we have used the fact that this set is non-empty: we have a distinguished "retract" $r\colon U[\mathcal{F}] \to U$, which when composed with the inclusion $U \to \mathbb{P}^n_S$, fills in the diagram.

Now we consider such diagrams from the point of view of the functor F which is represented by $\mathbb{P}^n_S$. As noted above, a map $T \to \mathbb{P}^n_S$ corresponds to a surjection of $\mathcal{O}_T$-modules $\mathcal{O}_T^{n+1} \to \mathcal{L}$, with $\mathcal{L}$ a line bundle on T. So the inclusion $U \to \mathbb{P}^n_S$ corresponds to a section $\mathcal{O}_U^{n+1}\mathcal{L}_U$, and since the inclusion commutes with the identity map $\mathbb{P}^n_S \to \mathbb{P}^n_S$, this line bundle $\mathcal{L}_U$ is actually just $\mathcal{O}_{\mathbb{P}^n_S}(1)\big|_U$, but we'll continue to denote it by $\mathcal{L}_U$ for now.

We also have our distinguished dotted arrow, the retract r (followed by inclusion into $\mathbb{P}^n_S$), which corresponds to a surjection $\mathcal{O}^{n+1}_{U[\mathcal{F}]} \to \mathcal{L}^\sim$. Notice that $\mathcal{O}_{U[\mathcal{F}]} \cong r^*\mathcal{O}_U$ since pullback sends the structure sheaf to the structure sheaf. So our second surjection really looks like $r^*\mathcal{O}_U^{n+1} \to \mathcal{L}^\sim$. But the fact that the square above commutes imposes some constraints on these two surjections. Namely, in light of our review of the functoriality of F, we must have a commuting square

$$\begin{tikzcd}
r^*\mathcal{O}_U^{n+1} \arrow[r, two heads, "r^*\pi"] \arrow[d] & r^*\mathcal{L}_U \arrow[d] \\
\mathcal{O}_U^{n+1} \arrow[r, two heads] & \mathcal{L}_U
\end{tikzcd}$$

Now we are asking whether there are other maps which fill in the square of schemes above, or equivalently, fill in this square of sheaves, and the functoriality therefore requires us to look for not just any surjection $r^*\mathcal{O}^{n+1} \to \mathcal{L}^\sim$, but specifically a surjection onto $r^*\mathcal{L}_U$. So what can be said about this $r^*\mathcal{L}_U$? Firstly, it fits into an exact sequence of $\mathcal{O}_U$-modules

$$0 \to \mathcal{L}_U \otimes \mathcal{F} \to r^*\mathcal{L}_U \to \mathcal{L}_U \to 0$$

The reason is that $r^*\mathcal{L}_U$ can be rewritten as follows:

$$r^*\mathcal{L}_U = \mathcal{O}_{U[\mathcal{F}]} \otimes_{\mathcal{O}_U} \mathcal{L}_U = \mathcal{L}_U \otimes_{\mathcal{O}_U} (\mathcal{O}_U \oplus \mathcal{F}\big|_U) = \mathcal{L}_U \oplus (\mathcal{L}_U \otimes_{\mathcal{O}_U} \mathcal{F}),$$

and this fits into the (split) exact sequence above. But the map on the right is just the right hand vertical arrow in our square of sheaves two paragraphs up, so we can fit the two diagrams together, giving

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_U \otimes \mathcal{F} \arrow[r] & r^*\mathcal{L}_U \arrow[r] & \mathcal{L}_U \arrow[r] & 0 \\
 & & \mathcal{O}_U^{n+1} \oplus \mathcal{F}^{n+1} \cong r^*\mathcal{O}_U^{n+1} \arrow[r] \arrow[u, two heads, "r^*\pi"'] & \mathcal{O}_U^{n+1} \arrow[u, two heads] &
\end{tikzcd}$$

All the maps here are $\mathcal{O}_U$-linear, and in addition, $r^*\pi$ is $\mathcal{O}_{U[\mathcal{F}]}$-linear. We ask: what other $\mathcal{O}_{U[\mathcal{F}]}$-linear maps are there that fill in this diagram? Observe first that by commutativity of the square any two such must agree when composed with the map $r^*\mathcal{L}_U \to \mathcal{L}_U$ killing $\mathcal{F}$. Thus by the exactness of the horizontal sequence their difference factors through $\mathcal{L}_U \otimes_{\mathcal{O}_U} \mathcal{F}$, i.e., is a map $\mathcal{O}^{n+1}_{U[\mathcal{F}]} \to \mathcal{L}_U \otimes \mathcal{F}$. So we can produce the desired

maps by adding to $r^*\pi$ an $\mathcal{O}_{U[\mathcal{F}]}$-linear map $\phi\colon \mathcal{O}_{U[\mathcal{F}]}^{n+1} \to \mathcal{L}_U \otimes \mathcal{F}$. This associates to each element of

$$\mathrm{Hom}_{\mathcal{O}_{U[\mathcal{F}]}}\left(\mathcal{O}_{U[\mathcal{F}]}^{n+1}, \mathcal{L}_U \otimes_{\mathcal{O}_U} \mathcal{F}\right)$$

a dotted arrow filling in our diagram. But by changing base to the category of $\mathcal{O}_U$-modules, this Hom set is in natural bijection with

$$\mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U^{n+1}, \mathcal{L}_U \otimes_{\mathcal{O}_U} \mathcal{F}\right),$$

where $\mathcal{L}_U \otimes_{\mathcal{O}_U} \mathcal{F}$ is regarded simply as an $\mathcal{O}_U$-module. Thus given any element of this set, we have produced a surjection $\mathcal{O}_{U[\mathcal{F}]}^{n+1} \to r^*\mathcal{L}_U$ of sheaves on $U[\mathcal{F}]$.

Now recall that the line bundle $\mathcal{L}_U$ is actually $\mathcal{O}_U(1)$, and that maps filling in the diagram correspond to elements of $\mathrm{Hom}_{\mathcal{O}_U}\left(\Omega^1_{\mathbb{P}^n_S}\big|_U, \mathcal{F}\right)$, so we have produced a map

$$\mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U^{n+1}, \mathcal{O}_U(1) \otimes_{\mathcal{O}_U} \mathcal{F}\right) \to \mathrm{Hom}_{\mathcal{O}_U}\left(\Omega^1_{\mathbb{P}^n_S}\big|_U, \mathcal{F}\right)$$

But with a little calculation, we can rewrite the domain as follows:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U^{n+1}, \mathcal{O}_U(1) \otimes_{\mathcal{O}_U} \mathcal{F}\right)\\
&\cong \mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U(-1)^{n+1} \otimes_{\mathcal{O}_U} \mathcal{O}_U(1), \mathcal{O}_U(1) \otimes_{\mathcal{O}_U} \mathcal{F}\right)\\
&\cong \mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U(-1)^{n+1}, \mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U(1), \mathcal{O}_U(1) \otimes_{\mathcal{O}_U} \mathcal{F}\right)\right)\\
&\cong \mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U(-1)^{n+1}, \mathcal{F}\right)
\end{aligned}$$

The final isomorphism being because $\mathcal{O}_U(1)$ is locally isomorphic to $\mathcal{O}_U$. For more detail, see the lemma below.

Summarizing, we now have a map

$$\mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{O}_U(-1)^{n+1}, \mathcal{F}\right) \to \mathrm{Hom}_{\mathcal{O}_U}\left(\Omega^1_{\mathbb{P}^n_S}\big|_U, \mathcal{F}\right)$$

for every open $U \subseteq \mathbb{P}^n_S$, which is compatible with restriction maps by construction. Thus it determines a morphism of sheaf-Homs

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}\left(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}, \mathcal{F}\right) \to \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}\left(\Omega^1_{\mathbb{P}^n_S}, \mathcal{F}\right)$$

The rest of the proof of Part 2 will be finished in the next lecture.

Here's a lemma which was referred to above, and which gets used in the next lecture also, but was neither explicitly stated nor proved in lecture.

Lemma 6.2. *Let $\mathcal{L}$ be a line bundle on X, and $\mathcal{F}$ a quasicoherent sheaf on X. Then there is an isomorphism of sheaves*

$$\sigma\colon \mathcal{H}om(\mathcal{L}, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\sim} \mathcal{F}$$

Proof. Choose a covering of X on which $\mathcal{L}$ is trivial and $\mathcal{F}$ is the sheaf associated to some module, i.e., consisting of open affines $U = \operatorname{Spec} A$, on which we have isomorphisms

$$\mathcal{L}\big|_U \cong \mathcal{O}_U \cong \widetilde{A} \text{ and } \mathcal{F}\big|_U \cong \widetilde{F},$$

for some A-module F. Then we have, by the equivalence of categories $\mathrm{QCoh}(U) \cong \mathrm{Mod}_A$,

$$\mathcal{H}om(\mathcal{L}, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F})(U) \cong \mathrm{Hom}_A(A, A \otimes_A F)$$

using which σ is given explicitly by the following: if $\phi \in \mathrm{Hom}_A(A, A \otimes_A F)$, and $\phi(1) = 1 \otimes f$ (we can always write it like this since the tensor product is taken over A), we define $\sigma(\phi) = f$. We need to check that this definition of σ agrees on the intersection W of two affines U and V. But in this case, we may restrict further if necessary to ensure that $(\mathcal{F}\big|_U)\big|_W \cong (\mathcal{F}\big|_V)\big|_W$. Moreover, the isomorphisms $\mathcal{L}\big|_U \cong \mathcal{O}_U$ and $\mathcal{L}\big|_V \cong \mathcal{O}_V$, when restricted to W, differ by multiplication by a unit in $\mathcal{O}_W$. Since this unit appears in both sides of the equation defining σ (specifically, we multiply the 1 in both sides by this unit), it may be cancelled, so that the equations defining σ on U and on V are the same when restricted down to W. Thus the definition of σ on open affines glues to give a morphism of sheaves, which is an isomorphism of sheaves since it is so locally: $\mathrm{Hom}_A(A, A \otimes_A F) \cong \mathrm{Hom}_A(A, F) \cong F$. □

□

7 Wednesday, February 1st: Conclusion of the Computation of $\Omega^1_{\mathbb{P}^n_S/S}$

7.1 Review of Last Lecture's Construction

By the end of the last lecture, we claimed to have produced, for each $\mathcal{F}$ in $\mathrm{QCoh}(\mathbb{P}^n_S)$, a morphism of sheaves

$$\gamma\colon \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1}, \mathcal{F}) \to \mathcal{H}om(\Omega^1_{\mathbb{P}^n_S/S}, \mathcal{F}).$$

Let's first clarify the construction of this map γ. Consider an affine open $\operatorname{Spec} R \subseteq \mathbb{P}^n_S$, with a sheaf $\mathcal{F} \cong \widetilde{F}$, for an R-module F. We are interested in dotted arrows

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r] \arrow[d] \arrow[dd, bend right, "\mathrm{Id}"'] & \mathbb{P}^n_S \arrow[d] \\
\operatorname{Spec} R[F] \arrow[r] \arrow[ur, dotted] \arrow[d, "r"] & S \\
\operatorname{Spec} R &
\end{tikzcd}$$

The inclusion $\operatorname{Spec} R \to \mathbb{P}^n_S$ corresponds to a surjection $\mathcal{O}_R^{n+1} \xrightarrow{\pi} \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $\operatorname{Spec} R$. [1] We observed that in fact $\mathcal{L} = \mathcal{O}(1)$, and that $r^*\mathcal{L} \cong \mathcal{L} \oplus (\mathcal{F} \otimes_{\mathcal{O}_R} \mathcal{L})$ as $\mathcal{O}_R$-modules. This implies that there is an exact sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_R} \mathcal{L} \to r^*\mathcal{L} \to \mathcal{L} \to 0,$$

[1] In the following, we will abbreviate $\mathcal{O}_{\operatorname{Spec} R}$ by $\mathcal{O}_R$, and similarly for $R[F]$.

in which the map $r^*\mathcal{L} \cong \mathcal{L} \oplus (\mathcal{F} \otimes \mathcal{L}) \to \mathcal{L}$ is projection onto the first factor; we fit this into the following diagram involving π and a potential lifting $\widetilde{\pi}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \otimes_{\mathcal{O}_R} \mathcal{L} \arrow[r] & r^*\mathcal{L} \arrow[r] & \mathcal{L} \arrow[r] & 0 \\
 & & \mathcal{O}_{R[F]}^{n+1} \arrow[r, two heads] \arrow[u, two heads, dotted, "\widetilde{\pi}"] & \mathcal{O}_R^{n+1} \arrow[u, two heads, "\pi"'] &
\end{tikzcd}$$

If e_i are basis elements of $\mathcal{O}_R^{n+1}$, then these lift to basis elements of $\mathcal{O}_{R[F]}$ [Reason: elements of $F \subseteq R[F]$ are nilpotent, hence contained in the nilradical, hence the radical of $R[F]$. Thus Nakayama says we can lift generators of $R[F]/F = R$ to generators of $R[F]$]. Abusing notation and denoting these lifts also by e_i, the commutativity of the square means that for any $\widetilde{\pi}$ filling in the diagram, we must have $\widetilde{\pi}(e_i) = (\pi(e_i), \alpha_i)$ for some α_i in $\mathcal{L} \otimes F$. Thus $\widetilde{\pi}$ is determined by these α_i, so the set of maps filling in the diagram is in bijection with n-tuples of sections of $\mathcal{L} \otimes \mathcal{F}$, which is in turn in bijection with $\mathrm{Hom}\left((\mathcal{L}^{-1})^{n+1}, \mathcal{F}\right)$,according to the following computation:

$$\begin{aligned}
(\mathcal{L} \otimes \mathcal{F})^{n+1} \cong \mathrm{Hom}(\mathcal{O}^{n+1}, \mathcal{L} \otimes \mathcal{F}) &\cong \mathrm{Hom}\left((\mathcal{L}^{-1})^{n+1} \otimes \mathcal{L}, \mathcal{L} \otimes \mathcal{F}\right) \\
&\cong \mathrm{Hom}\left((\mathcal{L}^{-1})^{n+1}, \mathcal{F}\right),
\end{aligned}$$

where we used Lemma 6.2 at the last step.

This construction is compatible with restriction, so this construction of $\widetilde{\pi}$ gives a map of sheaves

$$\mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}, \mathcal{F}) \to \{\text{maps } U \to \mathbb{P}^n \text{ filling in the diagram above}\}.$$

(We used $\mathcal{L}^{-1} \cong \mathcal{O}_{\mathbb{P}^n}(-1)$ here). Now the sheaf on the right is a priori just a sheaf of sets, but one can check that there is an $\mathcal{O}_{\mathbb{P}^n}$ module structure on it, in such a way that it is isomorphic as sheaves to $\mathcal{H}om(\Omega^1_{\mathbb{P}^n}, \mathcal{F})$.

Now we consider this map

$$\gamma\colon\ \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1}, \mathcal{F}) \to \mathcal{H}om(\Omega^1_{\mathbb{P}^n_S/S}, \mathcal{F}).$$

7.2 Conclusion of the Proof

To finish the proof of Step 2, it suffices to check that γ is surjective and that its kernel is $\mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}, \mathcal{F})$. This shows that $\mathcal{H}om(\Omega^1_{\mathbb{P}^n_S/S}, \mathcal{F})$ is the cokernel of $\mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}, \mathcal{F}) \to \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n_S}(-1)^{n+1}, \mathcal{F})$ because sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules form an abelian category, so every surjection (here, γ) is the cokernel of its kernel.

For surjectivity, we start with an open $U \subseteq \mathbb{P}^n$, and consider a surjection π' in the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{U[\mathcal{F}]^{n+1}} \arrow[r, two heads, "\pi'"] \arrow[d, two heads] & \mathcal{L}' \arrow[d, two heads] \\
\mathcal{O}_U^{n+1} \arrow[r, two heads] & \mathcal{L}
\end{tikzcd}$$

Remember that our construction began with the "canonical surjection $\mathcal{O}_U^{n+1} \to \mathcal{O}_U(1)$ and lifted it to a surjection $\mathcal{O}_{U[\mathcal{F}]}^{n+1} \to r^*\mathcal{L}$. Thus a map π' as above comes from our construction if and only if there exists an isomorphism $\rho\colon \mathcal{L}' \to r^*\mathcal{L}$ over $\mathcal{L}$, i.e., a diagram

$$\begin{tikzcd}
\mathcal{L}' \arrow[rr, "\cong"] \arrow[dr, two heads] & & r^*\mathcal{L} \arrow[dl, two heads] \\
& \mathcal{L} &
\end{tikzcd}$$

which must furthermore fit into the following:

$$\begin{tikzcd}
\mathcal{O}_{U[\mathcal{F}]}^{n+1} \arrow[r, two heads, "\pi'"] \arrow[d, two heads] \arrow[rr, bend left, "\phi+\pi'"] & \mathcal{L}' \arrow[r, "\rho"] \arrow[d, two heads] & r^*\mathcal{L} = \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{F}) \arrow[dl] \\
\mathcal{O}_U^{n+1} \arrow[r, two heads] & \mathcal{L} &
\end{tikzcd}$$

We exhibit this ρ locally: first find an open affine on which all three of $\mathcal{L}, \mathcal{L}'$, and $r^*\mathcal{L}$ are trivial. Choose a basis for $\mathcal{L}$, lift it via the surjections $\mathcal{L} \to \mathcal{L}$ and $r^*\mathcal{L}$ in such a way that the triangle above commutes (this can always be done, since locally, all three are isomorphic to some ring R). Then our choice of bases for $\mathcal{L}'$ and $r^*\mathcal{L}$ determines ρ locally. This proves surjectivity.

To determine the kernel, we ask: when do two maps $\phi_1, \phi_2\colon \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{F}$ determine the same $\pi\colon \mathcal{O}_{U[\mathcal{F}]}^{n+1} \to r^*\mathcal{L}$?. In other words, for which ϕ_1, ϕ_2 is there an isomorphism $\sigma\colon r^*\mathcal{L} \to r^*\mathcal{L}$ filling in the following diagram

$$\begin{tikzcd}
& r^*\mathcal{L} \arrow[dr] \arrow[dd, "\sigma"] & \\
\mathcal{O}_{U[\mathcal{F}]}^{n+1} \arrow[ur, "\pi+\phi_2"] \arrow[dr, "\pi+\phi_1"'] & & \mathcal{L} \\
& r^*\mathcal{L} \arrow[ur] &
\end{tikzcd}$$

Any isomorphism σ is given by multiplication by some unit in $\mathcal{O}_{U[\mathcal{F}]}^\times$; the requirement that it be an isomorphism *over* $\mathcal{L}$ forces this unit to map to 1 in $\mathcal{O}_U$ after collapsing $\mathcal{F}$. In other words, such units have the form $1 + x \in \mathcal{O}_{U[\mathcal{F}]}$, where $x \in \mathcal{F}$; in more otherer words, the sections of $\mathcal{F}\big|_U$ inject into the kernel of the map of sheaves of groups $\mathcal{O}_{U[\mathcal{F}]}^\times \to \mathcal{O}_U^\times$.

Now for any section x of $\mathcal{F}\big|_U$, we consider the maps

$$\begin{tikzcd}
\mathcal{O}_{U[\mathcal{F}]}^{n+1} \arrow[r, "\pi+\phi_2"] \arrow[dr, "\pi+\phi_1"'] & \mathcal{L}_U \oplus (\mathcal{L} \otimes \mathcal{F}) \arrow[d, "\cdot(1+x)"] \\
& \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{F})
\end{tikzcd}$$

Commutativity requires that for each basis element e_i of $\mathcal{O}_{U[\mathcal{F}]}$,

$$(\pi(e_i), \phi_1(e_i)) = (\pi(e_i), \pi(e_i) \cdot x + \phi_2(e_i)).$$

Thus the two maps agree if $\phi_1(e_i) - \phi_2(e_i) = \pi(e_i) \cdot x$. So the kernel of γ consists of these sections x of $\mathcal{F}$; that is, $\ker \gamma \cong \mathcal{F} \cong \mathcal{H}om(\mathcal{O}_U, \mathcal{F})$. We regard this latter as a subsheaf of $\mathcal{H}om(\mathcal{O}_u(-1)^{n+1}, \mathcal{F})$ by applying the functor $\mathcal{H}om(-, \mathcal{F})$ to the surjection $\mathcal{O}_U(-1)^{n+1} \to \mathcal{O}_U$, giving an inclusion $\mathcal{H}om(\mathcal{O}, \mathcal{F}) \hookrightarrow \mathcal{H}om(\mathcal{O}_U(-1)^{n+1}, \mathcal{F})$. Since our choice of U was arbitrary and the constructions are compatible with restriction, the conclusions hold with U replaced by $\mathbb{P}^n_S$, concluding the proof.

7.3 An Example

We'd like to compute $\Omega^1_{\mathbb{P}^1_k/k}$ for a field k. We know it's a line bundle, and we know what the line bundles on $\mathbb{P}^1$ are. But which line bundle is it? Theorem 5.1 gives us the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^1_k/k} \to \mathcal{O}_{\mathbb{P}^1_k}(-1)^2 \to \mathcal{O}_{\mathbb{P}^1_k/k} \to 0.$$

Now since $\mathcal{O}$ is free, the sequence splits, so the middle term is isomorphic to the direct sum of the outer two. Hence the second exterior power of the middle term is isomorphic to the tensor product of the outer two terms. This means that

$$\Lambda^2 \mathcal{O}(-1)^2 \cong \Omega^1_{\mathbb{P}^1_k/k} \otimes \mathcal{O} \cong \Omega^1_{\mathbb{P}^1_k/k},$$

but $\Lambda^2 \mathcal{O}(-1)^2 \cong \mathcal{O}(-2)$, so we have calculated

$$\Omega^1_{\mathbb{P}^1_k/k} \cong \mathcal{O}(-2)$$

More concretely, if we look on the affine open $\mathbb{A}^1 \subset \mathbb{P}^1$, we have $\Omega^1_{\mathbb{P}^1} \cong \Omega^1_{\mathbb{A}^1} \cong \mathcal{O}_{\mathbb{A}^1}$, generated by, say, $\mathrm{d}x$. Similarly on the other chart $\mathbb{A}^1 \cong D(y)$: there $\Omega^1_{\mathbb{P}^1}$ is generated by $\mathrm{d}y$. How are the two related? At $\infty \in D(y) \setminus D(x)$, the local ring $\mathcal{O}_{\mathbb{P}^1,\infty}$ is a DVR with uniformizer $x = 1/y$, giving $\mathrm{d}x = -1/y^2\, \mathrm{d}y$. Thus the transition map for Ω^1 is multiplication by $x^2 = 1/y^2$. The transition map for $\mathcal{O}(1)$ is multiplication by y. Thus $\Omega^1_{\mathbb{P}^n} \cong \mathcal{O}(-2)$.

7.4 Concluding Remarks

Before moving on to the study of curves, let's review the techniques we have available for calculating differentials on a scheme:

1. Try to use the calculation of $\Omega^1_{\mathbb{P}^n}$
2. Use the closed immersion exact sequence.
3. Use the relative cotangent exact sequence.

These ideas lead naturally into curve theory, where one of our main tasks will be to understand embeddings of curves into $\mathbb{P}^n$ using Ω^1.

8 Friday, February 3rd: Curves

Now we begin our study of curves by embedding them into projective space. An equivalent approach which we exploit is to study line bundles (and sections thereof) on a curve (especially Ω^1).

8.1 Separating Points and Tangent Vectors

Let k be an algebraically closed field, and X a proper k-scheme. Given a surjection $\mathcal{O}_X^{n+1} \to \mathcal{L}$, when is the corresponding map $\phi\colon X \to \mathbb{P}_k^n$ a closed immersion? First we fix some notation. Let $V = k^{n+1}$, with a fixed choice of basis $\{e_i\}$. We write $\mathcal{O}_X^{n+1} \cong V \otimes_k \mathcal{O}_X$, where in the tensor product we are really tensoring with the constant sheaf associated to V.

Let $\mathcal{L}$ be a line bundle on X, and $\mathcal{O}_X^{n+1} \to \mathcal{L}$ a surjection. For p a point of X, recall our notation $\mathcal{L}(p) = \mathcal{L}_p/\mathfrak{m}_p\mathcal{L}_p$. For any vector s in V, we obtain an element s_p of $\mathcal{L}(p)$ by the composite map

$$V \xrightarrow{v \mapsto v \otimes 1} \Gamma(X, V \otimes_k \mathcal{O}_X) \to \Gamma(X, \mathcal{L}) \to \mathcal{L}_p \to \mathcal{L}(p)$$

Recall also that by the Nullstellensatz, the residue field $k(p)$ at any closed point p of X is isomorphic to k (remember we assume k is algebraically closed).

With these observations in place, we answer the question above, viz. when do surjections onto line bundles give embeddings into projective space, by the following.

Proposition 8.1. *If X/k is proper over an algebraically closed field, and $\phi\colon X \to \mathbb{P}^n$ the map associated to the surjection $\mathcal{O}_X^{n+1} \xrightarrow{\pi} \mathcal{L}$, then ϕ is a closed immersion if and only if the following two conditions hold*

1. *(Separating Points) For every pair of distinct point p, q in X, there is an element s of V such that s_p (as defined above) is nonzero in $\mathcal{L}(p)$, whilst s_q is zero in $\mathcal{L}(q)$.*
2. *(Separating Tangent Vectors) For every closed point p of X, the map*

$$\ker\big(V \to \mathcal{L}(p)\big) \to \mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p\mathcal{L}_p^2$$

is surjective.

8.2 Motivating the Terminology

We'll prove this in the next lecture. First we motivate the conditions. As a warmup, write $V = kx_0 \oplus \ldots \oplus kx_n$, and consider the x_i as coordinates on $\mathbb{P}^n$. If we take, for example, $s = x_0$, then what is the set of closed points p in X such that s goes to zero in $\mathcal{L}(p)$?

First recall that we can write the element $\phi(p) \in \mathbb{P}_k^n$ as $[a_0 : \ldots : a_n]$ as follows. Choose a basis element for $\mathcal{L}(p)$, i.e., an isomorphism $\mathcal{L}(p) \xrightarrow{\sigma} k$. Then we get a map

$$k^{n+1} = V \to \mathcal{L}_p \xrightarrow{\sigma} k$$

which sends each basis element x_i to some a_i in k. Of course, it is not well-defined, since we had to choose a basis for $\mathcal{L}(p)$; but any other basis differs from our chosen one by a nonzero element of k, so our coordinates

a_i are well-defined up to nonzero scalars, giving a usual point of projective space (in the classical sense). Thus we see that the set of closed points of X for which x_0 maps to zero is just the preimage under ϕ of the set $\{[0 : a_1 : \ldots : a_n]\} \subset \mathbb{P}^n_k$.

More generally, if s is any linear form in the x_i (i.e., an element of V), then the points of X for which s maps to zero in $\mathcal{L}(p)$ are the preimage of the zero locus of s in $\mathbb{P}^n_k$. The zero locus of such a linear form is usually called a hyperplane in $\mathbb{P}^n_k$. So what condition 1 says is that for any $p, q \in X$, there exists a hyperplane passing through p but not q; succinctly, the map ϕ is injective on points.

Of course, this does not guarantee that ϕ is a closed immersion. For an example where this fails, let Y be the cusp $\operatorname{Spec} k[x, y]/(y^2 - x^3)$ and X its normalization $\operatorname{Spec} k[t]$. Then there is a morphism $\phi\colon X \to Y$ corresponding to the ring map $x \mapsto t^2, y \mapsto t^3$. This is a homeomorphism of topological spaces but not a closed immersion: the tangent space at the origin on Y is "too big".

To see why, look at the map on tangent spaces at the origin. On Y, we have $\mathfrak{m}_O/\mathfrak{m}_O^2 \cong k \cdot x \oplus k \cdot y$, and on X, we have $\mathfrak{m}_O/\mathfrak{m}_O^2 \cong k \cdot t$. But the induced map between them is zero, since the images of x and y are both in $\mathfrak{m}_O^2 \subset k[t]$. In order for this map ϕ to be a closed immersion, this map would need to be surjective.

Condition two says exactly this: the map $T_{\mathbb{P}^n}\big(\phi(p)\big)^* \to T_X(p)^*$ is surjective, or dually, the map on tangent spaces is injective.

8.3 A Preparatory Lemma

We'll need the following in our proof next time.

Lemma 8.1. *A map $\phi\colon X \to \mathbb{P}^n_k$ is a closed immersion if and only if, for every $i = 0, \ldots, n$,*

1. *The subscheme $X_i = \{p \in X \big| x_i \not\to 0 \text{ in } \mathcal{L}(p)\}$ is affine, and*
2. *the map $k[y_0, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ sending y_j to $\pi(x_j)/\pi(x_i)$ is surjective.*

We explain the second condition: π is our surjection $\mathcal{O}_X^{n+1} = V \otimes_k \mathcal{O}_X \to \mathcal{L}$, which we think of as determining the homogeneous coordinates of a point $p \in X$. Restricting to X_i, we get a map $\pi\big|_{X_i} : \Gamma(X_i, \mathcal{O}_{X_i}^{n+1}) \to \Gamma(X_i, \mathcal{L}\big|_{X_i})$. For each $p \in X_i$, if we restrict to the stalk at p, we find that $\pi\big|_{X_i}(x_i)$ is nonzero (it's nonzero in $\mathcal{L}(p)$, hence nonzero in $\mathcal{L}_p$ by Nakayama). But this holds for any $p \in X_i$, therefore $\pi(x_i)$ must have been a unit in $\Gamma(X_i, \mathcal{L}\big|_{X_i})$ (reason: it's a unit in every stalk, hence not contained in any prime ideal of $\Gamma(X_i, \mathcal{L}\big|_{X_i})$).

Now since $\mathcal{L}$ is a line bundle, we are guaranteed an open cover of X on which it trivializes. The above remarks show that in fact the X_i form such a cover, with the trivialization

$$\mathcal{O}_{X_i} \to \mathcal{L}\big|_{X_i}$$

given by $1 \mapsto \pi(x_i)$, since we just saw that $\pi(x_i)$ generates $\mathcal{L}$ over X_i. Therefore each global section $\pi(x_j)$ of $\mathcal{L}$ over X_i (some or all of which may

be zero), maps back under this isomorphism to an element $\pi(x_j)/\pi(x_i)$. Then of course since $k[y_0, \ldots, y_n]$ is free, we may define a map to $\Gamma(X_i, \mathcal{O}_{X_i})$ by sending each y_j to $\pi(x_j)/\pi(x_i)$. This is the map which appears in the second condition of Lemma 8.1, which we now prove.

Proof of Lemma 8.1. Since the condition of being a closed immersion is affine-local on the target, the map $\phi\colon X \to \mathbb{P}^n_k$ is a closed immersion if and only if, for each $i = 0, \ldots, n$, the restriction

$$\phi^{-1}\big(\mathbb{P}^n_k \setminus V(x_i)\big) \to \mathbb{P}^n_k \setminus V(x_i)$$

is a closed immersion. By definition, this morphism is just

$$X_i \to \operatorname{Spec} k[\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}].$$

Moreover, since the target now is affine, this morphism is a closed immersion if and only if X_i also is affine (a closed immersion must be an affine morphism) and the corresponding map of sheaves, which when X_i is affine corresponds to a ring map $k[\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}] \to \Gamma(X_i, \mathcal{O}_{X_i})$, is surjective.

□

9 Monday, February 6th: Proof of the Closed Embedding Criteria

9.1 Proof of Proposition 8.1

We need yet another lemma:

Lemma 9.1. *Let $f\colon A \to B$ be a local homomorphism of local Noetherian rings, such that the induced map on residue fields $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism, and the map on cotangent spaces*

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$$

is surjective. Then f itself is surjective.

The intuition behind the lemma is that if, say, $A = k[[x_1, \ldots, x_n]]$ and $B = k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_r)$, then surjectivity on cotangent spaces means that in each equation $f_i = 0$, we can solve for x_i in terms of elements of A.

Proof. The ideal $\mathfrak{m}_A B \subset B$ is contained in $\mathfrak{m}_B$ since f is a local homomorphism. Since $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, the images of generators of $\mathfrak{m}_A/\mathfrak{m}_A^2$ generate $\mathfrak{m}_B/\mathfrak{m}_B^2$, and we can lift these by Nakayama to generators of $\mathfrak{m}_B$. This implies that the images of generators of $\mathfrak{m}_A$ generate $\mathfrak{m}_B$, so we have $\mathfrak{m}_A B = \mathfrak{m}_B$. Thus by our first assumption, $A/\mathfrak{m}_A \cong B/\mathfrak{m}_A B$. In particular we have a surjection $A \to B/\mathfrak{m}_A B$, so the image of 1 generates $B/\mathfrak{m}_A B$ as an A-module. Applying Nakayama again, this time to B as an A-module, we can lift the image of 1 in $B/\mathfrak{m}_A B$ to a generator of B as an A-module, which means f is surjective.

□

Proof of Proposition 8.1. For the "if" direction, first observe that since X is proper over k, $X \to \mathbb{P}^n_k$ is also proper. Therefore the image $Z \subset |\mathbb{P}^n_k|$ of $|X|$ under ϕ is a closed set. The same holds with $|X|$ replaced by any closed subset of X and Z replaced by $\phi(V)$ (note closed immersions are proper and compositions of proper morphisms are still proper), so ϕ is a closed map.

Next, condition (1) implies that the map ϕ of topological spaces is injective on closed points. We claim that this implies ϕ is actually injective. I'll come back to this at some point.

Now a closed map that is also injective is a homeomorphism onto its image. Therefore (check!) ϕ is quasi-finite. But a proper and quasifinite map is in fact finite (check!) and a finite map is always affine, so $X_i = \phi^{-1}(U_i)$ is affine.

It remains to show that if $p \in X(k)$, then the map

$$\mathfrak{m}_{\mathbb{P}^n,\phi(p)}/\mathfrak{m}^2_{\mathbb{P}^n,\phi(p)} \to \mathfrak{m}_{X,p}/m^2_{X,p}$$

is surjective. [Recall our notation: $X(k)$ is the set of k-valued points, i.e., maps $\operatorname{Spec} k \to X$, which correspond to closed points of X since k is algebraically closed. So we're saying the above should be surjective for all closed points p]. This will suffice, because the condition that $\mathcal{O}_{\mathbb{P}^n} \to \phi_*\mathcal{O}_X$ be surjective can be checked locally, since the global sections functor is exact on affines.

For notational convenience, we make a linear change of coordinates so that $\phi(p) = [0 : \ldots : 0 : 1] \in \mathbb{P}^n$. Then

$$\mathfrak{m}_{\mathbb{P}^n,\phi(p)}/\mathfrak{m}^2_{\mathbb{P}^n,\phi(p)} \cong k \cdot \frac{x_0}{x_n} \oplus \ldots \oplus k \cdot \frac{x_{n-1}}{x_n},$$

the linear forms on the open affine $U_n = \operatorname{Spec} k[\frac{x_0}{x_n}, \ldots, \frac{x_{n-1}}{x_n}]$. Since $p \in X_n$, the global section $s_n = x_n$ maps to a basis of $\mathcal{L}_p$, i.e., gives a trivialization of $\mathcal{L}_p$ (the section s_n canonically trivializes $\mathcal{L}$ over X_n). So $\mathfrak{m}_{X,p}/m^2_{X,p} \cong \mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$, and this isomorphism of $\mathcal{O}_X$-modules fits into the following diagram

$$\begin{tikzcd}
\mathfrak{m}_{\mathbb{P}^n,\phi(p)}/\mathfrak{m}^2_{\mathbb{P}^n,\phi(p)} \arrow[r, "\cong"] \arrow[d] & k \cdot \frac{x_0}{x_n} \oplus \ldots \oplus k \cdot \frac{x_{n-1}}{x_n} \arrow[d] \\
\mathfrak{m}_{X,p}/m^2_{X,p} \arrow[r, "\cong"] & \mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p
\end{tikzcd}$$

The map on the left is the one we want to show is surjective, and the map on the right is the map of condition (2). This proves surjectivity of the map of sheaves locally at closed points. It is left as an exercise to check at the non-closed points (possible hint: use fiber products somehow).

Note: we hand-waved the other implication (closed immersion implies separates points and tangent vectors) in the statement of the Proposition in class last time. I'll try to come back and write it up clearly at some point. $\square$

9.2 Towards Cohomology

As before, consider a scheme X, proper over an algebraically closed field, of dimension one, normal, and connected. Such a curve has only two types of points: a generic point, and some closed points $X(k)$, at each of which the local ring is a DVR.

Remark. If P is a closed point of X, then $\mathcal{I}_P$, the (quasicoherent) ideal sheaf associated to the embedding $P \to X$, is invertible. In general, if $\mathcal{L}$ is an invertible sheaf on X, we may write $\mathcal{L}(P) = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{I}_P^{-1}$.

Now, when we are given a rank one surjection $\mathcal{O}_X^{n+1} \to \mathcal{L}$, how do we check whether conditions (1) and (2) of Proposition 8.1 hold? First we consider condition (1). Suppose given two distinct (closed) points P and Q of X. We hope to produce a section $s \in V$ such that $s(P) = 0$ but $s(Q) \neq 0$.

Consider the case when $V = \Gamma(X, \mathcal{L})$. Then the s we desire is, in particular, an element of the kernel of the map of k-vector spaces $\Gamma(X, \mathcal{L}) = V \to \mathcal{L}_P \otimes_{\mathcal{O}_{X,P}} k(P)$, and this kernel is just $\Gamma(X, \mathcal{L}(-P))$. To see this, note that the map in queston is obtained by tensoring the exact sequence

$$0 \to \mathcal{I}_P \to \mathcal{O}_X \to k(P) \to 0$$

with $\mathcal{L}$, and then taking global sections (which is left-exact on the left, so respects kernels). Then note that $\Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} k(P)) \cong \mathcal{L}_P \otimes_{\mathcal{O}_{X,P}} k(P)$ since $k(P)$ is a skyscraper sheaf [2].

In order to determine whether we can find an s such that, also, $s(Q) \neq 0$, we ask whether the map

$$\Gamma(X, \mathcal{L}(-P)) \to \mathcal{L}_Q \otimes k(Q)$$

is surjective. Here we regard $\mathcal{L}_Q \otimes k(Q) \cong \mathcal{L}_Q/\mathfrak{m}_Q\mathcal{L}_Q$ simply as a one-dimensional vector space; but we may also regard $\mathcal{L}_Q \otimes k(Q)$ as a skyscraper sheaf at Q, which fits into the following exact sequence of sheaves of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{L}(-P-Q) \to \mathcal{L}(-P) \to \mathcal{L}_Q \otimes k(Q) \to 0$$

Now the global sections functor is not right-exact, so the map

$$\Gamma(X, \mathcal{L}(-P)) \to \Gamma(X, \mathcal{L}_Q \otimes k(Q)) = \mathcal{L}_Q \otimes k(Q)$$

is not necessarily surjective. But since $\mathcal{L}_Q \otimes k(Q)$ is a one-dimensional k-vector space, there are only two possibilities: the map is zero, or else it's surjective. Condition (1) holds exactly when the map is not zero. Similar considerations apply to condition (2) by taking $P = Q$ in the above. So we are led to the following general problem. Given the short exact sequence

$$0 \to \mathcal{L}(-P-Q) \to \mathcal{L}(-P) \to \mathcal{L}_Q \otimes k(Q) \to 0,$$

when is the sequence

$$0 \to \Gamma(X, \mathcal{L}(-P-Q)) \to \Gamma(X, \mathcal{L}(-P)) \to \Gamma(X, \mathcal{L}_Q \otimes k(Q))$$

[2] This means that the sections over any open containing P are all the same, so taking the limit does nothing.

obtained by applying the global sections functor also exact? This is the question of cohomology. Once we have defined cohomology, we will see that there is a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathcal{L}(-P-Q)) \arrow[r] & \Gamma(X, \mathcal{L}(-P)) \arrow[r] & \Gamma(X, \mathcal{L}_Q \otimes k(Q)) \arrow[dll] \\
& H^1(X, \mathcal{L}(-P-Q)) \arrow[r] & H^1(X, \mathcal{L}(-P)) \arrow[r] & \cdots
\end{tikzcd}$$

and thus the vanishing of $H^1(X, \mathcal{L}(-P-Q))$ for all P and Q (not necessarily distinct) is sufficient to ensure that the associated map $X \to \mathbb{P}^n$ is a closed immersion.

10 Wednesday, Feb. 8th: Cohomology of Coherent Sheaves on a Curve

Before proving Riemann-Roch, we introduce, without proof of its existence, the cohomology theory we need, together with its relevant properties. The first section is a reproduction of a handout Professor Olsson distributed in class. The second and third contain my notes on his comments about the handout.

10.1 Black Box: Cohomology of Coherent Sheaves on Curves

For the results on curves that we will prove in this class, we will assume the existence of a cohomology theory satisfying the following properties. The existence of such a cohomology theory will be a consequence of the general development of cohomology which we will study later.

Fix an algebraically closed field k. In the following, a *curve* means a proper normal scheme X/k of dimension 1.

If X is a curve and F is a coherent sheaf on X, we write $H^0(X, F)$ for $\Gamma(X, F)$. As part of our black box we will include the following result:

(0) For every coherent sheaf F on a curve X, $H^0(X, F)$ is finite dimensional over k.

We assume given for every curve X a functor

$$H^1(X, -) : (\text{coherent sheaves on } X) \to (\text{finite dimensional } k\text{-vector spaces})$$

and for every short exact sequence of coherent sheaves on X

$$\mathscr{E} : 0 \to F' \to F \to F'' \to 0$$

a map

$$\delta_{\mathscr{E}} : \Gamma(X, F'') \to H^1(X, F')$$

such that the following conditions hold.

(i) The functor $H^1(X, -)$ is k-linear in the following sense. If $a \in k$ and F is a coherent sheaf, and if $m_a : F \to F$ denotes the multiplication by a map on F (which is an endomorphism in the category of coherent sheaves on X) then the induced map

$$H^1(m_a) : H^1(X, F) \to H^1(X, F)$$

is multiplication by a on the vector space $H^1(X, F)$.
(ii) For every short exact sequence $\mathscr{E}$ as above, the sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, F') \arrow[r] & H^0(X, F) \arrow[r] & H^0(X, F'') \arrow[dll, "\delta_{\mathscr{E}}"'] & \\
& H^1(X, F') \arrow[r] & H^1(X, F) \arrow[r] & H^1(X, F'') \arrow[r] & 0
\end{tikzcd}$$

is exact.
(iii) The maps $\delta_{\mathscr{E}}$ is functorial in the exact sequence $\mathscr{E}$ in the following sense. If

$$\begin{tikzcd}
0 \arrow[r] & F_1' \arrow[r] \arrow[d, "a"] & F_1 \arrow[r] \arrow[d, "b"] & F_1'' \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & F_2' \arrow[r] & F_2 \arrow[r] & F_2'' \arrow[r] & 0
\end{tikzcd} \tag{9}$$

is a commutative diagram of coherent sheaves on a curve X with exact rows, then the diagram

$$\begin{tikzcd}
H^0(X, F_1'') \arrow[r, "\delta_{\mathscr{E}_1}"] \arrow[d, "H^0(c)"] & H^1(X, F_1') \arrow[d, "H^1(a)"] \\
H^0(X, F_2'') \arrow[r, "\delta_{\mathscr{E}_2}"] & H^1(X, F_2')
\end{tikzcd}$$

commutes, where we write $\mathscr{E}_1$ (resp. $\mathscr{E}_2$) for the top (resp. bottom) short exact sequence in (9).
(iv) If F is a coherent sheaf on a curve X whose support is a finite set of points, then $H^1(X, F) = 0$.
(v) If X is a curve then there is an isomorphism $\mathrm{tr} : H^1(X, \Omega_X^1) \to k$ such that for any locally free sheaf E on X with dual $E^\vee$ the pairing

$$H^0(X, E) \times H^1(X, E^\vee \otimes \Omega_X^1) \longrightarrow H^1(X, \Omega_X^1) \xrightarrow{\mathrm{tr}} k$$

is perfect.

Remark. Axiom (i) implies that if F and G are coherent sheaves on X and if $z : F \to G$ is the zero map then the induced map $H^1(z) : H^1(X, F) \to H^1(X, G)$ is also the zero map. Indeed $z = m_0 \circ z$, where $m_0 : G \to G$ is multiplication by 0, and therefore $H^1(z) = H^1(m_0) \circ H^1(z)$. Since $H^1(m_0)$ is multiplication by 0 on $H^1(X, G)$ by (i) this implies that $H^1(z) = 0$.

Remark. Axiom (ii) implies that $H^1(X, -)$ commutes with finite direct sums. Indeed to show this it suffices by induction to consider two coherent sheaves F and G. In this case we have a split exact sequence

$$0 \to F \to F \oplus G \to G \to 0.$$

Since the projection map $F \oplus G \to G$ admits a section, then induced map $H^0(X, F \oplus G) \to H^0(X, G)$ is surjective, whence (ii) gives an exact sequence

$$0 \to H^1(X, F) \to H^1(X, F \oplus G) \to H^1(X, G) \to 0$$

split by the standard inclusion $G \hookrightarrow F \oplus G$. Since finite direct sums are isomorphic to finite direct products in both the category of coherent sheaves on X as well as the category of finite dimensional vector spaces, it follows that $H^1(X, -)$ also commutes with finite products. This has many consequences. For example, if F is a coherent sheaves and $\sigma : F \oplus F \to F$ is the summation map, then the induced map

$$H^1(X, F) \oplus H^1(X, F) \simeq H^1(X, F \oplus F) \xrightarrow{H^1(\sigma)} H^1(X, F)$$

is the summation map on the vector space $H^1(X, F)$. Similarly if $f, g : F \to G$ are two maps of coherent sheaves, then the two maps

$$H^1(f + g), H^1(f) + H^1(g) : H^1(X, F) \to H^1(X, G)$$

are equal.

10.2 Further Comments; Euler Characteristic

Axiom (v) is known as Serre duality. We explain how to obtain the first map, leaving the existence of the trace map tr as part of the axiom. For a perfect pairing, we would like to associate to each $e \in H^0(X, E)$ an operator $\langle e, -\rangle$ on $H^1(X, E^\vee \otimes \Omega^1_X)$, in such a way that this isomorphism identifies $H^0(X, E)$ with the dual space of $H^1(X, E^\vee \otimes \Omega^1_X)$. First observe that for each $e \in H^0(X, E) = \Gamma(X, E)$, we get a map $\mathcal{O}_X \to E$, which we tensor with $E^\vee$ to obtain maps of sheaves

$$\begin{tikzcd}
E^\vee \arrow[r, "\cong"] \arrow[drrr, dotted] & \mathcal{O}_X \otimes_{\mathcal{O}_X} E^\vee \arrow[r, "e\otimes 1"] & E \otimes E^\vee \arrow[d] \\
& & \mathcal{O}_x
\end{tikzcd}$$

where the vertical map is just evaluation. The dotted composite we now tensor with Ω^1 to give a map $E^\vee \otimes \Omega^1 \to \Omega^1$, to which we apply the functor $H^1(X-)$. So for each such global section e, we have a map $H^1(X, E^\vee \otimes \Omega^1_X) \to H^1(X, \Omega^1)$, and this defines the pairing. Note also that the existence of the trace map in the axiom should be expected if you are familiar with Hodge theory on complex manifolds.

We will state Riemann-Roch using Euler characteristic, and then rederive the more familiar form. The theorem is about calculating the dimensions of the k-vector spaces $H^0(X, \mathcal{L})$ for various line bundles $\mathcal{L}$. This can be quite hard, and it turns out to be easier to look instead at the Euler characteristic χ defined by, for $\mathcal{F} \in \mathrm{Coh}(X)$,

$$\chi(X, \mathcal{F}) = \dim_k H^0(X, \mathcal{F}) - \dim_k H^1(X, \mathcal{F})$$

One useful fact is that χ is additive on short exact sequences. More precisely, we have the following:

Lemma 10.1. *If the sequence* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is exact, then* $\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'')$.

Proof. From the given sequence we get a long exact sequence

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[dr] & 0 & & \\
 & & I_1 \arrow[d] \arrow[ur] & & I_3 \arrow[u] \arrow[dr] & & \\
0 \arrow[r] & H^0(\mathcal{F}') \arrow[r] & H^0(\mathcal{F}) \arrow[r] \arrow[ur] & H^0(\mathcal{F}'') \arrow[rr] \arrow[dr] & & H^1(\mathcal{F}') \arrow[r] & H^1(\mathcal{F}) \arrow[r] & H^1(\mathcal{F}'') \arrow[r] & 0 \\
 & & & & I_3 \arrow[ur] \arrow[dr] & & & & \\
 & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

Here the I_k are the kernels and images of the appropriate maps. This gives us four short exact sequences involving the cohomology groups and the I_k. Each short exact sequence relates the dimensions of three of the vector spaces. If you combine the four equations appropriately, you get the result. *Note: This diagram differs slightly from the one used in class, though the argument is essentially the same. This more symmetric form of the diagram was pointed out to me by Alex Kruckman.* $\square$

Remark. Let's briefly recall the relationship between divisors and line bundles. On a curve X, we can define a divisor as a formal sum of closed points, $D = \sum n_P \cdot P$. To such an object, we associate a line bundle, denoted by either $\mathcal{O}_X(D)$ or $\mathcal{L}(D)$, as follows. Each closed point P may be regarded as a closed subscheme of X, and thus has an associated ideal sheaf $\mathcal{I}_P$ [Recall further that a closed point is the same as a map $i\colon \operatorname{Spec} k \to X$, which induces a map $i^{\#}\colon \mathcal{O}_X \to i_*\mathcal{O}_{\operatorname{Spec} k}$, where the latter is most usefully regarded as the skyscraper sheaf of $k(P) = k$ at the point P. The ideal sheaf $\mathcal{I}_P$ is defined to be the kernel of $i^{\#}$]. Then we set

$$\mathcal{L}(D) = \mathcal{L}(\sum n_P \cdot P) = \bigotimes_P \mathcal{I}_P^{-n_P}$$

where here the superscript means tensor power.

Conversely, every invertible sheaf on X is of this form. Suppose given a line bundle $\mathcal{L}$ on X. Take an open subset $U \subset X$ on which we have a trivialization $\sigma\colon \mathcal{O}_U \xrightarrow{\sim} \mathcal{L}\big|_U$. Let $s = \sigma(1)$, a section of $\mathcal{L}$ over U. The complement of U is then a finite set of points $\{P_1, \ldots, P_r\}$ (possibly empty, if $\mathcal{L}$ itself happened to be trivial). Since X is a curve, the local ring at each P_i is a DVR. We want to use the valuations in each of these DVRs to produce the integers n_{P_i}. Now we proved last semester that there is an inclusion $\mathcal{L} \hookrightarrow \mathscr{K}$, where $\mathscr{K}$ is the constant sheaf associated to the function field $k(X)$. So we take s, and map it into $\mathscr{K}(U) \cong k(X)$. Then at each P_i we have a valuation ν_{P_i} on $k(X)$, and we define $n_{P_i} = \nu_{P_i}(s)$. These n_{P_i}, for each P_i, thus define a divisor $D = \sum n_{P_i} \cdot P_i$ on X. Of course, we still have to check that $\mathcal{L} \cong \mathcal{L}(D)$. For this, you can refer to the notes from last semester.

10.3 Statement of Riemann-Roch; Reduction to the More Familiar Version

From now on, we will use the notation $h^i(X, \mathcal{L}) = \dim_k H^i(X, \mathcal{L})$.

Theorem 10.1 (Riemann-Roch Theorem)**.** *If X is a curve, and $\mathcal{L}$ a line bundle on X, then*

$$\chi(X, \mathcal{L}) \cong \deg \mathcal{L} + \chi(X, \mathcal{O}_X).$$

We will prove this next lecture. First notice that

$$H^1(X, \mathcal{L}) = H^1\big(X, (\mathcal{L}^\vee \otimes \Omega^1_X)^\vee \otimes \Omega^1_X\big),$$

and by Serre Duality, this is dual to $H^0(X, \mathcal{L}^\vee \otimes \Omega^1_X)$. Hence $h^1(X, \mathcal{L}) = h^0(X, \mathcal{L}^\vee \otimes \Omega^1_X)$. This allows us to rewrite the left hand side of the equation in the theorem as $h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^\vee \otimes \Omega^1_X)$. Moreover, applying the same trick to $H^1(X, \mathcal{O}_X)$, we get $h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_X)$, which is by definition the genus, g, of X. Finally, we have $h^0(X, \mathcal{O}_X) = 1$, by either of two arguments presented at the beginning of the next lecture.

Thus we can rewrite Riemann-Roch as

$$h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^\vee \otimes \Omega^1_X) = \deg \mathcal{L} + 1 - g.$$

Remark. If $H^0(X, \mathcal{L}) \neq 0$, then $\deg \mathcal{L} \geq 0$. For if s is a nonzero global section, then as an element of the fraction field, its valuation at each point P is non-negative (i.e., it's in $\mathcal{O}_{X,P}$). Loosely, it's everywhere regular, so has no poles. In fact, you can calculate the degree of $\mathcal{L}$ explicitly by looking at the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}/\mathcal{O}_X \to 0,$$

where the term on the right is supported on a finite set of points.

11 Friday, Feb. 10th: Proof and Consequences of Riemann-Roch

11.1 Preparatory Remarks

Remark. Since $H^0(X, \mathcal{O}_X)$ is a k-algebra, $h^0(X, \mathcal{O}_X)$ is at least one. Here are two different reasons why it's equal to one.

Proof 1. Let $f \in H^0(X, \mathcal{O}_X)$. Then f defines a map $X \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$. Since $\mathbb{P}^1_k$ is proper, $f(X)$ is a closed subset of $\mathbb{P}^1_k$, hence a finite set of points in $\mathbb{A}^1_k$. Since X is connected, so is $f(X)$, which must therefore consist of a single point $\alpha \in \mathbb{A}^1_k$.[3] So the map to $\mathbb{A}^1_k$ defined by f agrees with the map defined by $\alpha \in k \subset H^0(X, \mathcal{O}_X)$. Therefore the global sections f and α differ by nilpotence, and since X is reduced, they are the same global section. Hence $H^0(X, \mathcal{O}_X) = k$.

□

[3] Recall that the map to $\mathbb{A}^1_k$ defined by f sends x to the residue of f in $k(x)$ for each $x \in X$. For closed points, $k(x) = k$, so in our case, we have that f maps to the same $\alpha \in k$ for every closed point, and we think of this α as living in $\mathbb{A}^1_k$ (i.e., as corresponding to the ideal $(t - \alpha) \subset k[t]$.

Proof 2. Since $H^0(X, \mathcal{O}_X)$ is finite-dimensional over k, it's a finitely generated integral (commutative) algebra over k. This implies it's a finitely generated field extension of k: to find the inverse to a nonzero α, use the monic polynomial cancelling α. That is, if $\alpha^n + b_{n-1}\alpha^{n-1} + \ldots + b_0 = 0$, then $\alpha(\alpha^{n-1} + b_{n-1}\alpha^{n-1} + \ldots + b_1) = -b_0$, which is nonzero since it's a domain, so α is a unit.

Then since k is algebraically closed, there are no nontrivial finitely generated extensions, hence $H^0(X, \mathcal{O}_X) = k$.

□

Remark. Recall from last time that if $\mathcal{L}$ is an invertible sheaf on X, and there is a nonzero global section $s \in H^0(X, \mathcal{L})$, then $\deg \mathcal{L} \geq 0$. Also, if $\deg \mathcal{L} = 0$, then $\mathcal{L} \cong \mathcal{O}_X$ (so $s \in k$). To see this, note that the section s defines a map $\mathcal{O}_X \to \mathcal{L}$. For over each open U, we can define $\mathcal{O}(U) \to \mathcal{L}\big|_U$ by sending 1 (i.e., the restriction of $1 \in H^0(X, \mathcal{O}_X)$) to $s\big|_U$. This obviously commutes with restriction maps, so it defines a morphism of sheaves. Of course, there's no reason for this to be an isomorphism, but it is injective since s is nonzero over every open U (using that X is integral).

We are interested in the cokernel of this map s. To measure the failure of s to be an isomorphism, we begin with an open U on which $\mathcal{L}$ is trivial; s itself furnishes a trivialization. The complement of U is a finite point set $\{P_1, \ldots, P_r\}$. The cokernel is trivial over U; looking at germs near P_i, $\mathcal{L}_{P_i}$ is a free rank one $\mathcal{O}_{X,P_i}$-module, and the cokernel of s locally has the form $\mathcal{L}_{P_i}/\mathfrak{m}_{P_i}^{n_i}\mathcal{L}_{P_i}$. Each n_i is non-negative since s is a global section. Thus the cokernel of s is the sheaf

$$\operatorname{coker}(s) = \bigotimes_{\{P_1,\ldots,P_r\}} \mathcal{L}_{P_i}/\mathfrak{m}_{P_i}^{n_i}\mathcal{L}_{P_i}$$

where each factor in the tensor product is a skyscraper sheaf at the appropriate P_i. The global sections of this sheaf forms a finite-dimensional k-vector space, whose dimension is the degree of $\mathcal{L}$, hence $\deg \mathcal{L} = \sum n_i$. It follows, then, that if $\deg \mathcal{L} = 0$, each n_i is zero, and the cokernel is trivial, so s defines an isomorphism $\mathcal{L} \cong \mathcal{O}_X$.

11.2 Proof of the Riemann-Roch Theorem

Proof. We will prove the relation

$$\chi(X, \mathcal{L}) = \deg \mathcal{L} + \chi(X, \mathcal{O}_X)$$

by induction on the degree of $\mathcal{L}$. For $\deg \mathcal{L} = 0$, it's a consequence of the above remarks, since we just saw that in this case, $\mathcal{L} \cong \mathcal{O}_X$.

For the inductive step, we prove that the equation holds for $\mathcal{L}$ if and only if it holds for $\mathcal{L}(P) = \mathcal{L} \otimes \mathcal{I}_P^{-1}$. To see this, tensor the sequence

$$0 \to \mathcal{I}_P \to \mathcal{O}_X \to k(P) \to 0$$

with $\mathcal{L}(P)$, giving

$$0 \to \mathcal{L} \to \mathcal{L}(P) \to k(P) \to 0$$

(note that the term on the right is a skyscraper sheaf, so tensoring with $\mathcal{L}(P)$ does nothing). Since the Euler characteristic is additive on exact sequences, this yields $\chi(X, \mathcal{L}(P) = \chi(X, \mathcal{L}) + 1$ (note that $h^1(X, k(P) = 0$ by one of our axioms for cohomology). Since $\deg \mathcal{L}(P) = \deg \mathcal{L} + 1$, this proves the equivalence.

This proves the theorem, for we have seen above that any line bundle can be obtained from $\mathcal{O}_X$ by adding or subtracting a finite number of points

□

11.3 Consequences

Corollary 11.1. $\deg \Omega^1_k = 2g - 2$.

This comes from setting $\mathcal{L} = \Omega^1_k$ in the statement of Theorem 10.1. It suggests that the study of curves should divide into three cases: $g = 0$, $g = 1$, or $g \geq 2$, according as the degree of Ω^1_k is negative, zero, or positive, respectively.

Corollary 11.2. *If* $\deg \mathcal{L} > 2g - 2$,

$$h^0(X, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

Corollary 11.3. *If* $\deg \mathcal{L} > 2g - 1$, *then* $\mathcal{L}$ *is generated by global sections.*

Remark. We could also express this by saying that the map $H^0(X, \mathcal{L}) \otimes_k \mathcal{O}_X \to \mathcal{L}$ is surjective. What map is this? Let $f\colon X \to \operatorname{Spec} k$ be the structure morphism. Since f^* is a left adjoint to f_*, there is a natural map $f^* f_* \mathcal{L} \to \mathcal{L}$. But f_* is just $H^0(X, \mathcal{L})$ sitting over the unique point of $\operatorname{Spec} k$. This pulls back to $f^{-1} H^0(X, \mathcal{L}) \otimes_{f^{-1}\mathcal{O}_{\operatorname{Spec} k}} \mathcal{O}_X$, and since f is actually an open map here, and $H^0(X, \mathcal{L})$ and $\mathcal{O}_{\operatorname{Spec} k}$ are constant sheaves f^{-1} just pulls them back to the corresponding constant sheaves on X, i.e., $f^{-1} H^0(X, \mathcal{L}) \otimes_{f^{-1}\mathcal{O}_{\operatorname{Spec} k}} \mathcal{O}_X \cong H^0(X, \mathcal{L}) \otimes_k \mathcal{O}_X$.

Proof of Corollary 11.3. We prove that the cokernel of the map $H^0(X, \mathcal{L}) \to \mathcal{L}$ described in the previous remark is zero. Since the support of the cokernel (and indeed any sheaf) is closed, it's enough to check that the cokernel is zero at closed points. Thus we have to show that for any closed point P, there is a global section f whose image in $\mathcal{L} \otimes k(P) \cong \mathcal{L}_P / \mathfrak{m}_P \mathcal{L}_P$ is nonzero (we are using Nakayama here). This is equivalent to finding a global section f which is in $\Gamma(X, \mathcal{L})$ but not in the subspace $\Gamma(X, \mathcal{L}(-P))$. This we can verify simply by comparing dimensions. Our degree assumption ensures that the h^1 term in Riemann-Roch vanishes for both sheaves (Corollary 11.2), so we have

$$h^0(X, \mathcal{L}) = \deg \mathcal{L} + 1 - g \quad \text{and} \quad h^0(X, \mathcal{L}(-P)) = \deg \mathcal{L} - 1 + 1 - g = \deg \mathcal{L} - g$$

This $\Gamma(X, \mathcal{L}(-P))$ is a proper subspace of $\Gamma(X, \mathcal{L})$, so we can find the desired global section f. □

Remark. (In response to a question) Recall that $\mathcal{L}(-P)$ has stalks equal to $\mathcal{L}$ away from P, and equal to $\mathfrak{m}_P \otimes \mathcal{L}$ at P.

Of course, it is not really necessary to require that $\deg \mathcal{L}$ be greater than $2g-1$, for if $\mathcal{L}$ has any positive degree, then we can take a high tensor power to obtain a sheaf whose degree is at least $2g-1$.

Definition 11.1. A line bundle $\mathcal{L}$ on X is *very ample* if the map $\pi\colon H^0(X,\mathcal{L})\otimes_k \mathcal{L} \to \mathcal{L}$ defines a closed immersion into $\mathbb{P}^n_k$. $\mathcal{L}$ is *ample* if there exists an $n>0$ such that $\pi\colon H^0(X,\mathcal{L}^{\otimes n})\otimes_k \mathcal{L} \to \mathcal{L}^{\otimes n}$ defines a closed immersion.

We will prove next time that if $\deg \mathcal{L} > 0$, then $\mathcal{L}$ is ample.

12 Monday, Feb. 13th: Applying Riemann-Roch to the Study of Curves

First we introduce some new terminology. Recall that we have constructed a map $H^0(X,\mathcal{L})\otimes_k \mathcal{O}_X \to \mathcal{L}$ of sheaves, for a line budle $\mathcal{L}$ on a curve X. We call the set of points at whose stalks this map is not surjective the *base locus* of $\mathcal{L}$, and if it surjective everywhere (i.e., $\mathcal{L}$ is generated by global sections), we say $\mathcal{L}$ is *base point-free*.

In this language, we saw last time that if the genus of X is g and $\deg \mathcal{L} > 2g-1$, then $\mathcal{L}$ is base point-free, in which case we get a morphism of schemes $X \to \mathbb{P}H^0(X,\mathcal{L})$.

12.1 Using Riemann-Roch to Detect Closed Immersions to $\mathbb{P}^n$

Any rank one quotient of $\mathcal{O}_X^{n+1}$ corresponds to some morphism $X \to \mathbb{P}^n_k$. When $\mathcal{L}$ is generated by global sections, we get a rank one quotient of the form $H^0(X,\mathcal{L})\otimes_k \mathcal{O}_X \to \mathcal{L}$, with corresponding morphism $X \to \mathbb{P}H^0(X,\mathcal{L})$. When is this a closed immersion?

By our earlier criteria, $\mathcal{L}$ must separate points and tangent vectors. We now rephrase these conditions in such terms as allow us to apply Riemann-Roch.

The condition of separating points is equivalent to requiring that for any two distinct closed poitns P and Q in X, there is a section s in $H^0(X,\mathcal{L}(-P)) \setminus H^0(X,\mathcal{L}(-P-Q))$. The condition of separating tangent vectors is equivalent to requiring that for any closed point P, there is a section s in $H^0(X,\mathcal{L}(-P)) \setminus H^0(X,\mathcal{L}(-2P))$. We can guarantee the satisfaction of both conditions at once by requiring that, for any P and Q in $X(k)$[4], not necessarily distinct, there is a section in $H^0(X,\mathcal{L}(-P)) \setminus H^0(X,\mathcal{L}(-P-Q))$. This can be checked with a simple dimension count, and Riemann-Roch gives the following criterion:

Proposition 12.1. *If $\deg \mathcal{L} > 2g$, then $\mathcal{L}$ is very ample.*

Proof. For any closed points P, Q (not necessarily distinct), our assumption on the degree of $\mathcal{L}$ ensures that h^1 is zero for both $\mathcal{L}(-P)$ and $\mathcal{L}(-P-Q)$, so Riemann-Roch computes the following:

$$h^0(X,\mathcal{L}(-P)) = (\deg \mathcal{L} - 1) + 1 - g$$
$$h^0(X,\mathcal{L}(-P-Q)) = (\deg \mathcal{L} - 2) + 1 - g,$$

[4]Recall that this is the set of k-valued points of X, namely the closed points.

This means the latter is a proper subspace of the former, and we can find the section we desire, as described in the preceding discussion. $\square$

We also have the following:

Corollary 12.1. *If* $\deg \mathcal{L} > 0$*,* $\mathcal{L}$ *is ample.*

12.2 Examples

Genus zero

Let X be a curve with $g = 0$. We will show that X is isomorphic to $\mathbb{P}^1_k$. For any $P \in X(k)$, let, as usual, $\mathcal{O}_X(P) = \mathcal{I}_P^{-1}$, where $\mathcal{I}_P$ is the ideal sheaf of $P \hookrightarrow X$. It has degree $1 > 2g - 2 = -2$, so

$$h^0(X, \mathcal{O}_X(P)) = \deg \mathcal{L} + 1 - g = 2$$

By choosing an isomorphism $H^0(X, \mathcal{O}_X(P)) \xrightarrow{\sim} k^2$, we get a closed immersion (by Proposition 12.1, since $\deg \mathcal{L} = 1 > 2g = 0$) $i\colon X \to \mathbb{P}^1_k$. We claim this is actually an isomorphism of schemes. Since it's a closed immersion, i is a closed map. Therefore its image is a closed set, hence all of $\mathbb{P}^1$. Since it's an immersion, it's injective, and since proper also, it's a homeomorphism (cf. a related statement in point-set topology - a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism.) Now we check the map of sheaves is an isomorphism. Since i is a closed immersion, it induces a surjection of sheaves, and hence a surjection of stalks $\mathcal{O}_{\mathbb{P}^1,P} \to \mathcal{O}_{X,P}$ (here we identify X with its image, namely P^1, and thus speak of the same P on both sides). The map i is a surjective morphism of schemes, hence in particular a dominant rational map, and so it induces a map of funtion fields $k(\mathbb{P}^1) \to k(X)$. This map is not the zero map, so it must be injective. Thus we obtain the following diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^1,P} \arrow[r, two heads] \arrow[d] & \mathcal{O}_{X,P} \arrow[d] \\
k(\mathbb{P}^1) \arrow[r] & k(X)
\end{tikzcd}$$

where the top arrow is surjective and the other three are injective. This forces the top arrow to be an isomorphism, like we wanted. This shows that X is isomorphic to $\mathbb{P}^1_k$.

Genus one

Now let X be a genus one curve[5] over k. Since $2g = 2$, we need a line bundle of degree at least three to embed in projective space. Pick a point $P \in X(k)$ and consider $\mathcal{O}_X(3P)$. After choosing a basis for the three-dimensional space $H^0(X, \mathcal{O}_X(3P))$, we get a closed immersion $i\colon X \to \mathbb{P}^2_k$. We will show next time that i identifies X with a cubic curve in $\mathbb{P}^2_k$, which can be given in characteristics different from 2 or 3 by an equation of the form $y^2 = x^3 + Ax + B$.

[5] This is not the same as saying that X is an elliptic curve - these are given by a genus one curve together with a chosen point.

Remark. We chose a degree 3 line bundle to satisfy the hypothesis of the theorem, but this is not necessarily the best we can do. For an example, take $X = \mathbb{P}^1_k$, and the line bundle $\mathcal{O}_X(2)$. Then $h^0(\mathbb{P}^1_x, \mathcal{O}(2))$, and we get an embedding $\mathbb{P}^1_k \to \mathbb{P}^2_k$. Exercise: find the equation of this embedding (hint: it's the Veronese map).

13 Wednesday, Feb. 15th: "Where is the Equation?"

Today we prove:

Theorem 13.1. *Any curve of genus one (over k) is isomorphic to a cubic in $\mathbb{P}^2_k$.*

Recall that a homogeneous polynomial F of degree d in the variables, say, $x_0, \ldots, x_n$, defines a closed subscheme of $\mathbb{P}^n_k$, which we denote $V(F)$. We ask, given a morphism $g\colon X \to \mathbb{P}^n_k$ corresponding to a rank one quotient $\mathcal{O}_X^{n+1} \to \mathcal{L}$, how can we detect whether g factors through $V(F)$? To answer this, let the generating global sections from the surjection $\mathcal{O}_X^{n+1} \to \mathcal{L}$ be $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Recall that the identity map $\mathbb{P}^n \to \mathbb{P}^n$ gives a surjection $\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$, with corresponding global sections $x_0, \ldots, x_n$, which we think of as the homogeneous coordinates on $\mathbb{P}^n_k$. Now if we have a morphism $g\colon \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$, we can pull back the surjection $\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$ over $\mathbb{P}^n_k$ to the surjection $\mathcal{O}_X^{n+1} \to g^*\mathcal{O}_{\mathbb{P}^n}(1)$ (recall that the pullback of the structure sheaf is again the structure sheaf). We will abuse notation slightly and refer to the global sections of this pulled-back surjection also as $x_0, \ldots, x_n$. The surjections fit into a diagram

$$\begin{tikzcd}
\mathcal{O}_X^{n+1} \arrow[r, two heads, "(s_0,\ldots,s_n)"] \arrow[dr, two heads, "(x_0,\ldots,x_n)"'] & \mathcal{L} \arrow[d, "\cong"] \\
& g^*\mathcal{O}_{\mathbb{P}^n}(1)
\end{tikzcd}$$

Here the vertical map is an isomorphism because both surjections correspond to the same morphism $X \to \mathbb{P}^n_k$.

Proposition 13.1. *The map g factors through $V(F)$ if and only if $F(s_0, \ldots, s_n)$ is zero in $\Gamma(X, \mathcal{L}^{\otimes d})$.*

Coarse Proof. The expression of a closed point P of $\mathbb{P}^n$ in terms of homogeneous coordinates $[x_0 : \ldots : x_n]$ is obtained by taking each of the global sections $x_0, \ldots, x_n$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ and evaluating them in $\mathcal{O}(1)_P/\mathfrak{m}_P\mathcal{O}(1)_P \cong k$. A point P is in $V(F)$ if and only if these evaluations satisfy $F = 0$, or put another way, if and only if $F(x_0, \ldots, x_n)$, regarded as an element of the sheaf $\mathcal{O}(1)^{\otimes d}$, maps to zero at P. This makes sense, since locally, $\mathcal{O}(1)^{\otimes d} \cong \mathcal{O}(1)$ via the multiplication map $a_0 \otimes \ldots \otimes a_n \mapsto a_1 \cdots a_n$ (of course, they're both isomorphic to $\mathcal{O}_{\mathbb{P}^n}$ locally). So the evaluation of F at P is obtained by taking F, restricting it to some open containing P on which $\mathcal{O}(1)$ is trivial, multiplying the sections occuring in F[6], and then

[6] It doesn't make sense to multiply, say, x_0x_1 over an arbitrary open since $\mathcal{O}(1)$ may not be a ring over any open - it only is once we trivialize the line bundle.

passing to $\mathcal{O}(1)_P/\mathfrak{m}_P\mathcal{O}(1)_P$. This gives an element of k, which we require to be zero in order for P to be in $V(F)$.

To ask whether a point of X maps into $V(F)$, note that the image of a point $P \in X$ under g is given by evaluating the sections $s_0, \ldots, s_n$ at P. So the discussion in the previous paragraph applies to (the images of) points of X, simply replacing x_i by s_i.

□

Let's look at some examples.

Example 13.1. Let X have genus zero, and pick a point $P \in X(k)$. The space $H^0(X, \mathcal{O}_X)$ is a proper subspace of $H^0(X, \mathcal{O}_X(P))$. We've already computed $\dim H^0(X, \mathcal{O}_X(P)) = 2$, so we can write $H^0(X, \mathcal{O}_X(P))$ as the span of two global sections 1 and x. Here 1 spans the constant sections, namely $H^0(X, \mathcal{O}_X)$, while x necessarily has a pole at P, or else it would be in $H^0(X, \mathcal{O}_X)$.

We've seen last time that a basis of global sections of $\mathcal{O}_X(P)$ determines a map $X \to \mathbb{P}^1$ (an isomorphism, in fact). Identifying X with $\mathbb{P}^1$ via this isomorphism, one thinks of the global sections x and 1 as dehomogenizations of the global sections (i.e., homogeneous coordinates) x and y on $\mathbb{P}^1$.

Example 13.2. Let X have genus one, and fix a point $P \in X(k)$. The invertible sheaf $\mathcal{L} = \mathcal{O}(3P)$ defines an embedding $X \to \mathbb{P}^2_k$, since $h^0(X, \mathcal{L}) = 3 > 2g = 2$. We have a chain of subspaces (writing $H^0(X, \mathcal{O}_X) = H^0(\mathcal{O}_X)$, etc., for short)

$$\begin{aligned} H^0(\mathcal{O}) \subseteq H^0(\mathcal{O}(P)) \subseteq H^0(\mathcal{O}(2P)) \subseteq H^0(\mathcal{O}(3P)) \subseteq H^0(\mathcal{O}(4P)) \\ \subseteq H^0(\mathcal{O}(5P)) \subseteq H^0(\mathcal{O}(6P)) \subseteq \ldots \end{aligned}$$

Riemann-Roch allows us to compute the dimensions: 1,1,2,3,4,5,6,.... Pick a basis $1, x$ for $H^0(\mathcal{O}(2P))$. Then x has a pole of order two at P, or else the previous space would have been two-dimensional. Similarly, we have a basis $1, x, y$ for $H^0(\mathcal{O}(3P))$, where y has a triple pole at P. Then x^2 extends this to a basis for $H^0(\mathcal{O}(4P))$, and xy extends it further to a basis for $H^0(\mathcal{O}(5P))$. But in $H^0(\mathcal{O}(6P))$, we have the seven elements $1, x, x^2, x^3, xy, y, y^2$, all of which have poles or order at most 6 at P. Since this space is six-dimensional, there must be a dependence relation amongst them, and this relations must involve both x^3 and y^2, or else one of these two would live in a previous space, but they both have poles of order exactly six. Thus, after rescaling the coordinates if necessary, we obtain a relation of the form

$$q(x, y) = y^2 + bxy + cy - x^3 - ex^2 - fx - g = 0$$

This implies that $X \subseteq Y = V(q(x, y)) \subseteq \mathbb{P}^2$. Now one argues that since X is a curve, $q(x, y)$ is irreducible, and Y is one-dimensional, X must be homeomorphic to Y. To check the map $X \to Y$ is an isomorphism, it's

enough to look at the stalks. For the stalk at P, we have a diagram

$$\begin{tikzcd} \mathcal{O}_{Y,P} \arrow[r, two heads] \arrow[d] & \mathcal{O}_{X,P} \arrow[d] \\ k(Y) \arrow[r, "\cong"] & k(X) \end{tikzcd}$$

where the vertical maps are inclusions, and the bottom arrow is an isomorphism since Y is integral (we mentioned above it's irreducible. Check it's reduced). Thus the map on stalks is an isomorphism at P.

In general, given X and $\mathcal{L}$, with corresponding embedding $X \hookrightarrow \mathbb{P}H^0(X, \mathcal{L})$, it possible to express

$$X \cong \operatorname{Proj} \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}).$$

We also have

$$\mathbb{P}^n_k = \operatorname{Proj} \bigoplus_{n \geq 0} S^n \Gamma(X, \mathcal{L}),$$

where $S^n V$ denotes the nth symmetric power. Then the embedding $X \hookrightarrow \mathbb{P}^n_k$ is given by a map of graded rings given in degree n by $S^n \Gamma(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. The equations defining X come from the kernel of this map of graded rings.

Going back to Example 2, the equation $q(x, y) = 0$ defines an open subset U of X inside the affine chart $\mathbb{A}^2_{x,y}$ of $\mathbb{P}^2_k$ (the point P is in the complement of U). We can regard this inclusion $Y \subset \mathbb{A}^2$ as being given by the coordinate functions x and y (this is basically the $\operatorname{Spec} - \Gamma$ adjunction for morphisms to affine targets). But the coordinate function x also defines a map $\mathbb{A}^2 \to \mathbb{A}^1$. Precomposing this with the inclusion $U \subseteq \mathbb{A}^2$ and then following with the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$ gives a map $U \to \mathbb{P}^1$. We have seen that such a map extends to a morphism $X \to \mathbb{P}^1$.

Now we ask: what is the line bundle and corresponding global sections associated to this map $X \to \mathbb{P}^1$? Since we mapped $\mathbb{A}^2$ to $\mathbb{A}^1$ by "forgetting" the y-coordinate, it is perhaps not surprising that the map to $\mathbb{P}^1$ comes from the space $H^0(X, \mathcal{O}(2P))$, which had as basis only 1 and x.

Remark. It is no accident that the degree of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$, when pulled back to X, is two. This is also the degree of

$$k(x)[y]/(y^2 - (x - \alpha)(x - \beta)(x - \gamma)).$$

Note also that, letting f be the map $X \to \mathbb{P}^1$, we have

$$\deg f^* \mathcal{O}_{\mathbb{P}^1}(1) = \deg \mathcal{I}_{f^{-1}(\infty)} = \deg \Big(\bigotimes_{P \,\big|\, f(P) = \infty} \mathcal{I}^{e_P} \Big) = - \left| f^{-1}(\infty) \right| = \sum e_P$$

Here e_P denotes the ramification index of f at P. We will discuss degree and ramification in the next lecture.

14 Friday, Feb. 17th: Ramification and Degree of a Morphism

Definition 14.1. Let $f\colon X \to Y$ be a nonconstant morphism of curves. The **degree** of f is the degree of the field extension $[k(Y) : k(X)]$. If P is a closed point of X, with $f(P) = Q$, the **ramification index** e_P of f at P is defined to be the ramification index of the induced map of DVRs $\mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$.

This isn't much of a definition until we have defined ramification indices for maps of DVRs. Let π_P, π_Q uniformizers of $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,Q}$, respectively. Then π_Q maps to a unit times $\pi_P^{e_P}$, where $e_P \geq 1$ since this is a local homomorphism. Then e_P is the degree of this map of DVRs. We say f is **unramified at** P if $e_P = 1$, and **ramified at** P if $e_P > 1$, in which case we also call Q a **branch point of** f.

The degree of the morphism f is, intuitively, the cardinality of the preimage of any point $Q \in Y$, but we have to be careful at the branch points.

Proposition 14.1. *For any closed point Q of Y,*

$$\sum_{P \in f^{-1}(Q)} e_P = \deg f.$$

Proof. Let $\mathscr{A} = f_*\mathcal{O}_X$. Then $X = \underline{\operatorname{Spec}}_Y \mathscr{A}$. The idea of the proof is to first prove that $\mathscr{A}$ is a locally free sheaf of finite rank on Y. Then we can compute the rank at any point of Y. Over the generic point, we show the rank is $[k(X) : k(Y)] = \deg f$, while over a closed point Q, we will show the rank is $\sum_{P \in f^{-1}(Q)} e_P$, giving the desired equality.

First let's prove $\mathscr{A}$ is locally free of finite rank. Since f is proper and quasifinite, f is finite, so we will have finite rank if we can show $\mathscr{A}$ is locally free. For this, it is enough to check at closed points $Q \in Y(k)$, since the generic point is a localization of any closed point. First we observe that $\mathscr{A}_Q$ is a flat $\mathcal{O}_{Y,Q}$-module. Flatness is equivalent to the map $I \otimes \mathscr{A}_Q \to \mathscr{A}$ being injective for any ideal I of $\mathcal{O}_{Y,Q}$. Since $\mathcal{O}_{Y,Q}$ is a DVR, it's enough to check that the action of the uniformizer π_Q on $\mathscr{A}_Q$ is nonzero. But over an open U, $\mathscr{A}(U) = \mathcal{O}_X(f^{-1}(U))$, and the action of π_Q is given by mapping π_Q into $\mathcal{O}_X(f^{-1}(U))$ and multiplying. But $\mathcal{O}_X(f^{-1}(U))$ is a domain, so this action is nonzero. Now the stalks of $\mathscr{A}$ (which are stalks of $\mathcal{O}_X$), are finitely generated over the stalks of $\mathcal{O}_Y$ since f is finite. But flat and finitely generated impies free, so $\mathscr{A}$ has free stalks. We conclude that $\mathscr{A}$ is locally free by the following.

Lemma 14.1. *If $\mathcal{F}$ is a coherent sheaf on Y such that $\mathcal{F}_Q$ is a free $\mathcal{O}_Q$-module for all Q, then $\mathcal{F}$ is locally free.*

Proof. At each Q we have an isomorphism $\mathcal{O}_{Y,Q}^r \to \mathcal{F}_Q$. Let $f_1', \ldots, f_r' \in \mathcal{F}_Q$ be the image of the generators under this isomorphism, and $f_1, \ldots, f_r$ their representatives on some neighborhood U containing Q. Then these f_i define a map of sheaves on U (in fact, of $\mathcal{O}_U$-modules) $\sigma_U\colon \mathcal{O}_U^r \to \mathcal{F}\big|_U$ which is an isomorphism at the stalk at Q. Let $\mathcal{K}$ and $\mathcal{G}$ be the kernel and cokernel, respectively, of σ_U.

Since $\mathcal{O}_U^r$ and $\mathcal{F}$ are coherent, so are $\mathcal{K}$ and $\mathcal{G}$. Thus their supports are closed subsets of U not containing Q. In particular, there is an open neighborhood $V \subseteq U$ containing Q such that $\mathcal{K}_P$ and $\mathcal{G}_P$ are both zero for all $P \in V$. Since isomorphisms can be checked at stalks, this means that $\sigma_U\big|_V$ is an isomorphism over V, and hence $\mathcal{F}\big|_V$ is free. $\square$

Returning to the proof, we have now that $\mathscr{A}$ is locally free, and want to compute its rank at both a closed point and the generic point η of Y.

First, letting ξ be the generic point of X, we note that

$$\mathscr{A}_\eta = \lim_{\eta \in U} \mathcal{O}_X(f^{-1}(U)) = \lim_{\xi \in V} \mathcal{O}_X(V) = k(X),$$

the equality in the middle coming from the fact that for any open V of X, we can find $U \subseteq Y$ such that $f^{-1}(U) \subseteq V$ ("preimages of opens in Y are cofinal amongst opens in X"). Namely, if $V = X \setminus \{P_i\}$, set $U = Y \setminus \{f(P_i)\}$. We use this to compute the rank of $\mathscr{A}$ at η: it's $[k(X) : k(Y)]$, i.e., $\deg f$.

Now let Q be a closed point of Y. We will compute the rank of $\mathscr{A}$ at Q. This is by definition the rank of $\mathscr{A}_Q$ as an $\mathcal{O}_{Y,Q}$-module, which is also equal to the dimension of $\mathscr{A}_Q \otimes_{\mathcal{O}_{Y,Q}} k(Q)$ as a $k(Q)$-vector space. First we claim that

$$\mathscr{A}_Q \otimes_{\mathcal{O}_{Y,Q}} k(Q) \cong \Gamma(X \times_Y \operatorname{Spec} k(Q), \mathcal{O}_{X \times \operatorname{Spec} k(Q)})$$

To see this, we can replace X and Y by affines $\operatorname{Spec} R$ and $\operatorname{Spec} S$, respectively, where Q is contained in $\operatorname{Spec} S$ and $f(\operatorname{Spec} R) = \operatorname{Spec} S$ since f is affine. The fiber product in question corresponds to the diagram

$$\begin{tikzcd}
R \otimes_S k(Q) & k(Q) \arrow[l] \\
R \arrow[u] & S \arrow[l] \arrow[u]
\end{tikzcd}$$

which we can expand into a double fibered diagram

$$\begin{tikzcd}
R \otimes_S k(Q) & k(Q) \arrow[l] \\
R \otimes_S \mathcal{O}_{Y,Q} \arrow[u] & \mathcal{O}_{Y,Q} \arrow[l] \arrow[u] \\
R \arrow[u] & S \arrow[l, "f^\#"'] \arrow[u]
\end{tikzcd}$$

Thus the claim follows from the observation that in fact $\mathscr{A}_Q \cong R \otimes_S \mathcal{O}_{Y,Q}$, since

$$R \otimes_S \mathcal{O}_{Y,Q} = \lim_{Q \in D(g)} R \otimes_S S_g = \lim_{Q \in D(g)} R_{f^\# g} = \lim_{Q \in D(g)} \mathcal{O}_X(f^{-1}(D(g))),$$

which is $(f_* \mathcal{O}_X)_Q = \mathscr{A}_Q$.

We must therefore compute the dimension over $k(Q)$ of the global sections of $X \times_Y k(Q)$. Since f is finite, $X \times_Y \operatorname{Spec} k(Q)$ is a disjoint

union of points: $f^{-1}(Q) = \{P_1, \ldots, P_s\}$, and therefore it is an affine scheme:

$$X \times_Y \operatorname{Spec} k(Q) \cong \coprod_{1=1}^{s} \operatorname{Spec} \mathcal{O}_{X,P_i} \otimes_{\mathcal{O}_{Y,Q}} k(Q) \cong \operatorname{Spec} \prod_{i=1}^{s} \mathcal{O}_{X,P_i} \otimes_{\mathcal{O}_{Y,Q}} k(Q).$$

Now, $\mathcal{O}_{X,P_i} \otimes_{\mathcal{O}_{Y,Q}} k(Q) \cong \mathcal{O}_{X,P_i}/\pi_i^{e_i}$, since the $\mathcal{O}_{Y,Q}$-algebra structure on $\mathcal{O}_{X,P_i}$ is given by sending the uniformizer of $\mathcal{O}_{Y,Q}$ to $\pi_i^{e_i}$, where π_i is the uniformizer of $\mathcal{O}_{X,P_i}$, and e_i the ramification index of f at P_i. Thus we conclude that

$$\mathscr{A}_Q \otimes_{\mathcal{O}_{Y,Q}} k(Q) \cong \prod_{i=1}^{s} \mathcal{O}_{X,P_i}/\pi_i^{e_i},$$

which has dimension $\sum_{i=1}^{s} e_i$. This is therefore the rank of $\mathscr{A}$ at Q, and equating it with the rank at the generic point ($\deg f$) gives the result. $\square$

Example 14.1. If X is a curve of genus 1, and P a closed point, then $h^0(X, \mathcal{O}_X(2P)) = 2$, so we get a map $f\colon X \to \mathbb{P}^1$. On the complement of P, we have a map to $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ given by $f^{\#}\colon k[t] \to \Gamma(X \setminus P, \mathcal{O}_X)$. The section $f^{\#}(t)$ extends to a global section of $\mathcal{O}_x(2P)$ with a pole of order 1 or 2 at P. Thus $\deg f \leq 2$. But the degree cannot be one, or else f would be an isomorphism, which is impossible since X has genus 1. So the divisor $2P$ defines a degree two map to $\mathbb{P}^1$.

15 Wednesday, Feb. 22nd: Hyperelliptic Curves

So far in our study of curves, we have seen that genus one curves are isomorphic to $\mathbb{P}^0$ while genus one curves are cut out by cubic equations in $\mathbb{P}^2$. Today we study curves of genus greater than one. We will see that they split into two camps: those that are "canonically embedded" in some $\mathbb{P}^n$, and hyperelliptic curves.

15.1 Canonical Embeddings

For a curve X of genus $g \geq 2$, we have $\deg \Omega^1_X > 0$, so we can ask whether the line bundle Ω^1_X defines a morphism, or better yet, an embedding, into $\mathbb{P}^n$. Since $h^0(X, \Omega^1_X) = g$, we will get a morphism to $\mathbb{P}^{g-1}$; in the case that it's an embedding, we will call this the *canonical embedding*.

To get a morphism at all, we need Ω^1_X to be base point free, which you recall means that there is no point of X at which all global sections of Ω^1_X vanish. Since the dimension $h^0(X, \Omega^1_X(-P))$ is always one less than or equal to $h^0(X, \Omega^1_X)$, it is equivalent to require that that for all $P \in X(k)$, $h^0(X, \Omega^1_X(-P)) < h^0(X, \Omega^1_X)$, i.e., there is some global section not vanishing at P.

If furthermore we ask that this morphism be an embedding, Ω^1_X must separate points and tangent vectors, which we have seen earlier is equivalent to requiring that for all $P, Q \in X(k)$, not necessarily distinct $h^0(X, \Omega^1_X(-P-Q)) < h^0(X, \Omega^1_X(-P))$. Thus we obtain easily the following criterion.

Lemma 15.1. *Ω^1_X defines a closed embedding $X \hookrightarrow \mathbb{P}^{g-1}$ if and only if, for all closed points P, Q of X (not necessarily distinct), $h^0(X, \mathcal{O}^1_X(-P-Q)) = g-2$.*

Proof. This follows from the above discussion: the dimension of $H^0(X, \mathcal{O}^1_X)$ is g. Being base point free is equivalent to requiring that the dimension drop by one when we subtract a point from $\mathcal{O}^1_X$; separating points and tangent vectors happens exactly when the dimension drops one further after subtraction of another point. $\square$

When is the condition of the lemma satisfied? We apply Riemann-Roch to the line bundle $\Omega^1_X(-P-Q)$:

$$h^0(X, \Omega^1_X(-P-Q)) - h^0(X, \mathcal{O}_X(P+Q)) = g-3.$$

So we get a closed embedding exactly when, for all P, Q in X, $h^0(X, \mathcal{O}_X(P+Q)) = 1$. Of course, there is always an inclusion $k \hookrightarrow H^0(X, \mathcal{O}_X(P+Q))$, so we are really asking that there be no nonconstant functions with simple poles at one or both of P and Q, for any choice of P and Q. But the case when there is such a function is also interesting, for such functions give degree two morphisms to $\mathbb{P}^1$. We now turn our attention to this latter situation.

15.2 Hyperelliptic Curves

Definition 15.1. A curve X is **hyperelliptic** if there exists a degree two map $X \to \mathbb{P}^1$.

With this terminology, we summarize our results thus far by saying that any curve of genus $g \geq 2$ can either be (canonically) embedded in $\mathbb{P}^{g-1}$ or else is hyperelliptic. We now ask for a classification of hyperelliptic curves. This will be accomplished by explicitly writing down an equation for such curves.

So suppose $X \to \mathbb{P}^1$ is a degree two map. Then we have a separated degree two, hence Galois, field extension $k(\mathbb{P}^1) \hookrightarrow k(X)$. This gives a $\mathbb{Z}/2$ action on $k(X)$, which lifts to an action of $\mathbb{Z}/2$ on X over $\mathbb{P}^1$. For example, if X were defined by an equation of the form $y^2 = (x-a_1)\cdots(x-a_n)$, then the action would be given by sending y to $-y$.

How can we obtain the $\mathbb{Z}/2$ action on X from that on $k(X)$? In general, since the map $f\colon X \to \mathbb{P}^1$ is affine, $\mathscr{A} = f_*\mathcal{O}_X$ is a locally free sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras of rank two. For an open affine U of $\mathbb{P}^1$, we have maps

$$\begin{tikzcd} k(X) & \mathscr{A}(U) \arrow[l] \\ k(\mathbb{P}^1) \arrow[u] & \mathcal{O}_{\mathbb{P}^1}(U) \arrow[l] \arrow[u] \end{tikzcd}$$

The maps $\mathcal{O}_{\mathbb{P}^1}(U) \hookrightarrow k(\mathbb{P}^1) \hookrightarrow k(X)$ are inclusions, and hence identify $\mathcal{O}_{\mathbb{P}^1}(U)$ with a subring of $k(X)$. We claim that $\mathscr{A}(U)$ is the integral closure of $\mathcal{O}_{\mathbb{P}^1}(U)$ in $k(X)$. Since the map f is finite, $\mathscr{A}(U)$ is a finite $\mathcal{O}_{\mathbb{P}^1}(U)$-module, and hence $\mathscr{A}(U)$ is integral over $\mathcal{O}_{\mathbb{P}^1}(U)$. This shows

that we can recover X over $\mathbb{P}^1$ from the inclusion of fields $k(\mathbb{P}^1) \hookrightarrow k(X)$ (since $X = \underline{\operatorname{Spec}}\,\mathscr{A}$). By the functoriality of this description of $\mathscr{A}$, we get a $\mathbb{Z}/2$-action on $\mathscr{A}$ over $\mathcal{O}_{\mathbb{P}^1}$. This breaks $\mathscr{A}$ into eigenspaces; we write $\mathscr{A} = \mathscr{A}_+ \oplus \mathscr{A}_-$, where the generator σ of $\mathbb{Z}/2$ acts trivially on $\mathscr{A}_+$ and by -1 on $\mathscr{A}_-$. Both "eigensheaves" are locally free of rank one, for this can be checked locally, say at the generic point, where it's true because $k(X)/k(\mathbb{P}^1)$ is Galois.

Also, $\mathcal{O}_{\mathbb{P}^1}$ injects into $\mathscr{A}_+$, and in fact this is an isomorphism (check), so there is some line bundle $\mathcal{L}$ on $\mathbb{P}^1$ (so $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(d)$) such that $\mathscr{A} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}$ as $\mathcal{O}_{\mathbb{P}^1}$-modules. How does the algebra structure of $\mathscr{A}$ look under this isomorphism? We already know how to multiply two elements of $\mathcal{O}_{\mathbb{P}^1}$, or an element of $\mathcal{O}_{\mathbb{P}^1}$ and an element of $\mathcal{L}$, so the algebra structure on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}$ is determined by a map $\rho\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_{\mathbb{P}^1}$.

Pick an open $\operatorname{Spec} k[t] = \mathbb{A}^1 \subset \mathbb{P}^1$; then $\mathcal{O}_{\mathbb{A}^1} \cong \widetilde{k[t]}$. Since $\operatorname{Pic}\mathbb{A}^1$ is trivial, $\mathcal{L}\big|_{\mathbb{A}^1} \cong \mathcal{O}_{\mathbb{A}^1} \cong \widetilde{k[y]}$. The map $\rho\big|_{\mathbb{A}^1}\colon \mathcal{O}_{\mathbb{P}^1}^{\otimes 2} \cong \widetilde{k[y^2]} \to \widetilde{k[t]} \cong \mathcal{O}_{\mathbb{A}^1}$ sends y^2 to an element $g(t) \in k[t]$. Thus

$$\mathscr{A}\big|_{\mathbb{A}^1} \cong \mathcal{O}_{\mathbb{A}^1} \oplus \mathcal{L}\big|_{\mathbb{A}^1} \cong k[t,y]/(y^2 - g(t)),$$

with the $\mathbb{Z}/2$ action given by $y \mapsto -y$. In conclusion, hyperelliptic curves X are given (in an open affine $\mathbb{A}^2 \subset \mathbb{P}^2$) by equations of the form $y^2 = g(t)$. We will see next lecture how to (almost) obtain the degree of $g(t)$ solely from the genus of X.

16 Friday, Feb. 24th: Riemann-Hurwitz Formula

Today we show how to use a morphism $f\colon X \to Y$ of curves to relate the genus g_X of X to the genus g_Y of Y. The key ingredient in this computation is the familiar exact sequence

$$f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0.$$

We will use the following terminology: the morphism f is **separable** if the corresponding field extension $k(Y) \hookrightarrow k(X)$ is separable.

16.1 Preparatory Lemmata

Lemma 16.1. *The map $f^*\Omega^1_{Y/k} \xrightarrow{\alpha} \Omega^1_{X/k}$ is injective if and only if f is separable.*

Proof. Injectivity of α can be checked at stalks. Both X and Y are integral schemes, so the vertical maps in the following diagram are injective:

$$\begin{tikzcd}
\left(f^*\Omega^1_Y\right)_x \arrow[r] \arrow[d] & \Omega^1_{X,x} \arrow[d] \\
\left(f^*\Omega^1_Y\right)_\eta \arrow[r] & \Omega^1_{X,\eta}
\end{tikzcd}$$

where η is the generic point of X. Thus the top arrow is injective if and only if the bottom one is, so it suffices to check injectivity at the generic

point. The inclusions into X and Y of the respective generic points gives a fibered diagram

$$\begin{tikzcd} \operatorname{Spec} k(X) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k(Y) \arrow[r] & Y \end{tikzcd}$$

This implies that $\left(\Omega^1_{X/Y}\right)_\eta \cong \Omega^1_{k(X)/k(Y)}$, and this is zero if and only if f is separable, as we checked on a previous HW assignment.

□

Lemma 16.2. *The diagram*

$$\begin{tikzcd} \operatorname{Pic}(X) \arrow[r, "\deg_X"] & \mathbb{Z} \\ \operatorname{Pic}(Y) \arrow[r, "\deg_Y"] \arrow[u, "f^*"] & \mathbb{Z} \arrow[u, "\cdot \deg f"'] \end{tikzcd}$$

commutes.

Proof. Recall that any line bundle on Y has the form $\mathcal{O}_Y(\sum n_P P)$, and the degree of such a line bundle is just $\sum n_P$. For the proof, it is enough to check it on the generators of $\operatorname{Pic}(Y)$, namely those line bundles of the form $\mathcal{O}_Y(P)$ for $P \in Y$. Since such a line bundle has degree one, we need to show that $\deg f = \deg_X(f^*\mathcal{O}_Y(P))$. Let $\mathcal{I}_P$ be the ideal sheaf of the inclusion $P \hookrightarrow Y$. Then $f^*\mathcal{I}_P$ is an ideal sheaf on X, isomorphic to $\bigotimes_{f(Q)=P} \mathcal{I}_Q^{e_Q}$. Therefore

$$\begin{aligned} \deg_X(f^*\mathcal{O}_Y(P)) = -\deg_X(f^*\mathcal{I}_P) &= -\deg_X\Big(\bigotimes_{f(Q)=P} \mathcal{I}_Q^{e_Q}\Big) \\ &= -\Big(\sum_{f(Q)=P} -e_Q\Big) = \deg f, \end{aligned}$$

where the last equality is from Proposition 14.1.

□

Our final lemma concerns the local structure of the module of differentials.

Lemma 16.3. *Let π_X be a uniformizer of the local ring $\mathcal{O}_{X,P}$. Then $\Omega^1_{X,P} \cong \mathcal{O}_{X,P} \cdot d\pi_X$.*

Proof. $\Omega^1_{X,P}$ is generated by elements of the form $\mathrm{d}f$, for $f \in \mathcal{O}_{X,P}$. But $\mathrm{d}f = \mathrm{d}(f - f(P))$, since $\mathrm{d}(f(P)) = 0$. Since $f - f(P)$ is in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,P}$, we can write it as $g\pi_X$. Then

$$\mathrm{d}f = \mathrm{d}(g\pi_X) = g\,\mathrm{d}\pi_X + \pi_X\,\mathrm{d}g.$$

Thus the image of $\mathrm{d}\pi_X$ generates $\Omega^1_{X,P}/\mathfrak{m}\Omega^1_{X,P}$, and therefore $\mathrm{d}\pi_X$ generates $\Omega^1_{X,P}$ by Nakayama. □

16.2 Riemann-Hurwitz Formula

We now assume that f is separable, so by Lemma 16.1 the sequence

$$0 \to f^*\Omega^1_{Y/k} \xrightarrow{\alpha} \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0.$$

is exact. In particular, α is injective; we want to understand its image. For this we look at the stalk, where the sequence looks like

$$0 \to \Omega^1_{Y,f(P)} \otimes_{\mathcal{O}_{Y,f(P)}} \mathcal{O}_{X,P} \xrightarrow{\alpha_P} \Omega^1_{X,P} \to \Omega^1_{X/Y,P} \to 0.$$

Let π_Y be a uniformizer for $\mathcal{O}_{Y,f(P)}$ and π_X a uniformizer for $\mathcal{O}_{X,P}$. We can write the image of π_Y in $\mathcal{O}_{X,P}$ as $u\pi_X^{e_P}$ for some unit u and integer $e_P \geq 1$. Since $\Omega^1_{Y,f(P)}$ is generated as an $\mathcal{O}_{Y,P}$-module by $\mathrm{d}\pi_Y$, $f^*\Omega^1_{Y,f(P)} \cong \mathrm{d}\pi_Y\mathcal{O}_{X,P}$, so using this isomorphism, α_P sends $\mathrm{d}\pi_Y$ to $\mathrm{d}(u\pi_X^{e_P}) = \pi_X^{e_P}\,\mathrm{d}u + e_p u\pi_X^{e_P-1}\,\mathrm{d}\pi_X$.

If it happens that the characteristic of k divides e_P, the situation is considerably more complicated, and gets a name, but we will not treat this case.

Definition 16.1. The morphism f **has wild ramification at** P if the characteristic of k divides e_P.

So assume that the characteristic of k does not divide e_P. Then following α_P with the quotient map to $\Omega^1_{X,P}/\mathfrak{m}^{e_P}\Omega^1_{X,P}$, the term $\pi_X^{e_P}\,\mathrm{d}u$ in the above vanishes, so the image is generated by $\pi_X^{e_P-1}\,\mathrm{d}\pi_X$. By Nakayama, then, the image of α_P itself is generated by $\pi_X^{e_P-1}\,\mathrm{d}\pi_X$. Thus $f^*\Omega^1_{Y,f(P)} \cong \pi_X^{e_P-1}\Omega^1_{X,P}$. Allowing P to vary, we obtain the main theorem of the lecture.

Theorem 16.1. *If f is separable and has no wild ramification, then*

$$f^*\Omega^1_Y\big(\sum_{P\in X}(e_P-1)\cdot P\big) \cong \Omega^1_X$$

Theorem 16.2 (Riemann-Hurwitz)**.** *If f is separable and has no wild ramification, then*

$$2g_X - 2 = \deg f(2g_Y - 2) + \sum_{P\in X}(e_P - 1)$$

Proof. Take degrees in Theorem 16.1. □

Example 16.1. Let $f\colon X \to \mathbb{P}^1$ be a degree two map, i.e., X be a hyperelliptic curve. By the previous lecture, we can write the equation for X as $y^2 = (x-a_1)\cdots(x-a_r)$, so f is ramified at the points a_k and possibly at ∞. For those points at which f is ramified, its ramification index is two, so using 16.2 we find

$$2g_X - 2 = 2(-2) + r + \epsilon,$$

where ϵ comes from the ramification at ∞: it is 1 if f is ramified there, and 0 if not. By parity, we see that if r is odd, $\epsilon = 1$, so f is ramified at ∞ and $g_X = (r-1)/2$. If r is even, f is unramified at ∞ and $g_X = (r-2)/2$. This makes good on the promise made at the end of the previous lecture.

Example 16.2. As a reality check, consider the case where $r = 3$, i.e., X is an elliptic curve. It's ramified at infinity, and the computation above agrees with our knowledge that $g_X = 1$. Phew!

17 Monday, Feb. 27th: Homological Algebra I

Lecture and typesetting by Piotr Achinger

17.1 Motivation

17.1.1 Algebraic topology

was the main motivation for the concepts of homological algebra. The term *homology* itself comes from Poincaré's first attempts on algebraic topology. Given a nice topological space[7] X and a commutative ring R, one can associate to it the following sequence of maps between R-modules:

$$0 \to C^0(X) \xrightarrow{d} C^1(X) \xrightarrow{d} C^2(X) \xrightarrow{d} \dots$$

where $C^i(X)$ is the dual of the free R-module spanned by the set of all continuous maps $f : [0,1]^n \to X$ („singular cubes in X") and the maps d are dual to the maps taking such „singular cubes" to their „boundaries". It is intuitively clear that *the boundary of the boundary is zero*, i.e., $d \circ d = 0$. We can therefore look at the *cohomology groups*[8] :

$$H^i(X,R) := \frac{\ker(d : C^i(X) \to C^{i+1}(X))}{\operatorname{im}(d : C^{i-1}(X) \to C^i(X))}.$$

These encode important topological information of X, and have many useful properties, to note a few:

(a) $H^0(X,R)$ is a free R-module spanned by the (path-)connected components of X.

(b) The $H^i(X,R)$ are functorial in both X and R – contravariantly in X and covariantly in R (the map $H^i(f)$ induced by $f : X \to Y$ is usually denoted by f^*),

(c) given a nice inclusion[9] $i : Y \hookrightarrow X$, one gets a long cohomology exact sequence

$$\to H^{i-1}(Y,R) \xrightarrow{\delta} H^i(Z,R) \xrightarrow{p^*} H^i(X,R) \xrightarrow{i^*} H^i(Y,R) \xrightarrow{\delta} H^{i+1}(Z,R) \to \dots \tag{10}$$

where $Z = X/Y$ is X with Y contracted to a point, and $p : X \to Z$ is the projection.

17.1.2 Algebraic geometry – a.k.a. the Black Box

Our main goal is to define and study the *sheaf cohomology* functors $H^i(X,\mathscr{F})$, where X is a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. We already know some of their properties, having used them as a „black box" to study algebraic curves:

(a) $H^0(X,\mathscr{F}) = \Gamma(X,\mathscr{F})$,

[7] e.g. a CW-complex

[8] If we didn't take duals here, we would get *homology groups* – but cohomology serves a better motivation for our purposes.

[9] e.g. a cofibration

(b) The $H^0(X, \mathscr{F})$ are functorial in both X and $\mathscr{F}$ – contravariantly in X and covariantly in $\mathscr{F}$ (contravariance in X is to be understood as follows: if $f : X \to Y$ is a continuous map and $\mathscr{G}$ is a sheaf on Y, we get a map $H^i(Y, \mathscr{G}) \to H^i(X, f^{-1}\mathscr{G})$).

(c) Given a short exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ of sheaves on X, one gets a long cohomology exact sequence

$$\to H^{i-1}(X, \mathscr{H}) \xrightarrow{\delta} H^i(X, \mathscr{F}) \to H^i(X, \mathscr{G}) \to H^i(X, \mathscr{H}) \xrightarrow{\delta} H^{i+1}(X, \mathscr{F}) \to \tag{11}$$

Propaganda 17.1. Our goal, motivated by algebraic topology, is to develop a general notion of a ,,cohomology theory", encompassing (17.1.1) and (17.1.2) above[10] and many other examples in a uniform fashion. Our second goal is to develop a theory which would handle the non-exactness of the functor $\Gamma(X, -)$ (and other non-exact functors).

17.2 Complexes, cohomology and snakes

We start with a rather lengthy definition. I hope most readers will be familiar with most of the notions presented here.

Definition 17.2. Let R be a commutative ring.

1. A *complex of R-modules* is a pair $C^\bullet = (C^n, d^n)$ where C^n $(n \in \mathbb{Z})$ are R-modules and $d^n : C^n \to C^{n+1}$ are maps (called *differentials*) satisfying the condition $d^{n+1} \circ d^n = 0$. (In practice, one usually writes d instead of d^n, so our condition can be written succintly as $d^2 = 0$).
2. A *morphism of complexes* $f^\bullet : C^\bullet \to D^\bullet$ is a family of maps $f^n : C^n \to D^n$ commuting with the differentials, i.e., $d^n_D \circ f^n = f^{n+1} \circ d^n_C$. One therefore gets a category $Ch(R - mod)$ of complexes of R-modules.
3. The *cohomology groups* of a complex $C^\bullet$ are the groups $H^i(C^\bullet) := \ker d^n / \operatorname{im} d^{n-1}$. One easily checks that they are covariant functors $H^i : Ch(R - mod) \to R - mod$. Elements of $\ker d^n$ are called *cycles* and denoted by $Z^n(C)$ and elements of $\operatorname{im} d^{n-1}$ are called *boundaries* and denoted by $B^i(C)$. Therefore $H^i(C) = Z^i(C)/B^i(C)$.
4. The kernel, image and cokernel of a morphism of complexes are defined in the usual way. One can then talk about subcomplexes, quotient complexes, exact sequences of complexes, complexes of complexes, cohomology complexes of complexes of complexes and so on.

Propaganda 17.3. Here is one principle of our approach to homological algebra: *to study an object A, associate to it a complex $C^\bullet(A)$*. This complex will depend on the ,,cohomology theory". In other words – to construct a cohomology theory $\mathscr{H}^*$, one first describes a way to attach a complex $C^\bullet(A)$ to A and define $\mathscr{H}^i(A)$ as $H^i(C^\bullet(A))$. Usually $C^\bullet(A)$ is *not* functorial in A (but its cohomology is).

[10] In fact (again for ,,nice" spaces), (1) is a special case of (2): $H^i(X, R) = H^i(X, \mathscr{R})$ where $\mathscr{R}$ is the constant sheaf associated to R.

How do we get long exact sequences? We extend our principle as follows: if an object X fits into a ,,short exact sequence" $Y \hookrightarrow X \twoheadrightarrow Z$, then there should be an exact sequence of complexes $0 \to C^\bullet(Y) \to C^\bullet(X) \to C^\bullet(Z) \to 0$. Then we get a long exact sequence from the Snake Lemma (below). This encompasses both long exact sequences (10) and (11).

Lemma 17.1 (Snake Lemma[11]). *(a) Given a commutative diagram of R-modules with exact rows*

$$\begin{tikzcd} . & C \arrow[r] \arrow[d, "f"] & D \arrow[r] \arrow[d, "g"] & E \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' & . \end{tikzcd}$$

there exists an arrow $\delta : \ker g \to \operatorname{coker} f$ such that the sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

is exact. Here is the whole picture:

$$\begin{tikzcd} & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\ & C \arrow[r] \arrow[d, "f"] & D \arrow[r] \arrow[d, "g"] & E \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & C' \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d] & E' \arrow[d] & \\ & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h & \end{tikzcd}$$

(b) The arrow δ is functorial, that is, given two diagrams as above and a commutative system of arrows between them, the two δ-arrows keep the system commutative.

(c) Given a short exact sequence of complexes $0 \to C^\bullet \xrightarrow{i} D^\bullet \xrightarrow{p} E^\bullet \to 0$, there is a long exact sequence of cohomology

$$\ldots \to H^{i-1}(D^\bullet) \xrightarrow{\delta} H^i(C^\bullet) \xrightarrow{i_*} H^i(D^\bullet) \xrightarrow{p_*} H^i(E^\bullet) \xrightarrow{\delta} H^{i+1}(C^\bullet) \to \ldots$$

(d) *The arrows δ above are functorial, that is, given two short exact sequences as above and a commutative system of arrows between them, one gets a commutative ladder between the two long cohomology exact sequences.*

Proof. You should prove (a) and (b) yourself. For (c), apply (a) to the diagram

$$\begin{tikzcd}
. & C^i/B^i(C) \arrow[r] \arrow[d, "d_C^i"] & D^i/B^i(D) \arrow[r] \arrow[d, "d_D^i"] & E^i/B^i(E) \arrow[r] \arrow[d, "d_E^i"] & 0 \\
0 \arrow[r] & Z^{i+1}(C) \arrow[r] & Z^{i+1}(D) \arrow[r] & Z^{i+1}(E) & .
\end{tikzcd}$$

(it is straightforward to check that the rows are exact) obtaining

$$\begin{tikzcd}
 & H^i(C) \arrow[r] \arrow[d] & H^i(D) \arrow[r] \arrow[d] & H^i(E) \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
 & C^i/B^i(C) \arrow[r] \arrow[d, "d_C^i"] & D^i/B^i(D) \arrow[r] \arrow[d, "d_D^i"] & E^i/B^i(E) \arrow[r] \arrow[d, "d_E^i"] & 0 \\
0 \arrow[r] & Z^{i+1}(C) \arrow[r] \arrow[d] & Z^{i+1}(D) \arrow[r] \arrow[d] & Z^{i+1}(E) \arrow[d] & \\
 & H^{i+1}(C) \arrow[r] & H^{i+1}(D) \arrow[r] & H^{i+1}(E) &
\end{tikzcd}$$

Then (d) follows from (b) and (c). $\square$

17.3 Homotopy

In Section 1, we discussed cohomology groups $H^i(X, R)$ of a topological space X. In fact, these are *homotopy invariants* of X: that two maps $f, g : X \to Y$ are *homotopic* if there exists a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. One proves that homotopic maps induce the same maps on cohomology groups. To prove this, one first constructs a family of maps $s^i : C^i(Y) \to C^{i-1}(X)$ such

that $ds + sd = f^* - g^*$, as in the diagram below.

$$\begin{tikzcd}
\dots \arrow[r] & C^{i-1}(Y) \arrow[r, "d_Y^{i-1}"] \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] & C^i(Y) \arrow[r, "d_Y^i"] \arrow[dl, "s^{i+1}"'] \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] & C^{i+1}(Y) \arrow[r] \arrow[dl, "s^{i+1}"'] \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] & \dots \\
\dots \arrow[r] & C^{i-1}(X) \arrow[r, "d_X^{i-1}"'] & C^i(X) \arrow[r, "d_X^i"'] & C^{i+1}(X) \arrow[r] & \dots
\end{tikzcd}$$

Then, the statement follows formally from the equation $ds+sd = f^*-g^*$[12].

Definition 17.4. Let $C^\bullet$, $D^\bullet$ be complexes and let $f, g : C^\bullet \to D^\bullet$ be two maps of complexes. We say that f and g are *homotopic* (denoted $f \sim g$) if there exists a family of maps $s^i : C^i \to D^{i-1}$ such that

$$f^i - g^i = d_D^{i-1} \circ s^i + s^{i+1} \circ d_C^i.$$

Here is the diagram for your convenience:

$$\begin{tikzcd}
\dots \arrow[r] & C^{i-1} \arrow[r] & C^i \arrow[r, "d"] \arrow[dl, red, "s"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C^{i+1} \arrow[r] \arrow[dl, red, "s"'] & \dots \\
\dots \arrow[r] & D^{i-1} \arrow[r, "d"'] & D^i \arrow[r] & D^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

Lemma 17.2. *If $f \sim g$, then $H^i(f) = H^i(g)$ for all i.*

Proof. Let $z \in Z^i(C)$. We want to prove that there is an $x \in D^{i-1}$ such that $f(z) - g(z) = dx$. But

$$f(z) - g(z) = d(s(z)) + s(d(z)) = d(s(z)) + 0$$

since $dz = 0$ by assumption. Therefore we can take $x = s(z)$. $\square$

Some more vocabulary:

Definition 17.5. The family of maps s^i is called a *homotopy* between f and g. A map which is homotopic to zero is called *nullhomotopic*. We call an $f : C^\bullet \to D^\bullet$ a *homotopy equivalence* if there exists a $g : D^\bullet \to C^\bullet$ such that $f \circ g$ is homotopic to id_D and $g \circ f$ is homotopic to id_C.

Remark. By the Lemma, a homotopy equivalence induces an isomorphism on cohomology groups (such a map is called a *quasi-isomorphism*). The problem with quasi-isomorphisms is that they are not preserved under additive functors, for example a short exact sequence of sheaves $0 \to \mathscr{F}' \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{F}'' \to 0$ can be seen as a quasi-isomorphism as in the diagram below:

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathscr{F}' \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathscr{F} \arrow[r, "\beta"'] & \mathscr{F}'' \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

[12]For the construction of s^i, see e.g. A. Hatcher *Algebraic Topology*, Theorem 2.10

This map of complexes will not however remain a quasi-isomorphism after applying $\Gamma(X, -)$. On the other hand, homotopy equivalences are preserved by additive functors, by the nature of their definition. This means that homotopy equivalence is a much better behaved notion. In fact, they will play a key role in our construction of ,,cohomology theories" (i.e., derived functors).

18 Wednesday, Feb. 29th: Homological Algebra II

Lecture and typesetting by Piotr Achinger
Here is what we know after Lecture 1:

- we know what is a complex (of abelian groups or R-modules), what are its cohomology groups, what it means for it to be exact,
- we know the Snake Lemma and that a short exact sequence of complexes gives us a long cohomology exact sequence,
- we know the notion of homotopy of maps between complexes and that homotopic maps induce the same map on cohomology.

18.1 Additive and abelian categories

A model example of a category in which one can make homological considerations is the category R-Mod of (left) modules over a ring R.

Observation 18.1. In the category R-Mod there exist:

(a) a zero (initial and terminal) object 0.

(b) a structure of an abelian group on the sets of morphisms $\mathrm{Hom}(M, N)$ satisfying the distributivity laws of left and right composition with repect to addition (i.e. for $f : M \to N$ the functions

$$x \mapsto x \circ f : \mathrm{Hom}(N, N') \to \mathrm{Hom}(M, N'),$$

$$x \mapsto f \circ x : \mathrm{Hom}(M', M) \to \mathrm{Hom}(M', N)$$

are group homomorphisms).

(c) direct sums $M \oplus N$ which are simultaneously products (i.e. we have both the projections $p_M : M \oplus N \to M$, $p_N : M \oplus N \to N$ and the inclusions $i_M : M \to M \oplus N$, $i_N : N \to M \oplus N$ satisfying $p_M \circ i_M = id_M$, $p_N \circ i_N = id_N$, $i_M \circ p_M + i_N \circ p_N = id_{M \oplus N}$).

Definition 18.2. A category satisfying (a)–(c) above is called *additive*[13]. An *additive functor* is a functor $F : \mathscr{C} \to \mathscr{D}$ between additive categories taking 0 to 0, direct sums to direct sums and such that the induced maps $\mathrm{Hom}(M, N) \to \mathrm{Hom}(FM, FN)$ are group homomorphisms.

In an additive category, it makes sense to talk about complexes, but not cohomology. Therefore we need additional properties.

[13] One may think that ,,additive" is an extra structure on a category. In fact, whenever a category is additive, the group structure on the sets $\mathrm{Hom}(M, N)$ is uniquely determined. Therefore ,,additive" is a property of a category

Observation 18.3. In the category R-Mod there also exist:

(d) kernels, i.e. equalizers of any morphisms with the zero morphism *(that is, for any $f : M \to N$ there is an object* $\ker f = K$ *together with a map $i_f = i : K \to M$ with the following universal property: given $g : M' \to N$ such that $f \circ g = 0$, there is a unique $\bar{g} : M' \to K$ satisfying $g = i \circ \bar{g}$),*

(e) cokernels, i.e. coequalizers of any morphisms with the zero morphism *(that is, for any $f : M \to N$ there is an object* $\operatorname{coker} f = C$ *together with a map $p_f = p : N \to C$ with the following universal property: given $g : N \to N'$ such that $g \circ f = 0$, there is a unique $\bar{g} : C \to N'$ satisfying $g = \bar{g} \circ p$),*

(f) canonical isomorphims $\ker(p_f) = \operatorname{coker}(i_f)$, that is ,,the cokernel of the kernel is the kernel of the cokernel" *(by the universal property of kernel and cokernel, there is a canonical map* $\ker(p_f) \to \operatorname{coker}(i_f)$ *– we require it to be an isomorphism).*

Definition 18.4. An additive category satisfying additionally (d)–(f) is called *abelian*. In an abelian category, the usual notions of cohomology of a complex or an exact sequence make sense. An additive functor $F : \mathscr{C} \to \mathscr{D}$ between two abelian categories is called *exact* if it takes kernels to kernels and cokernels to cokernels (or equivalently – takes exact sequences to exact sequences).

Observation 18.5. In the category R-Mod we have the *Snake Lemma*, which gives us *long cohomology exact sequences for short exact sequences of complexes.*

At the first glance, it is not obvious how the proof of the Snake Lemma should go in any abelian category – recall that our proof used ,,elements" of the objects in the diagram. However, the Snake Lemma and its conclusions hold in any abelian category, thanks to the following result:

Theorem 18.1 (Freyd-Mitchell, 1964)**.** *Let $\mathscr{A}$ be a* small[14] *abelian category. Then $\mathscr{A}$ is a full abelian subcategory of the category R-*Mod *of modules over a certain ring R. More precisely, there exists a ring R and a fully faithful exact additive functor $F : \mathscr{A} \to S - Mod$.*

Remark. In fact, the Freyd-Mitchell theorem allows us to perform *diagram chasing* in any abelian category – given a diagram (whose objects form a set) we can pass to the smalles abelian subcategory containing the objects of the diagram (which should be small). I will ignore set theoretic issues of this type from now on.

Remark. The category $Ch(\mathscr{A})$ of complexes of objects of an abelian category $\mathscr{A}$ is abelian (we take sums, kernels and cokernels ,,levelwise"). The functors

$$H^i : Ch(\mathscr{A}) \to \mathscr{A}, \quad H^i(A^\bullet) = ker(d^i)/im(d^{i-1})$$

are additive, but obviously not exact.

Our considerations concerning R-Mod and the Freyd-Mitchell theorem show that

[14] A category is small if its objects form a set.

Theorem 18.2. *If $\mathscr{A}$ is an abelian category and $\xi : 0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ is a short exact sequence of complexes in $\mathscr{A}$, then there exists the long cohomology exact sequence*

$$\ldots \to H^{i-1}C^\bullet \xrightarrow{\delta_\xi^{i-1}} H^iA^\bullet \xrightarrow{H^if} H^iB^\bullet \xrightarrow{H^ig} H^iC^\bullet \to \ldots$$

which is natural in the following sense: if $\xi' : 0 \to A'^\bullet \xrightarrow{f} B'^\bullet \xrightarrow{g} C'^\bullet \to 0$ is another such sequence and we are given a morphism of short exact sequences[15] *$\phi : \xi \to \xi'$, then the diagrams*

$$\begin{tikzcd} H^iC^\bullet \arrow[r, "\delta_\xi^i"] \arrow[d, "\phi_*"] & H^{i+1}A^\bullet \arrow[d, "\phi_*"] \\ H^iC'^\bullet \arrow[r, "\delta_{\xi'}^i"] & H^{i+1}A'^\bullet \end{tikzcd}$$

commute.

Examples 18.6. The following additive categories are abelian:

- abelian groups, R-modules,
- sheaves and presheaves of abelian groups on a topological space X, sheaves of $\mathscr{O}_X$-modules on a ringed space $(X, \mathscr{O}_X)$,
- coherent and quasi-coherent $\mathscr{O}_X$-modules on a scheme X,
- complexes in an abelian category $\mathscr{A}$.

and the following are not:

- finitely generated free abelian groups,
- vector bundles (i.e., locally free $\mathscr{O}_X$-modules of finite rank) on a scheme X.

18.2 δ-functors

Propaganda 18.7. Recall that our slogan was ,,replace the objects we study by complexes". One can find manifestations of this idea on every level of homological algebra, the most refined of them being the notion of a *derived category*, developed by J.-L. Verdier in 1960s.

Trying to realize our motto in a naive way, *imagine* that to a given object A of an abelian category $\mathscr{A}$ there is assigned a complex $X^\bullet$ in some other abelian category $\mathscr{B}$. For simplicity (and better analogy with our motivational examples coming from topology) we will assume that $X^i = 0$ for $i < 0$.

With this picture there should be an associated *cohomology theory*, that is, a sequence of functors $H^i(A) = H^i(X^\bullet)$ and for every short exact sequence $0 \to A \to B \to C \to 0$ a sequence of arrows $\delta^i : H^i(C) \to H^{i+1}(A)$ (as if $0 \to A \to B \to C \to 0$ gave a short exact sequence of complexes), giving a long cohomology exact sequence.

The notion realizing the above picture is called a *δ-functor*.

[15] This is simply a commutative ,,ladder" between ξ and ξ'.

Definition 18.8. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. By a (covariant, cohomological) *δ-functor* $T^\bullet$ from $\mathscr{A}$ to $\mathscr{B}$ we mean the following data

- a sequence of additive functors $T^i : \mathscr{A} \to \mathscr{B}$ $(i \geq 0)$,
- a sequence of morphisms $\delta^i_\xi : T^iC \to T^{i+1}A$ $(i \geq 0)$ for any short exact sequence $\xi : 0 \to A \to B \to C \to 0$ in $\mathscr{A}$,

satisfying the following conditions

1. for any exact sequence $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathscr{A}$ the sequence

$$0 \to T^0A \xrightarrow{T^0f} T^0B \xrightarrow{T^0g} T^0C \xrightarrow{\delta^0_\xi} \ldots \to T^iA \xrightarrow{T^if} T^iB \xrightarrow{T^ig} T^iC \xrightarrow{\delta^i_\xi} T^{i+1}A \to \ldots$$

   is exact in $\mathscr{B}$,
2. for any two exact sequences $\xi : 0 \to A \to B \to C \to 0$ and $\xi' : 0 \to A' \to B' \to C' \to 0$ and a morphism $\phi : \xi \to \xi'$ the diagram

$$\begin{tikzcd} T^iC \arrow[r, "\delta^i_\xi"] \arrow[d, "\phi_*"] & T^{i+1}A \arrow[d, "\phi_*"] \\ T^iC' \arrow[r, "\delta^i_{\xi'}"] & T^{i+1}A' \end{tikzcd}$$

   commutes

Remark. There also exist notions of a contravariant δ-functor and of a homological δ-functor.

Propaganda 18.9. What we would also like to have (this condition may not sound very well motivated to a person without background from say algebraic topology) is for this ,,cohomology theory" to be ,,determined by its coefficients" – where by ,,coefficients" we mean the functor T^0. Indeed, the cohomology functors $H^i(X, R)$ satisfy

1. $H^0(X, R) = R$ for X connected,
2. if $H^i(X)$ is an *ordinary cohomology theory*, then $H^0(pt)$ determines the whole theory (at least for CW-complexes),
3. a morphism $R \to R'$ of coefficient rings extends uniquely to a ,,morphism of theories", i.e., a system of natural transformations $H^i(-, R) \to H^i(-, R')$, compatible with the connecting homomorphisms δ in the long exact sequences.

Another motivation comes from our Black Box – we want *universal* functors H^i satisfying the properties we need.

To understand, what 3. above should mean in the context of δ-functors, we should introduce the notion of a morphism of δ-functors (,,natural transformation of cohomology theories" – in fact the motivating example for Eilenberg and MacLane's definition of a natural transformation of functors).

Definition 18.10. Let $T^\bullet$, $T'^\bullet$ be two δ-functors from an abelian category $\mathscr{A}$ to an abelian category $\mathscr{B}$. A *morphism of δ-functors* $\tau : T^\bullet \to T'^\bullet$ is a sequence of natural transformations $\tau^i : T^i \to T'^i$ which commute with δ^i, δ'^i, that is, for every short exact sequence $\xi : 0 \to A \to B \to C \to 0$ the diagram

$$\begin{tikzcd} T^iC \arrow[r, "\delta^i_\xi"] \arrow[d, "\tau^i_C"] & T^{i+1}A \arrow[d, "\tau^{i+1}_A"] \\ T'^iC \arrow[r, "\delta'^i_\xi"] & T'^{i+1}A \end{tikzcd}$$

commutes.

We can now say what it means that a δ-functor $T^\bullet$ is determined by its coefficients T^0.

Definition 18.11. Let $T^\bullet$ be a δ-functor from an abelian category $\mathscr{A}$ to an abelian category $\mathscr{B}$. We say that $T^\bullet$ is *universal* if for every other δ-functor $S^\bullet$ from $\mathscr{A}$ to $\mathscr{B}$ and a natural transformation $\sigma : T^0 \to S^0$ there is a unique morphism of δ-functors $\tau : T^\bullet \to S^\bullet$ with $\tau^0 = \sigma$.

As usual, the universal property implies that whenever a universal δ-functor with a fixed T^0 exists, it is unique.

18.3 Left- and right-exact functors

What additive functors $F : \mathscr{A} \to \mathscr{B}$ does there exist a (not necessarily universal) δ-functor $T^\bullet$ with $T^0 = F$? From the ,,long exact sequence" property it follows that for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence

$$0 \to FA \to FB \to FC$$

(with no 0 on the right!) is exact.

Definition 18.12. Let $\mathscr{A}$, $\mathscr{B}$ be abelian categories and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. We call F *left-exact* if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, the sequence

$$0 \to FA \to FB \to FC$$

is exact; F is *right-exact* if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, the sequence

$$FA \to FB \to FC \to 0$$

is exact[16].

Remark. There also is a notion of a contravariant left- or right-exact functor. We just consider such a functor as a covariant functor $\mathscr{A}^{op} \to \mathscr{B}$.

[16] That is, F is left/right exact iff it preserves kernels/cokernels – or equalizers/coequalizers – or (because F is additive) finite limits/colimits.

Examples 18.13. For any object $A \in \mathscr{A}$, the functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, A)$ are left-exact. If $\mathscr{A} \in R\text{-}\mathrm{Mod}$, then $A \otimes_R -$ is right-exact. If $f : X \to Y$ is a map of topological spaces, then $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is left-exact and $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is right exact (where Sh are the categories of sheaves of abelian groups) and similarly for f_* and f^* for sheaves of modules if X and Y are ringed spaces. In particular, $\Gamma : \mathrm{Sh}(X) \to \mathscr{A}b$ is left-exact. More generally, if $F : \mathscr{A} \to \mathscr{B}$ is left adjoint to $G : \mathscr{B} \to \mathscr{A}$ then F is right exact and G is left exact (Problem 4).

A few observations are in order. First, a functor is exact if and only if it is both left- and right-exact. Secondly, a functor extends to a δ-functor only if it is left-exact. Finally, if $F : \mathscr{A} \to \mathscr{B}$ is exact, then $T^0 = F$, $T^i = 0$ for $i > 0$ defines a *universal* δ-functor. Therefore F can give an interesting universal δ-functor only if it is left-, but not right-exact.

Propaganda 18.14. We now change our paradigm a little: we consider (universal) δ-functors as a tool to study not (or not only) objects, but the non-exactness of left-exact functors. That is, we focus on the second goal mentioned in the first lecture: *to develop a theory which would handle the non-exactness of the functor* $\Gamma(X, -)$ *(and other non-exact functors).*

Our main goal is now the construction of a universal δ-functor extending a given left-exact functor (the construction for right-exact functors is analogous) – we shall call them (left) *derived functors.*

18.4 Universality of cohomology

To make sure that our approach may be right and imagine how the general construction should work, we solve the following ,,toy" problem: since the main prototype of the notion of a δ-functor are the cohomology functors $H^i : Ch_{\geq}(\mathscr{A}) \to \mathscr{A}$ ($Ch_{\geq}(\mathscr{A})$ is the category of complexes concentrated in degrees ≥ 0), do they form a universal δ-functor?

Proposition 18.1. *The functors $H^i : Ch_{\geq}(\mathscr{A}) \to \mathscr{A}$ together with the morphisms δ from Theorem 2 form a universal δ-functor.*

Proof (sketch). Let there be given a δ-functor $S^\bullet$ from $Ch_{\geq}(\mathscr{A})$ to $\mathscr{A}$ and a natural transformation $\tau^0 : H^0 \to S^0$. We shall construct the $\tau^i : H^i \to S^i$ inductively.

Suppose that τ_{i-1} has already been constructed. From the Lemma below it follows that for any complex $A^\bullet$ there is an injection $A^\bullet \to I^\bullet$ into an *exact* complex $I^\bullet$. This gives us a short exact sequence of complexes $0 \to A^\bullet \to I^\bullet \to B^\bullet \to 0$. Looking at the long cohomology exact sequences gives us the diagram

$$\begin{tikzcd}
H^{i-1}A^\bullet \arrow[r] \arrow[d, "\tau^{i-1}"] & H^{i-1}I^\bullet \arrow[r] \arrow[d, "\tau^{i-1}"] & H^{i-1}B^\bullet \arrow[r] \arrow[d, "\tau^{i-1}"] & H^iA^\bullet \arrow[r] \arrow[d, dotted, "\tau^i"] & H^iI^\bullet = 0 \\
S^{i-1}A^\bullet \arrow[r] & S^{i-1}I^\bullet \arrow[r] & S^{i-1}B^\bullet \arrow[r] & S^iA^\bullet \arrow[r] & S^iI^\bullet
\end{tikzcd}$$

By some easy diagram chasing we convince ourselves that this determines $\tau^i : H^iA \to S^iA$. We then have to check that τ does not depend on the choice of $A^\bullet \to I^\bullet$, that it is a natural transformation etc. $\square$

Lemma 18.1 (Problem 3)**.** *For any $A^\bullet \in Ch(\mathscr{A})$ there is an injection $A^\bullet \to I^\bullet$ where $I^\bullet \in Ch(\mathscr{A})$ and $H^i(I^\bullet) = 0$ for all i.*

Remark for algebraic topologists. We can look at the above proof in the following way: we embed a given space A into the cone $I = CA$, which is contractible. Then $B = I/A = \Sigma A$ is the suspension of A. By the suspension axiom we get $H^i(A) = H^{i-1}(B)$ for any cohomology theory $H^\bullet$. But on H^{i-1} everything is already defined, hence the inductive step.

An important observation here is that the proof of Proposition 18.1 essentially shows the following

Lemma 18.2 (Effaceable δ-functors are universal. See Problem 5)**.** *Let $T^\bullet : \mathscr{A} \to \mathscr{B}$ be a δ-functor which is* effaceable*, that is, for every $A \in \mathscr{A}$ there exists an injection $A \hookrightarrow J$ into a $T^\bullet$-acyclic object J (J is called acyclic for $T^\bullet$ if $T^i(J) = 0$ for $i > 0$). Then $T^\bullet$ is universal.*

19 Friday, March 2nd: Homological Algebra III

Lecture and Typesetting by Piotr Achinger
Here is what we know after Lecture 2:

- what is an additive or abelian category, what is an additive or exact functor,
- what is a δ-functor and what it means for it to be universal,
- what is a left- or right-exact functor,
- that an effaceable[17] δ-functor is universal.

Our goal is to extend a given left-exact functor to a universal δ-functor.

19.1 Derived functors – a construction proposal

Let us think about what was crucial in our proof of ,,universality of cohomology". The key idea was that any object A could be embedded into an object I on which the given functor was ,,exact" (that is, the ,,coordinates" of the δ functor for $i > 0$ vanished on I).

Lemma 19.1 (Effaceable δ-functors are universal)**.** *Let $T^\bullet : \mathscr{A} \to \mathscr{B}$ be a δ-functor which is* effaceable*, that is, for every $A \in \mathscr{A}$ there exists an injection $A \hookrightarrow J$ into a $T^\bullet$-acyclic object J (J is called acyclic for $T^\bullet$ if $T^i(J) = 0$ for $i > 0$). Then $T^\bullet$ is universal.*

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and let $F : \mathscr{A} \to \mathscr{B}$ be a left-exact functor. We are looking for a universal δ-functor $T^\bullet$ with $T^0 = F$. Motivated by the above Lemma, we want to find a good class $\mathscr{I}$ of objects of $\mathscr{A}$, such that

0. every object $A \in \mathscr{A}$ injects into an object $I(A) \in \mathscr{I}$,
1. objects of $\mathscr{I}$ are acyclic for F.

[17] A δ-functor $T^\bullet$ is *effaceable* if any object injects into an object on which $T^1, T^2, \ldots$ vanish.

It is not clear what it should mean for an $I \in \mathscr{I}$ to be acyclic, since we don't have a δ-functor yet, but only its zero component $T^0 = F$. But in any case, the following should hold: whenever $0 \to I \to M \to N \to 0$ is a short exact sequence, $0 \to FI \to FM \to FN \to 0$ is also exact (because the next term after FN should be $T^1 I = 0$). This of course holds when $0 \to I \to M \to N \to 0$ *splits*, since a split short exact sequence stays split (therefore exact) after applying F. Let us be greedy here and ask for the class of objects I for which all such sequences split.

Definition 19.1. An object $I \in \mathscr{A}$ is called *injective* if any short exact sequence $0 \to I \to M \to N \to 0$ splits. An object $P \in \mathscr{A}$ is called *projective* if any short exact sequence $0 \to M \to N \to P \to 0$ splits (that is, if P is injective in $\mathscr{A}^{op}$).

Problem 6. What are the injective and projective objects in the category of abelian groups?

The class of projective objects plays the same role as injective objects when one wants to study right-exact functors (or contravariant left-exact functors). Then one has to consider surjections $P \twoheadrightarrow A$ and the whole program carries over.

With these assumptions in place, set $\mathscr{I}$ to be the class of injective objects in $\mathscr{A}$ and let us try to construct T^i using the principle of ,,induction by dimension shifting" as in the proof of Lemma 19.1. Given $A \in \mathscr{A}$, find $A \hookrightarrow I$, with $I \in \mathscr{I}$ and let B be the cokernel:

$$0 \to A \xrightarrow{\alpha} I \xrightarrow{\beta} B \to 0.$$

After applying F, we get

$$0 \to FA \xrightarrow{F\alpha} FI \xrightarrow{F\beta} FB$$

which we want to continue to the right

$$0 \to FA \xrightarrow{F\alpha} FI \xrightarrow{F\beta} FB \to T^1 A \to T^1 I \to \dots,$$

and since we *wish* that $T^i I = 0$ for $i > 0$, we should put $T^1 A = \operatorname{coker} F\beta$ and $T^{i+1} A = T^i B$ for $i \geq 1$.

Thus to compute say $T^2 A$, we should find an short exact sequence $0 \to B \to J \xrightarrow{\gamma} C \to 0$ and compute $\operatorname{coker} F\gamma$. This seems already cumbersome, but we can make a long exact complex $0 \to A \to I \to J \to \dots$ (an ,,injective resolution" of A) by ,,splicing" all the short exact sequences.

Construction. Given $A \in \mathscr{A}$, construct an injective resolution $A \to$

$I^\bullet$ of I as follows

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & \\
 & & & & B^2 \arrow[ur] \arrow[dr, "i_{B^2}"] & & & \\
0 \arrow[r] & A \arrow[r, "i_A"] & I^0 \arrow[rr] \arrow[dr] & & I^1 \arrow[ur] \arrow[rr] & & I^2 \arrow[rr] \arrow[dr] & & \dots \\
 & & & B^1 \arrow[ur, "i_{B^1}"] \arrow[dr] & & & & B^3 \arrow[ur, "i_{B^3}"] \arrow[dr] & \\
 & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

(where $I^0 = I(A)$, $B^1 = \operatorname{coker} i_A$, $I^i = I(B^i)$ for $i > 0$, $B^{i+1} = \operatorname{coker}(B^i \to I^i)$). Then apply F to $I^\bullet$:

$$F(I^\bullet) : F(I^0) \to F(I^1) \to F(I^2) \to \dots$$

and define

$$T^i A = H^i(F(I^\bullet)).$$

By left-exactness of F, since $0 \to A \to I^0 \to I^1$ is exact, $0 \to FA \to FI^0 \to FI^1$ is exact, hence $T^0 A = FA$. Obviously this agrees with our previous approach.

Remark. It is convenient to view $0 \to A \to I^\bullet \to \dots$ as a quasi-isomorphism $A \to I^\bullet$ (where A is identified with the complex $\dots \to 0 \to A \to 0 \to \dots$). This means that in the derived category of $\mathscr{A}$ (the category of complexes where quasi-isomorphisms are made to be actual isomorphisms), A and $I^\bullet$ are the same.

There are several obvious problems here:

2. Why should $T^i A$ be well-defined (i.e., independent of the choice of $A \to I^\bullet$)?
3. Why should it be functorial in A?
4. And how to define the connecting homomorphisms δ to get a δ-functor?

Suppose that (2)-(4) were resolved. Then (1) follows, since we can choose $0 \to I \to I \to 0$ to be the injective resolution of I, which gives us $T^i I = 0$ for $i > 0$. If in additions (0) would hold, then by Lemma 19.1 we have constructed a universal δ-functor.

19.1.1 Derived functors – resolution of technical issues

Regarding (2) – $T^i A$ are well-defined

Note that T^i constructed as above will not depend on the choice of $0 \to A \to I^\bullet$ if any two such resolutions will be homotopic (since homotopic maps

1. remain homotopic after applying any additive functor,

2. induce the same map on cohomology).

What we need is the following

Theorem 19.1. *1. for any two resolutions $0 \to A \to I^\bullet$ amd $0 \to A \to J^\bullet$ there exist ϕ, ψ – morphisms between $0 \to A \to I^\bullet$ i $0 \to A \to J^\bullet$ (i.e. $d\phi = \phi d$ oraz $d\psi = \psi d$) giving the identity on A:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] \arrow[d, bend left, "\phi"] & I^1 \arrow[r] \arrow[d, bend left, "\phi"] & I^2 \arrow[r] \arrow[d, bend left, "\phi"] & \dots \\
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] \arrow[u, bend left, "\psi"] & J^1 \arrow[r] \arrow[u, bend left, "\psi"] & J^2 \arrow[r] \arrow[u, bend left, "\psi"] & \dots
\end{tikzcd}$$

2. any *two such ϕ, ψ be the homotopy inverses of each other – i.e. there exist $s^i : I^i \to I^{i-1}$ and $t^i : J^i \to J^{i-1}$ ($i \geq 0$, we put $I^{-1} = A = J^{-1}$) such that $\psi\phi - id = ds + sd$ and $\phi\psi - id = dt + td$:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] \arrow[d, "\psi\phi - id"] \arrow[dl, "s"'] & I^1 \arrow[r] \arrow[d, "\psi\phi - id"] \arrow[dl, "s"'] & I^2 \arrow[r] \arrow[d, "\psi\phi - id"] \arrow[dl, "s"'] & \dots \arrow[dl, "s"'] \\
0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \dots
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & J^0 \arrow[r] \arrow[d, "\phi\psi - id"] \arrow[dl, "t"'] & J^1 \arrow[r] \arrow[d, "\phi\psi - id"] \arrow[dl, "t"'] & J^2 \arrow[r] \arrow[d, "\phi\psi - id"] \arrow[dl, "t"'] & \dots \arrow[dl, "t"'] \\
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \dots
\end{tikzcd}$$

Let us try to construct ϕ inductively – the first step looks as follows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] \arrow[dr, dotted] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \dots \\
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \dots
\end{tikzcd}$$

We would like to extend the dotted arrow $A \to J^0$ to I^0. This follows from

Problem 7. An object $I \in \mathscr{A}$ is injective iff any morphism $A \to I$ defined on $A \subseteq B$ can be extended to B (or, equivalently, iff the functor $\mathrm{Hom}(-, I)$ is exact).

Regarding (3) – Functoriality

In fact, a more general version of Theorem 19.1 holds:

Theorem 19.2. *Let $f : A' \to A$ be a map in $\mathscr{A}$ and let $A \to I^\bullet$, $A' \to I'^\bullet$ be resolutions of A and A', respectively, with I^i injective (the objects I'^i do not have to be injective). Then there exists a chain map $\bar{f} : I^\bullet \to I'^\bullet$, inducing f on H^0, which is unique up to homotopy.*

Problem 8. Prove Theorem 19.2 and deduce Theorem 19.1.

Thus, the exercise above resolves both our issues with (2) and (3).

Regarding (4) – Construction of δ^i

This is called the Horseshoe Lemma:

Lemma 19.2. *Given a short exact sequence* $0 \to A \to B \to C \to 0$ *and already chosen injective resolutions* $A \to I_A^\bullet$ *and* $C \to I_C^\bullet$ *one can find an injective resolution* $B \to I_B^\bullet$ *so that one obtains a short exact sequence of complexes* $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ *which is* $0 \to A \to B \to C \to 0$ *on* H^0.

Problem 9. Prove the above statement. Hint: set $I_B^0 = I_A^0 \oplus I_C^0$ and construct an arrow $B \to I_B^0$. Use the Snake Lemma, obtaining a short exact sequence $0 \to I_A^0/A \to I_B^0/B \to I_C^0/C \to 0$. Proceed by induction.

Thus, given a short exact sequence $0 \to A \to B \to C \to 0$, we obtain the morphisms $\delta^i : T^iC \to T^{i+1}A$ from the long cohomology exact sequence associated to the short exact (why?) sequence of complexes $0 \to F(I_A^\bullet) \to F(I_B^\bullet) \to F(I_C^\bullet) \to 0$ where $I_A^\bullet, I_B^\bullet, I_C^\bullet$ are as in the Horseshoe Lemma.

We need to check that the morphisms δ are well defined and natural, which is also left to the reader.

Regarding (0) – Sufficiently many injective objects

The only problem may happen with **(0)**. For example, if $\mathscr{A}$ is the category of all finitely generated abelian groups, then $\mathscr{A}$ has no nonzero injective objects!

Definition 19.2. An abelian category $\mathscr{A}$ is said to have *sufficiently many injective objects* if for any object there exists an injection into an injective one. We say that $\mathscr{A}$ has *sufficiently many projective objects* if for any object there is a surjection from a projective one.

Theorem 19.3. *For any ring* R*, the category* R-Mod *has sufficiently many injective and projective objects.*

19.2 Derived functors – conclusion

Theorem 19.4. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be abelian categories and let* $T^0 : \mathscr{A} \to \mathscr{B}$ *be a left-exact functor. If* $\mathscr{A}$ *has sufficiently many injective objects, then* T^0 *extends to a universal* δ*-functor* $T^\bullet$ *with* $T^0 = F$.

Definition 19.3. We denote the functors T^i by R^iT^0 and call the *right derived functors of* T^0.

A similar Theorem holds for right-exact functors and projective objects (then one has to use homological δ-functor). The left-derived functors of a right-exact functor T_0 are denoted by L_iT_0.

Examples 19.4. Let $A \in \mathscr{A}$, then $\mathrm{Ext}^i(A, -) := R^i \operatorname{Hom}(A, -)$ and $\mathrm{Ext}^i(-, A) = R^i \operatorname{Hom}(-, A)$ (these two coincide, so one can compute $\mathrm{Ext}^i(A, B)$ from an injective resolution of B or a projective resolution of A). For $A \in R - Mod$, $\to r_i(A, B) := L_i(A \otimes_R B)$ (here one proves

that $\to r_i(A,B) \simeq \to r_i(B,A)$). For a topological space X, $H^i(X,-) := R^i\Gamma : \mathrm{Sh}(X) \to \mathscr{A}b$ are called the *sheaf cohomology* functors (one checks first that the category of abelian sheaves on X has sufficiently many injective objects). For a continuous map $f : X \to Y$, the derived functors $R^i f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ are called the *higher direct image functors*. For a group G, the functor of G-invariants $(-)^G : G - Mod \to \mathscr{A}b$ is left-exact and we call its derived functors $H^i(G,-) = R^i(-)^G$ the *group cohomology* functors. One checks easily that $M^G = \mathrm{Hom}(\mathbb{Z}, G)$ where $\mathbb{Z}$ has the trivial $\mathbb{Z}$-action. Therefore $H^i(G,M) = \mathrm{Ext}^i(\mathbb{Z}, M)$ can be computed from a projective resolution of $\mathbb{Z}$, which is useful since we do not have to pick a new resolution whenever M changes, and also because projective objects are generally easier to work with than injective objects.

19.3 Two useful lemmas

Lemma 19.3 (Acyclic resolutions suffice)**.** *Let $F : \mathscr{A} \to \mathscr{B}$ be left-exact, where $\mathscr{A}$ has sufficiently many injective objects. Let $A \in \mathscr{A}$ and let $0 \to A \to J^0 \to J^1 \to \dots$ be an exact sequence with all J^n acyclic for F (that is, $R^iF(J^n) = 0$ for $i > 0$ and all $n \geq 0$). Then there are natural isomorphisms $H^i(F(I^\bullet)) \simeq R^iF(A)$.*

Lemma 19.4. *Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be abelian categories, $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be left-exact functors. Suppose that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives (so that the derived functors R^iF, R^iG and $R^i(G \circ F)$ exist) and that F maps injective objects of $\mathscr{A}$ to G-acyclic objects of $\mathscr{B}$ (that is, if $I \in \mathscr{A}$ is injective, then $(R^iG)(F(A)) = 0$ for $i > 0$). Let $A \in \mathscr{A}$. Then*

(a) if $(R^iF)(A) = 0$ for $i > 0$, then $R^i(G \circ F)(A) \simeq (R^iG)(F(A))$,

(b) if $(R^iG)(F(A)) = 0$ for $i > 0$, then $R^i(G \circ F)(A) \simeq G(R^iF(A))$.

In general, under the assumptions of the lemma, there is a *spectral sequence*

$$E_2^{pq} = (R^pG)(R^qF(A)) \quad \Rightarrow \quad R^{p+q}(G \circ F)(A).$$

20 Monday, March 5th: Categories of Sheaves Have Enough Injectives

We have seen last week how to construct the right derived functors of an additive left-exact functor, so long as the categories we are interested in have enough injectives. Today we verify that this is the case for the categories we are interested in, namely sheaves of $\mathcal{O}_X$-modules $\mathrm{Mod}_{\mathcal{O}_X}$, or perhaps just sheaves of abelian groups $\mathrm{Sh}(X)$. The functor whose right derived functors we construct is f_*, where $f\colon X \to Y$ is a morphism of schemes. We will be especially interested in the case where $Y = \mathrm{Spec}\, k$; in this case f_* is the just the global sections functor $\Gamma(X,-)\colon \mathrm{Sh}(X) \to \mathrm{Ab}$.

Remark. In some sense, it won't really matter what categories between which we regard f_* as a functor. In fact this is reflected in the notation: R^if_* does not specify $\mathrm{Sh}(X)$ or $\mathrm{Mod}_{\mathcal{O}_X}$, etc. Of course, some care needs to be taken. For instance, we will not consider categories of coherent sheaves, because over $\mathrm{Spec}\,\mathbb{Z}$, there aren't any (except for the zero sheaf):

the reason is that injective $\mathbb{Z}$-modules are things like $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$, which are not finitely-generated.

The goal of today's lecture is to establish the following, which holds even for ringed spaces with noncommutative rings:

Theorem 20.1. *If $(X, \mathcal{O}_X)$ is a ringed space, then $\mathrm{Mod}_{\mathcal{O}_X}$ has enough injectives.*

Remark. Note that by taking $\mathcal{O}_X = \underline{\mathbb{Z}}$, we obtain as a corollary that $\mathrm{Sh}(X)$ has enough injectives.

Proof. We first show that categories of modules over a ring have enough injectives, and then globalize. The following simple lemma is at the heart of both steps of the proof.

Lemma 20.1. *If $F\colon \mathscr{A} \to \mathscr{B}$ is an additive functor which has an exact*[18] *left adjoint $L\colon \mathscr{B} \to \mathscr{A}$, then F takes injectives to injectives.*

Proof. Let $I \in \mathscr{A}$ be an injective object. To show that $F(I)$ is an injective object of $\mathscr{B}$ we must show there exists a dotted arrow in the following commutative diagram in $\mathscr{B}$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl, dotted] \\ & FI & \end{tikzcd}$$

By the adjunction, this is equivalent to filling in the dotted arrow in

$$\begin{tikzcd} 0 \arrow[r] & LM \arrow[r] \arrow[d] & LN \arrow[dl, dotted] \\ & I & \end{tikzcd}$$

which can be done because I is injective. Note that the map $LM \to LN$ is still injective because L is left exact. $\square$

Now let R be a ring (commutative for simplicity). Then Mod_R has enough injectives. This follows easily from the following lemma.

Lemma 20.2. *Let R be a ring. Then the forgetful functor $L\colon \mathrm{Mod}_R \to \mathrm{Ab}$ has a right adjoint $F\colon \mathrm{Ab} \to \mathrm{Mod}_R$, and if $M \in \mathrm{Mod}_R$, and $LM \hookrightarrow I$ is a monomorphism in Ab, then the composite $M \to FL(M) \to FI$ is a monomorphism.*

Proof. We define, for A any abelian group, $FA = \mathrm{Hom}_{\mathrm{Ab}}(R, A)$ which has an R-module structure given by $(r \cdot f)(x) = f(rx)$. To see this gives an adjunction we need to establish a natural bijection, for $M \in \mathrm{Mod}_R$, $A \in \mathrm{Ab}$,

$$\mathrm{Hom}_{\mathrm{Ab}}(LM, A) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathrm{Ab}}(R, A))$$

[18] We actually only use left-exactness of L, but right exactness comes for free because left-adjoints preserve colimits

The map going to the right sends $f\colon M \to A$ to the map $m \mapsto (r \mapsto f(rm))$. The map going to the left sends $\phi\colon M \to \mathrm{Hom}_{\mathrm{Ab}}(R, A)$ to the map $m \mapsto \phi_m(1)$, where ϕ_m is the image of m under ϕ. We omit the verification that these constructions are inverse and functorial in M and A.

Now we must check that given an inclusion $LM \hookrightarrow I$, the map $M \to FL(M) \to FI$ is also an inclusion. But this map is exactly the map coming from the adjunction $\mathrm{Hom}_{\mathrm{Ab}}(LM, I) \simeq \mathrm{Hom}_R(M, FI)$. We need to check that given ϕ such that

$$\begin{tikzcd} K \arrow[r, "\phi"'] \arrow[rr, bend left, "0"] & M \arrow[r] & FL(M) \end{tikzcd}$$

then actually ϕ is zero. Applying L and using the adjunction gives

$$\begin{tikzcd} LK \arrow[r, "L\phi"'] \arrow[rr, bend left, "0"] & LM \arrow[r, "\mathrm{Id}"'] & LM \end{tikzcd}$$

Thus $L\phi = 0$, hence $\phi = 0$, since L is faithful ($L\phi$ is the same map as ϕ, it just ignores the R-module structure). This finishes the proof. $\square$

To complete the proof of the theorem, we now consider $\mathcal{O}_X$-modules. Fix $x \in X$, i.e., a morphism $j\colon x \hookrightarrow X$, and for ease of notation, let $\mathscr{A} = \mathrm{Mod}_{\mathcal{O}_X}$, the category of $\mathcal{O}_X$-modules, and $\mathscr{B} = \mathrm{Mod}_{\mathcal{O}_{X,x}}$, the category of modules over the local ring $\mathcal{O}_{X,x}$ at x. Then we have an adjoint pair of functors

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "j^{-1}"] & \mathscr{B} \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Now j_* is a right adjoint, so if $I \in \mathscr{B}$ is injective, then so is j_*I, by the lemma[19].

For the remainder of the discussion, fix $\mathcal{F} \in \mathscr{A}$, and since $\mathscr{B}$ has enough injectives, pick for each x an inclusion $\mathcal{F}_x \to I^{(x)}$ for some injective $\mathcal{O}_{X,x}$-module $I^{(x)}$. Then we have a map

$$\mathcal{F} \xrightarrow{\alpha} \prod_{x \in X} j_{x*} I^{(x)},$$

where each map j_x is the inclusion of the point x. A product of injectives is still injective, coarsely speaking, because both products and injectives are defined by maps in. Moreover, α is a monomorphism. We check this at the stalk at an arbitrary point $z \in X$:

$$\mathcal{F}_z \xrightarrow{\alpha_z} \Big(\prod_{x \in X} j_{x*} I^{(x)} \Big)_z \to j_{z*} I^{(z)} \cong I^{(z)},$$

[19] Note that j^{-1} is exact: in general, for $f\colon X \to Y$, and $\mathcal{F}$ a sheaf on Y, the stalk of $f^{-1}\mathcal{F}$ at $x \in X$ is the same as the stalk of $\mathcal{F}$ at $f(x) \in Y$.

where the isomorphism on the right is because j_{z*} and j_z^{-1} are an adjoint pair and j^{-1} is exact. The composite map here is just the chosen inclusion $\mathcal{F}_z \hookrightarrow I^{(z)}$. Since this is injective, so is α_z. Thus we have take an arbitrary sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, and embedded it into an injective object of $\mathscr{A}$, which shows that $\mathscr{A}$ has enough injectives.

□

21 Wednesday, March 7th: Flasque Sheaves

Having shown that $\mathrm{Mod}_{\mathcal{O}_X}$ has enough injectives, we can define , for $\mathcal{F}$ an $\mathcal{O}_X$-module

$$H^i(X, \mathcal{F}) = H^i\big(\Gamma(X, I^0) \to \Gamma(X, I^1) \to \Gamma(X, I^2) \to \cdots\big),$$

where $\mathcal{F} \to I^\bullet$ is an injective resolution of $\mathcal{F}$. However, in practice it is easier to work with flasque sheaves, which we now define, rather than injectives.

Definition 21.1. A sheaf $\mathcal{F}$ on a topological space is **flasque** if for every inclusion, $U \subset V$, $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

This notion is useful because, in consideration of whether a sheaf is flasque, it does not matter whether we regard it as a sheaf of $\mathcal{O}$-modules or abelian groups. Moreover, every injective sheaf is flasque. In the proof, we will use the following.

Definition 21.2. Let $j\colon U \hookrightarrow X$ be an open subset, and $\mathcal{F}$ be a sheaf of $\mathcal{O}_U$-modules. Then the **extension by zero** of $\mathcal{F}$ to X is defined to be the sheafification of the presheaf

$$W \mapsto \begin{cases} \mathcal{F}(W) & \text{if } W \subset U \\ 0 & \text{if not.} \end{cases}$$

It is denoted $j_!\mathcal{F}$.

Note that $j_!\mathcal{F}$ has stalks equal to those of $\mathcal{F}$ inside U, and zero elsewhere. Also, $j_!\mathcal{F}$ is a left adjoint to j^{-1}[20].

Lemma 21.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $I \in \mathrm{Mod}_{\mathcal{O}_X}$ is injective, then I is flasque.*

Proof. Let $V \subset U$ be two opens in X, with inclusions $i\colon U \hookrightarrow X$, $j\colon V \hookrightarrow U$. By the adjunction, we have for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules,

$$\mathrm{Hom}_{\mathcal{O}_X}(i_!\mathcal{O}_U, \mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, i^{-1}\mathcal{F}) \simeq \mathcal{F}(U)$$

Now there is an inclusion of $\mathcal{O}_U$-modules $j_!j^{-1}\mathcal{O}_V \to \mathcal{O}_U$, which gives a map

$$\mathrm{Hom}_{\mathcal{O}_X}(i_!\mathcal{O}_U, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X}(i_!(j_!j^{-1}\mathcal{O}_V), \mathcal{F})$$

by restricting along the above inclusion. But $\mathrm{Hom}_{\mathcal{O}_X}(i_!\mathcal{O}_U, \mathcal{F}) \simeq \mathcal{F}(U)$, and similarly $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, i^{-1}\mathcal{F}) \simeq \mathcal{F}(V)$. So we have a natural map $\mathcal{F}(U) \to \mathcal{F}(V)$. This is for any $\mathcal{F}$, but now take $\mathcal{F} = \mathcal{I}$ an injective

[20] Remember that j^{-1} is itself a left adjoint to j_*.

$\mathcal{O}_X$-module. Then since $i_!$ is exact[21], the inclusion $j_!j^{-1}\mathcal{O}_V \to \mathcal{O}_U$ gives an inclusion $i_!j_!j^{-1}\mathcal{O}_V \to i_!\mathcal{O}_U$, and then injectivity implies the map $\mathrm{Hom}_{\mathcal{O}_X}(i_!\mathcal{O}_U, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(i_!(j_!j^{-1}\mathcal{O}_V), \mathcal{I})$ is surjective, i.e., $\mathcal{I}(V) \to \mathcal{I}(U)$ is surjective.

□

Our next goal is to show that flasque sheaves are acyclic. For this we need a(nother) lemma.

Lemma 21.2. *Let $\mathcal{F}$ be a flasque sheaf of $\mathcal{O}_X$-modules, and embed $\mathcal{F} \hookrightarrow \mathcal{I}$ into an injective sheaf $\mathcal{I}$, with cokernel $\mathcal{G}$. Then for all opens U of X, $\mathcal{I}(U) \to \mathcal{G}(U)$ is surjective.*

Proof. We have an exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$. Fix a section $g \in \mathcal{G}(U)$. By exactness at the stalk, for each point of U, there is a neighborhood V over which we can lift $g\big|_V$ to some $\widetilde{g}_V \in \mathcal{I}(V)$. Suppose we have also some other neighborhood W, and a lifting $\widetilde{g}_W$ of $g\big|_W$ to $\mathcal{I}(W)$. We would like to glue these to a lift in $\mathcal{I}(V \cup W)$. Since $\widetilde{g}_V\big|_{W\cap V}$ and $\widetilde{g}_W\big|_{W\cap V}$ both map to $g\big|_{W\cap V}$ in $\mathcal{G}(W \cap V)$, the exactness of the above sequence implies that their difference $\widetilde{g}_V\big|_{W\cap V} - \widetilde{g}_W\big|_{W\cap V}$ lies in $\mathcal{F}(W \cap V)$[22]. Since $\mathcal{F}$ is flasque, $\mathcal{F}(W)$ surjects onto $\mathcal{F}(W \cap V)$, so there is some $\epsilon \in \mathcal{F}(W)$ which maps to $\widetilde{g}_V\big|_{W\cap V} - \widetilde{g}_W\big|_{W\cap V}$ in $\mathcal{F}(W \cap V)$. Since ϵ maps to zero in $\mathcal{G}(W)$, $\widetilde{g}_W + \epsilon \in \mathcal{I}(W)$ is a lift of $g\big|_W$ to $\mathcal{I}(W)$. Moreover, $\widetilde{g}_V\big|_{W\cap V} - (\widetilde{g}_W + \epsilon)|_{W\cap V} = 0$ in $\mathcal{I}(W \cap V)$, so they glue to give a lifting of g over $W \cup V$.

Thus we have shown that we can lift sections of $\mathcal{G}$ "one open at a time". We now apply Zorn's lemma to the set

$$\{(V, \widetilde{g}_V \mid \widetilde{g}_V \text{ lifts } g \text{ over } V\},$$

partially ordered by $(V, g_V) \leq (V', g_{V'})$ if $V \subset V'$ and $g_{V'}\big|_V = g_V$. Any increasing chain of such objects has an upper bound by taking the union of the V. Thus we have a maximal such pair, which must be a lifting of g over all of U: if it were not, we could extend further by the above and violate maximality.[23]

□

We can now prove

Proposition 21.1. *Let $\mathcal{F}$ be a flasque sheaf of $\mathcal{O}_X$-modules. Then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof. First we pick an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$ of $\mathcal{F}$ into an injective $\mathcal{I}$. Then $\mathcal{I}$ is flasque by Lemma 21.1. The quotient $\mathcal{G}$ is also flasque, because if $V \subset U$ is an inclusion, then we have a diagram

$$\begin{tikzcd} \mathcal{I}(U) \arrow[r, two heads] \arrow[d, two heads] & \mathcal{G}(U) \arrow[d] \\ \mathcal{I}(V) \arrow[r, two heads] & \mathcal{G}(V) \end{tikzcd}$$

[21] This follows from the description of the stalks.

[22] We are really using the left-exactness of the functor $\Gamma(W \cap V, -)$ here.

[23] Cf. also Hartshorne Ex II.1.16

in which the horizontal arrows are surjective by Lemma 21.2. This implies that the vertical arrow on the right is surjective also, hence $\mathcal{G}$ is flasque. Now $H^1(X, \mathcal{F}) = 0$ since it is the cokernel of the map $\Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G})$, which is surjective by the previous lemma. The short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ gives rise to a long exact sequence in cohomology, in which the middle terms $H^r(X, \mathcal{I})$ are zero because $\mathcal{I}$ is injective. Thus we obtain isomorphisms $H^i(X, \mathcal{G}) \simeq H^{i+1}(X, \mathcal{F})$. Since $\mathcal{G}$ is flasque, the result follows by induction. □

22 Friday, March 9th: Grothendieck's Vanishing Theorem

22.1 Statement and Preliminary Comments

Ther next few lectures will be devoted to the proof of the following

Theorem 22.1. *If X is a Noetherian topological space of dimension n, and $\mathcal{F}$ a sheaf of abelian groups on X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Our primary application of this theorem will be when $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. In order to do so, we will need the fact that computations of cohomology in the categories of abelian groups give the same result. More precisely, we have a universal δ-functor $\{H^i_M(X, -)\}$ from $\mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$ (the subscript "M" is for module), and also a δ-functor $\{H^i_A(X, -)\}$ from $\mathrm{Sh}(X) \to \mathrm{Ab}$ ("A" is for abelian group), which is defined by regarding a sheaf of abelian groups $\mathcal{F} \in \mathrm{Sh}(X)$ as a sheaf of $\underline{\mathbb{Z}}$-modules, where $\underline{\mathbb{Z}}$ is the constant sheaf[24] There is also the forgetful functor $\epsilon\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Sh}(X)$. Since ϵ is exact, the composition $\{H^i_A(X, -) \circ \epsilon\}$ is a δ-functor. We ask whether it is isomorphic, as δ-functors, to $\{H^i_M(X, -)\}$.

Theorem 22.2. *If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module, then $H^i_A(X, \epsilon\mathcal{F}) \simeq H^i_M(X, \mathcal{F})$.*

Proof. We have an isomorphism

$$H^0_M(X, -) \simeq H^0_A(X, \epsilon(-))$$

which, by the universality of H^i_M, induces a morphism of δ-functors

$$\{H^i_M(X, -)\} \to \{H^i_A(X, -) \circ \epsilon\}.$$

This is an isomorphism because $\{H^i_A(X, -) \circ \epsilon\}$ is also universal. This follows because it is effaceable: for $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, pick an embedding into an injective $\mathcal{O}_X$-module $\mathcal{I}$. This injective is also flasque, and so $\epsilon\mathcal{I}$ is flasque in $\mathrm{Sh}(X)$. Flasque sheaves in $\mathrm{Sh}(X)$ are acyclic for H^i_A by applying Lemma 21.1 with $\mathcal{O}_X = \underline{\mathbb{Z}}$. □

[24] Note that the same argument as in the case of $\mathrm{Mod}_{\mathcal{O}_X}$ shows that $\mathrm{Sh}(X)$ has enough injectives, by taking $\mathcal{O}_X = \underline{\mathbb{Z}}$.

22.2 Idea of the Proof; Some lemmas

The proof of the vanishing theorem begins by reducing first to the case when X is irreducible, and then reducing to the case where $\mathcal{F} = j_! \underline{\mathbb{Z}}$ for $j\colon U \hookrightarrow X$ and open. Then we use induction on the dimension of X. To make the reduction on $\mathcal{F}$, we will need to know how flasque sheaves and cohomology interact with direct limits.

Lemma 22.1. *On a Noetherian topological space, a direct limit of flasque sheaves is flasque.*

Proof. HW exercise. □

The next lemma says that "cohomology commutes with direct limits". More precisely, if $(\mathcal{F}_\alpha)$ is a directed system in $\mathrm{Sh}(X)$, then for each α we have a map $\mathcal{F}_\alpha \to \varinjlim \mathcal{F}_\alpha$. Applying H^i we get maps $H^i(X, \mathcal{F}_\alpha) \to H^i(X, \varinjlim \mathcal{F}_\alpha)$ for each α. By the universal property of direct limit, these induce a map

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \to H^i(X, \varinjlim \mathcal{F}_\alpha)$$

Lemma 22.2. *If X is a Noetherian topological space, the map $\varinjlim H^i(X, \mathcal{F}_\alpha) \to H^i(X, \varinjlim \mathcal{F}_\alpha)$ is an isomorphism.*

Proof. Let A be a directed set, which we can regard as a category. We denote by $\mathrm{Sh}(X)^A$ the category of functors $A \to \mathrm{Sh}(X)$. Note that it is an abelian category; kernels, cokernels, etc., are defined "index by index". It also has arbitrary products, and enough injectives. From the following diagram

$$\begin{tikzcd}
\mathrm{Sh}(X)^A \arrow[r, "\varinjlim"] \arrow[d, "\Gamma"'] & \mathrm{Sh}(X) \arrow[d, "\Gamma"] \\
\mathrm{Ab}^A \arrow[r, "\varinjlim"'] & \mathrm{Ab}
\end{tikzcd}$$

we obtain two δ-functors $\mathrm{Sh}(X)^A \to \mathrm{Ab}$, namely the right-derived functors of the two different compositions in the square. More precisely, the two δ-functors in question are

1. $(\mathcal{F}_\alpha) \mapsto H^i(X, \varinjlim \mathcal{F}_\alpha)$
2. $(\mathcal{F}_\alpha) \mapsto \varinjlim H^i(X, \mathcal{F}_\alpha)$

We will be done if we can show that these are both universal, for then the universality gives a canonical isomorphism between them. We show that they are both effaceable. The reason is that for any sheaf $\mathcal{F}$, there is a functorial embedding $\mathcal{F} \hookrightarrow \widetilde{\mathcal{F}}$ of $\mathcal{F}$ into a flasque sheaf, namely take $\widetilde{\mathcal{F}}$ to be the so-called *sheaf of discontinuous sections of* $\mathcal{F}$, given by

$$\widetilde{\mathcal{F}}(U) = \{s\colon U \bigcup_{p \in U} \mathcal{F}_P \mid s(p) \in \mathbb{F}_p \text{ for each } p\}.$$

Then the first δ-functor is effaceable since this is functorial, i.e., given an object $(\mathcal{F}_\alpha) \in \mathrm{Sh}(X)^A$, the construction produces a system $\widetilde{\mathcal{F}}_\alpha$ of flasque sheaves, whose direct limit is flasque by Lemma 22.1, hence $H^i(X, \varinjlim \mathcal{F}_\alpha) =$

0 for $i > 0$. The second is effaceable since each $H^i(X, \widetilde{\mathcal{F}}) = 0$ for $i > 0$, so $\varinjlim H^i(X, \mathcal{F}_\alpha) = 0$. $\square$

23 Monday, March 12th: More Lemmas

We stated without proof last time that the functor $\varinjlim\colon \mathrm{Sh}(X) \to \mathrm{Ab}$ is exact. Let's prove that now.

Proposition 23.1. *For A a directed set, the functor $\varinjlim\colon \mathrm{Sh}(X)^A \to \mathrm{Sh}(X)$ is exact.*

Proof. Let

$$0 \to (\mathcal{F}'_\alpha) \to (\mathcal{F}_\alpha) \to (\mathcal{F}''_\alpha) \to 0$$

be a short exact sequence in $\mathrm{Sh}(X)^A$. For each $x \in X$, and each $\alpha \in A$, the sequence

$$0 \to \mathcal{F}'_{\alpha,x} \to \mathcal{F}_{\alpha,x} \to \mathcal{F}''_{\alpha,x} \to 0$$

is exact, and the functor $\varinjlim\colon \mathrm{Mod}^A_{\mathcal{O}_{X,x}} \to \mathrm{Mod}_{\mathcal{O}_{X,x}}$ is exact, so we have short exact sequences

$$0 \to \varinjlim \mathcal{F}'_{\alpha,x} \to \varinjlim \mathcal{F}_{\alpha,x} \to \varinjlim \mathcal{F}''_{\alpha,x} \to 0$$

for each $x \in X$. Hence we will be done if we can show that $\varinjlim$ commutes with stalks, i.e.,

$$\varinjlim (\mathcal{F}_\alpha)_x = \varinjlim \mathcal{F}_{\alpha,x}$$

But stalk is a special case of pullback, so it suffices to prove more generally that for $f\colon Y \to X$, the functor $f^{-1}\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ commutes with direct limits. For this we use Yoneda: let $(\mathcal{F}_\alpha) \in \mathrm{Sh}(X)$ and $\mathcal{G} \in \mathrm{Sh}(Y)$. Then

$$\begin{aligned}
\mathrm{Hom}_Y(f^{-1} \varinjlim \mathcal{F}_\alpha, \mathcal{G}) &= \mathrm{Hom}_X(\varinjlim \mathcal{F}_\alpha, f_*\mathcal{G}) \\
&= \varinjlim \mathrm{Hom}_X(\mathcal{F}_\alpha, f_*\mathcal{G}) \\
&= \varinjlim \mathrm{Hom}_Y(f^{-1}\mathcal{F}_\alpha, \mathcal{G}) \\
&= \mathrm{Hom}_Y(\varinjlim \mathcal{F}_\alpha, \mathcal{G}),
\end{aligned}$$

so $f^{-1} \varinjlim \mathcal{F}_\alpha = \varinjlim \mathcal{F}_\alpha$.[25] $\square$

The next lemma says that if we have a sheaf $\mathcal{F}$ on a closed subset $Z \subset X$, we can compute the cohomology of $\mathcal{F}$ directly on Z, or push it forward to X and compute there, and we will get the same result.

Lemma 23.1. *Let $i\colon Z \hookrightarrow X$ be a closed subset; then for any $\mathcal{F} \in \mathrm{Sh}(Z)$,*

$$H^i(Z, \mathcal{F}) \simeq H^i(X, i_*\mathcal{F})$$

[25] The argument is a common one: "colimits commute with left adjoints". Also note that this isomorphism holds for $\mathcal{O}_X$-modules, although our proof only works in $\mathrm{Sh}(X)$, because f^{-1} is not well-defined for $\mathcal{O}_X$-modules.

Proof. First note that since i is a closed immersion, i_* is exact.[26] Also, i_* has an exact left-adjoint, so by Lemma 20.1, it takes injectives to injectives. Thus taking an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$ in $\mathrm{Sh}(Z)$, we get an injective resolution $i_*\mathcal{F} \to i_*\mathcal{I}^\bullet$ of $i_*\mathcal{F}$ in $\mathrm{Sh}(X)$. But for any sheaf $\mathcal{G}$ on Z, $\Gamma(X, i_*\mathcal{G}) = \Gamma(Z, \mathcal{G})$, so applying the global sections functor to these two injective resolutions gives the same sequence of abelian groups, hence the result. □

Remarks. 1. To see that i_* is exact when i is a closed immersion, note that the stalk of $i_*\mathcal{F}$ is

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in Z \\ 0 & \text{if not} \end{cases}.$$

Since exactness can be checked at stalks, it is easy to see in this case that i_* is exact. However, this fails when i is not a closed immersion: consider the open immersion

$$i\colon Y = \mathbb{A}^2_k \setminus \{(0,0)\} \hookrightarrow \mathbb{A}^2_k = X$$

Then the stalk $(i_*\mathcal{O}_Y)_{(0,0)}$ is just $\mathcal{O}_{X,(0,0)}$. Intuitively, an open neighborhood U of the origin, and the punctured neighborhood $U \setminus \{(0,0)\}$ have the same sections, because the point has codimension two.

2. In applying the lemma, we will frequently use the following fact. If $\mathcal{F}$ is a sheaf on a space X, whose support is contained in a closed subset $i\colon Z \hookrightarrow X$, then $\mathcal{F} = i_*i^{-1}\mathcal{F}$.

At this point we began the proof of Theorem 22.1, but I'll consolidate the whole argument into the notes for the next lecture.

24 Wednesday, March 14th: Proof of the Vanishing Theorem

We recall first the statement: If X is a Noetherian topological space of dimension n, and $\mathcal{F}$ a sheaf of abelian groups on X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Proof. **24.1 Base step**

We proceed by induction on $n = \dim X$. For the case $n = 0$, the theorem says exactly that the global sections functor is exact. But here X is a disjoint union of points, finite by the Noetherian hypothesis. A sheaf on a point is just an abelian group, so a sheaf on X is just a finite product of abelian groups. Thus $\Gamma(X, -)$ is exact - it sends a finite product of abelian groups to ... that same finite product of abelian groups.

Now we fix $n > 0$, and assume the theorem holds for any space of dimension less than n. Fix a sheaf $\mathcal{F}$ of abelian groups on X.

[26] It's a right adjoint, so it's always left exact (even if i is not a closed immersion). For right-exactness in the case of a closed immersion, see the remark following the proof.

24.2 Reduction to X irreducible

Here we argue that it is enough, under our inductive hypothesis, to assume that X is irreducible. Suppose it were not. Then pick an irreducible closed subset X' of X, let U be the complement in X' of the other irreducible components of X, and let $Y = X \setminus U$. Then we have inclusions

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y$$

with j open and i closed. Let $\mathcal{F}_U = j_!(\mathcal{F}\big|_U)$ and $\mathcal{F}_Y = i_* i^{-1}\mathcal{F}$. We obtain[27] an exact sequence of sheaves on X:

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Y \to 0,$$

which gives rise to a long exact sequence of cohomology

$$H^0(X, \mathcal{F}_U) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}_Y) \to H^1(X, \mathcal{F}_U) \to H^1(X, \mathcal{F}) \to \cdots$$

By Lemma 23.1, we have $H^i(X, \mathcal{F}_Y) = H^i(Y, i^{-1}\mathcal{F})$. Also, since $\mathcal{F}_U$ is supported on X', we may regard it as a sheaf on X', and we have $H^i(X, \mathcal{F}_U) = H^i(X', \mathcal{F}_U)$, again by Lemma 23.1 [28]. So the long exact sequence looks like

$$\cdots \to H^{i-1}(Y, i^{-1}\mathcal{F}) \to H^i(X', \mathcal{F}_U) \to H^i(X, \mathcal{F}) \to H^i(Y, i^{-1}\mathcal{F}) \to \cdots$$

Since X' is irreducible, and Y' is a closed subset of X with less irreducible components than X, this sequence will prove that $H^i(X, \mathcal{F}) = 0$ for $i > n$ by induction on the number of irreducible components, so long as we can establish this in the case where X is irreducible (i.e., the base step for the induction on the irreducible components).

24.3 Reduction to the case where $\mathcal{F} = \mathbb{Z}_U$.

Now we assume X is irreducible. For $j\colon U \hookrightarrow X$ any open subset, we let as above $\underline{\mathbb{Z}}_U = j_!\,\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the constant sheaf $\mathbb{Z}$ on U. We will simply write $\mathbb{Z}_U$ from now on. Define

$$\mathscr{S} = \coprod_{U \subset X} \mathcal{F}(U),$$

and let $\mathscr{A}$ be the set of *finite* subsets of $\mathscr{S}$, partially ordered by inclusion. A typical element of $\mathscr{A}$ looks like

$$\alpha = \{(U_1, f_1), \ldots, (U_r, f_r)\}, \quad f_k \in \mathcal{F}(U_k)$$

Such a thing defines a map

$$\bigoplus_{i=1}^{r} \mathbb{Z}_{U_i} \overset{(f_i)}{\to} \mathcal{F}$$

[27] Cf. Hartshorne, Ex II.1.19(c).

[28] Abusing notation slightly: should be $H^i(X, \mathcal{F}_U) = H^i(X', s_* s^{-1}\mathcal{F}_U)$, where $s\colon X' \hookrightarrow X$ is the inclusion. See remark 2 following Lemma 23.1.

For each α as above, we let $\mathcal{F}_\alpha$ be the image of this map. Then the inclusions $\mathcal{F}_\alpha \hookrightarrow \mathcal{F}$ induce a map $\varinjlim \mathcal{F}_\alpha \to \mathcal{F}$ which is an isomorphism. It's injective since the $\mathcal{F}_\alpha$ are subsheaves of $\mathcal{F}$. It's surjective on each open, since if $f \in \mathcal{F}(U)$, then take $\alpha = (U, f)$; the image of 1 under $\mathbb{Z}_U(U) \to \mathcal{F}_\alpha(U) \to (\varinjlim \mathcal{F}_\alpha)(U)$ is a preimage for f over U.

Since $H^i(X, \varinjlim \mathcal{F}_\alpha) \simeq \varinjlim H^i(X, \mathcal{F}_\alpha)$, it is now enough to prove that when $i > n$, $H^i(X, \mathcal{F}_\alpha) = 0$. With $\alpha = \{(U_1, f_1), \ldots, (U_r, f_r)\}$ as above, and setting $\alpha' = \{(U_1, f_1), \ldots, (U_{r-1}, f_{r-1})\}$, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\alpha'} \arrow[r] & \mathcal{F}_\alpha \arrow[r] & \mathscr{Q} \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^{r-1} \mathbb{Z}_{U_i} \arrow[r] \arrow[u, two heads] & \bigoplus_{i=1}^{r} \mathbb{Z}_{U_i} \arrow[r] \arrow[u, two heads] & \mathbb{Z}_{U_r} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the horizontal rows are exact by construction (here $\mathscr{Q}$ is just the cokernel of $\mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha$), and the left two vertical arrows are surjective by definition. It follows that the rightmost vertical arrow is also surjective. Now from the long exact sequence in cohomology induced by the top sequence here, and using induction on the cardinality of α, it suffices to show that $H^i(X, \mathscr{Q}) = 0$ for $i > n$, where $\mathscr{Q}$ is any quotient of a sheaf of the form $\mathbb{Z}_U$.

For such a sheaf $\mathscr{Q}$, we have an exact sequence

$$0 \to \mathscr{K} \to \mathbb{Z}_U \to \mathscr{Q} \to 0$$

and again using the long exact sequence, we see it will be enough to show that $H^i(X, \mathscr{K}) = 0$ for $i > n$, where $\mathscr{K}$ is any subsheaf of $\mathbb{Z}_U$ (including $\mathbb{Z}_U$ itself). So fix some subsheaf $\mathscr{K}$ of $\mathbb{Z}_U$; then the stalk $\mathscr{K}_x$ at $x \in X$ is a principal ideal $(d_x) \subset \mathbb{Z}$, where we may assume that $d_x \geq 0$. If all integers d_x are zero, then $\mathscr{K}$ is the zero sheaf, whose higher cohomology vanishes, so there's nothing to show. Otherwise, set $d = \inf\{d_x\}$, where the infimum is taken over all x such that $d_x \neq 0$. Then we will have $d_x = d$ for at least one point of X. Take some neighborhood $V \subset U$ containing this x, and a section $s \in \mathscr{K}(V)$ representing d on this neighborhood. The section s defines an injective map $\mathbb{Z}_V \hookrightarrow \mathscr{K}$ sending 1 to d. Composing with the inclusion $\mathscr{K} \hookrightarrow \mathbb{Z}_U$ gives a map

$$\mathbb{Z}_V \to \mathscr{K} \to \mathbb{Z}_U$$

of sheaves on U, which is multiplication by d at stalks of V, and the zero map elsewhere. Now we claim that in fact $\mathscr{K}_V$ is isomorphic to $\mathbb{Z}_V$. We check this at the stalk at a point y of V. There the inclusions above look like

$$\mathbb{Z}_{V,y} \hookrightarrow \mathscr{K}_y \hookrightarrow \mathbb{Z}_{U,y},$$

where $\mathbb{Z}_{V,y} \simeq \mathbb{Z}_{U,y} \simeq \mathbb{Z}$ and the composite map is multiplication by d. But this multiplication must factor through the image of $\mathscr{K}_y \hookrightarrow \mathbb{Z}_{U,y}$, which is just the subgroup generated by d_y. So $(d) \subseteq (d_y)$; but d is minimal amongst the d_x as x varies, therefore $d_y = d$ and $\mathscr{K}_y \simeq (d) \simeq \mathbb{Z}_{V,y}$. In particular, $\mathbb{Z}\big|_V \simeq \mathscr{K}\big|_V$.

This isomorphism extends to an inclusion $\mathbb{Z}_V \to \mathscr{K}_V \hookrightarrow \mathscr{K}$. Let the cokernel of $\mathbb{Z}_V \hookrightarrow \mathscr{K}$ be $\mathscr{D}$. Then since $\mathbb{Z}_V \simeq \mathscr{K}$ over V, $\mathscr{D}$ is supported on $U \setminus V$, and hence on $\overline{U \setminus V}$, which has lower dimension than X. For if $Z_0 \subset \cdots Z_r$ is a chain of irreducible closed subsets in $\overline{U \setminus V}$, then the inclusion $Z_r \subset X$ extends this chain, since X has been assumed irreducible. Thus the long exact sequence in cohomology coming from

$$0 \to \mathbb{Z}_V \to \mathscr{K} \to \mathscr{D} \to 0$$

shows that $H^i(X, \mathbb{Z}_V) \simeq H^i(X, \mathscr{K})$ for $i > n$, so it will be enough to prove $H^i(X, \mathbb{Z}_V) = 0$ for $i > n$ (note that the long exact sequence even shows $H^{n+1}(X, \mathbb{Z}_V) \simeq H^{n+1}(X, \mathscr{K})$, since the cohomology of $\mathscr{D}$ vanishes in degrees greater than *or equal to* n).

24.4 Conclusion of the Proof

Now we have finished our reduction. To show that $H^i(X, \mathbb{Z}_V) = 0$ in degrees greater than n, consider the inclusion $\mathbb{Z}_V \hookrightarrow \underline{\mathbb{Z}}$ (the constant sheaf on X), and let $\mathscr{Q}$ be the cokernel. Since X is irreducible, $\underline{\mathbb{Z}}$ is flasque, so its cohomology vanishes in positive degree. The long exact sequence coming from

$$0 \to \mathbb{Z}_V \to \underline{\mathbb{Z}} \to \mathscr{Q} \to 0$$

thus shows that $H^i(X, \mathscr{Q}) \simeq H^{i+1}(X, \mathbb{Z}_V)$ for all $i > 0$. But $\mathscr{Q}$ is supported on $X \setminus V$, which as above has lower dimension than X since X is irreducible[29]. So $H^i(X, \mathscr{Q}) = 0$ for $i \geq n$ by induction[30]. Thus $H^i(X, \mathbb{Z}_V) = 0$ for all $i \geq n+1$, finishing the proof. $\square$

25 Friday, March 16th: Spectral Sequences - Introduction

In the following few lectures, we will state without proof the results about spectral sequences which we will use, along with some common applications. Our aim is to obtain the following two results

1. If X is affine and $\mathcal{F}$ a quasicoherent sheaf on X, then the higher cohomology of $\mathcal{F}$ vanishes (we saw the vanishing of H^1 last semester).
2. If $f\colon X \to Y$ is an affine morphism and $\mathcal{F} \in \mathrm{QCoh}(X)$, then the higher direct image sheaves $R^i f_* \mathcal{F}$ are zero for $i > 0$. This generalizes the previous result by taking Y =point.

25.1 Motivating Examples

Examples 25.1. 1. Let X be a topological space and $\mathfrak{U} = \{U_i\}$ an open cover. If $\mathcal{F} \in \mathrm{Sh}(X)$, we ask: how can we compute $H^s(X, \mathcal{F})$

[29] The V from above was non-empty by construction. If we were proving this for general V, the case when V is empty implies that $\mathbb{Z}_V$ is the zero sheaf, so there's nothing to show.

[30] We're using Lemma 23.1 and Remark 2 following it again.

in terms of the $H^s(U_i, \mathcal{F}\big|_{U_i})$? Consider first the case when $s = 0$. Then by the sheaf property, $H^0(X, \mathcal{F})$ is the kernel of the map

$$\prod_i H^0(U_i, \mathcal{F}) \rightrightarrows \prod_{i,j} H^0(U_{ij}, \mathcal{F}).^{31}$$

For the case $s = 1$, let

$$K = \ker\big(\prod_i H^1(U_i, \mathcal{F}) \rightrightarrows \prod_{i,j} H^1(U_{ij}, \mathcal{F})\big).$$

We also define groups (the "Čech cohomology groups")

$$\check{H}^1(\mathfrak{U}, \mathcal{F}) = \frac{\ker\big(\prod_{i,j} H^0(U_{ij}, \mathcal{F}) \rightrightarrows \prod_{i,j,k} H^0(U_{ijk}, \mathcal{F})\big)}{\operatorname{im}\big(\prod_i H^0(U_i, \mathcal{F}) \rightrightarrows \prod_{i,j} H^0(U_{ij}, \mathcal{F})\big)}$$
$$\check{H}^2(\mathfrak{U}, \mathcal{F}) = \frac{\ker\big(\prod_{i,j,k} H^0(U_{ijk}, \mathcal{F}) \rightrightarrows \prod_{i,j,k,l} H^1(U_{ijkl}, \mathcal{F})\big)}{\operatorname{im}\big(\prod_{ij} H^0(U_{ij}, \mathcal{F}) \rightrightarrows \prod_{i,j,k} H^0(U_{ijk}, \mathcal{F})\big)}$$

At this point you should ask "what actually *are* all those maps $\rightrightarrows$?" We'll come to that later. We will see that these groups fit into an exact sequence

$$0 \to \check{H}^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \check{H}^2(\mathfrak{U}, \mathcal{F})$$

In the case mentioned above, namely when X is affine and $\mathcal{F}$ quasi-coherent, the $\prod_i H^1(U_i, \mathcal{F}) = 0$, so $K = 0$, and we get $\check{H}^1(\mathfrak{U}, \mathcal{F}) \simeq H^1(X, \mathcal{F})$.

2. Consider continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, and a sheaf $\mathcal{F}$ on X. We would like to relate the composite right derived functors $R^q g_*(R^p f_* \mathcal{F})$ with the right derived functors of the composite $R^n(fg)_*\mathcal{F}$. This is accomplished by the Leray spectral sequence, which we will see soon.

25.2 Definition

Definition 25.2. Let $\mathscr{A}$ be an abelian category, and $a \in \mathbb{N}$. A **spectral sequence starting at page** a consists of the following data:

1. Objects E_r^{pq} of $\mathscr{A}$, for $p, q \in \mathbb{Z}$ and $r \geq a$ some integer. The collection E_r^{pq} for fixed r is called the "rth page".
2. Morphisms $d_r^{pq} \colon E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d^2 = 0$ (whenever d^2 makes sense). The maps go "r to the right and $r-1$ down".
3. Isomorphisms $E_{r+1}^{pq} \simeq H^*(E_r^{pq})$.

Usually we will consider situations where $p, q \geq 0$, in which case for each p, q there exists a page r beyond which the differentials d are all zero (this r will depend on p, q), so $E_r^{pq} = E_{r+1}^{pq} = \ldots$. We denote this common value by E_∞^{pq}. We say then that the spectral sequence **converges**.

[31] As usual, $U_{ij} = U_i \cap U_j$, and in the following we will denote triple intersections by $U_{ijk} = U_i \cap U_j \cap U_k$, etc. Also, we will carelessly write $\mathcal{F}$ even where we should actually be writing $\mathcal{F}\big|_{U_i}$, etc.

Definition 25.3. Let $H^* = \{H^n\}_{n\in\mathbb{Z}}$ be a collection of objects of $\mathscr{A}$. We say that E_r^{pq} **converges to** H^* if a) it converges in the sense described above, and b) each H^n has a filtration[32] $F^\bullet$ such that $F^p/F^{p+1} \simeq E_\infty^{n,n-p}$. We use the notation $E_r^{pq} \Rightarrow H^*$ in this case.

In other words, $E_r^{pq} \Rightarrow H^*$ means that the nth antidiagonal of E_∞ gives the graded pieces of a filtration on H^n.

Remark. Note that this information is not always enough to determine H^n. It is, for example when $\mathscr{A}$ is the category of k-vector spaces. On the other hand, the two complexes of abelian groups

$$0 \to \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0 \text{ and}$$
$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

Both have filtrations with quotients $\mathbb{Z}/2$, but are not isomorphic complexes.

25.3 Spectral Sequence of a Filtered Complex

Let

$$K^\bullet \ldots 0 \to K^r \to K^{r+1} \to \ldots$$

be a complex in $\mathscr{A}$, with a filtration by subcomplexes F^i, so that

$$0 = F^s \subseteq F^{s-1} \subseteq \ldots \subseteq F^1 \subseteq F^0 = K^\bullet.$$

We say that the filtration is **finite** because the F^i are eventually zero, and **exhaustive** because $F^0 = K^\bullet$. We will use frequently the following result:

Theorem 25.1. *In the situation above, there is a spectral sequence*

$$E_1^{pq} = H^{p+q}(\mathrm{gr}^p K^\bullet) \Rightarrow H^{p+q}(K\bullet).$$

A word on notation. When we write $E_1^{pq} = H^{p+q}(\mathrm{gr}^p K^\bullet) \Rightarrow H^{p+q}(K\bullet)$, we mean that the spectral sequence begins at page 1, with the first page being $H^{p+q}(\mathrm{gr}^p K^\bullet)$. Here $\mathrm{gr}^p K^\bullet$ means the pth graded piece of the given filtration on $K^\bullet$, namely F^p/F^{p+1} (note that this, too, is a complex, so we can take cohomology). The notation for the spectral sequence gives no indication of what the differentials are, not even on the first page. To understand what they are, you must study the proof of the theorem. See Weibel's book, section 5.4, for instance.

One thing we can say is that the map $H^{p+q}(\mathrm{gr}^p K^\bullet) \to H^{p+1+q}(\mathrm{gr}^{p+1} K^\bullet)$ is the boundary map coming from application of the Snake Lemma to the short exact sequence

$$0 \to F^{p+1}/F^{p+2} \to F^p/F^{p+2} \to F^p/F^{p+1} \to 0.$$

[32] For us a filtration will always mean a decreasing filtration, i.e., a sequence of inclusions $\ldots \subset F^{i+1} \subset F^i \subset \ldots H^n$.

Example 25.1. The "filtration bête" on a complex

$$K^\bullet = K^0 \to K^1 \to K^2 \to \dots$$

is as follows:

$$\begin{tikzcd}
F^2: & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & \cdots \\
F^1: & 0 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & \cdots \\
K^\bullet: & K^0 \arrow[r] & K^1 \arrow[r] & K^2 \arrow[r] & \cdots
\end{tikzcd}$$

In this case $\operatorname{gr}^p K^\bullet = F^p/F^{p+1}$ is the complex with the object K^p in degree p and zeroes elsewhere, which we denote $K^p[-p]$[33] Therefore

$$H^{p+q}(\operatorname{gr}^p K^\bullet) = \begin{cases} K^p & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

Thus the first page of our spectral sequence is

$$\begin{tikzcd}
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
K^0 \arrow[r] & K^1 \arrow[r] & K^2 \arrow[r] & K^3 \arrow[r] & \cdots
\end{tikzcd}$$

which converges by the second page since the differentials are zero there. But on the second page the objects are $E_2^{p0} = H^p(K^\bullet)$. So the Theorem tells us that the objects $H^{p+q}(K^\bullet)$ have a filtration whose pth graded piece is 0 unless $q = 0$, in which case it is $H^p(K^\bullet)$. In other words, H^n is filtered by H^n. Bête, indeed.

26 Monday, March 19th: Spectral Sequence of a Double Complex

26.1 Double Complex and Total Complex

Today we discuss another important example of a spectral sequence. Suppose we have a double complex $I^{\bullet,\bullet}$; that is, a collection of objects I^{pq}

[33] In general, whenever we write an object as standing for a complex, we mean the complex containing that object in degree zero with zeroes elsewhere. Here we have shifted the degree to make the object sit in degree p.

and two differentials d and δ, such that the following diagram commutes

$$\begin{tikzcd} I^{p,q+1} \arrow[r, "\delta"] & I^{p+1,q+1,} \\ I^{p,q} \arrow[u, "d"] \arrow[r, "\delta"'] & I^{p+1,q} \arrow[u, "d"'] \end{tikzcd}$$

Then we can form a new complex, the **total complex** $\mathrm{tot}(I)$ **of** $I^{\bullet,\bullet}$, defined as follows:

$$(\mathrm{tot}\, I)^n = \bigoplus_{i+j=n} I^{i,j}$$

The differential of this complex is

$$D\big|_{I^{ij}} = d + (-1)^j \delta$$

There are other sign conventions in the literature - other authors take $D = d + \delta$ but insist that the squares in the original double complex anticommute; the important point is that in order to get a complex, the square in the following diagram must anticommute

$$\begin{tikzcd} I^{i,j+2} & & \\ I^{i,j+1} \arrow[u, "d"] \arrow[r, "\mp\delta"'] & I^{i+1,j+1} & \\ I^{i,j} \arrow[u, "d"] \arrow[r, "\pm\delta"'] & I^{i+1,j} \arrow[u, "d"'] \arrow[r, "\delta"'] & I^{i+2,j} \end{tikzcd}$$

D^2 is the sum of all the paths from the lower left to the three objects on the upper right antidiagonal - these should sum to zero.

There is a filtration on $\mathrm{tot}(I^{\bullet,\bullet})$ whose pth term is

$$(F^p)^n = \bigoplus_{\substack{i+j=n \\ i \geq p}} I^{ij}$$

In other words, the pth term F^p of the filtration is that part of the nth antidiagonal of $I^{\bullet,\bullet}$ which lies to the right of the $p-1$ column. From this description, we see that the quotient F^p/F^{p+1} is a complex whose objects are precisely the pth colum of $I^{\bullet,\bullet}$. But I^{pq} sits in degree $n = p+q$ according to the above definition, so the complex is graded by q, but with a shift down in degree by p. I.e.,

$$F^p/F^{p+1} = I^{p,\bullet}[-p]^{34}$$

Theorem 25.1 in this situation can be stated as:

Theorem 26.1. *In the situation above, there is a spectral sequence*

$$E_1^{pq} = H^{p+q}(I^{p,\bullet}[-p]) \Rightarrow H^{p+q}(\mathrm{tot}\, I^{\bullet,\bullet})$$

[34]Recall our convention about grading shifts: $(C[r])^n = C^{r+n}$.

Note that $H^{p+q}(I^{p,\bullet}[-p]) = H^q(I^{p,\bullet})$.

Corollary 26.1. *Let $f\colon I^{\bullet,\bullet} \to J^{\bullet,\bullet}$ be a morphism of double complexes such that for each p, $I^{p,\bullet} \to J^{p,\bullet}$ is a quasi-isomorphism. Then* $\operatorname{tot} I^{\bullet,\bullet} \to \operatorname{tot} J^{\bullet,\bullet}$ *is a quasi-isomorphism.*

Proof. The morphism f induces a morphism of spectral sequences, as follows. By assumption, the maps $H^q(I^{p,\bullet}) \to H^q(J^{p,\bullet})$ induced by f are isomorphisms between the first pages of the two spectral sequences. Thus when we take cohomology we get isomorphisms between the 2nd pages, and so on. So isomorphisms between the first pages give rise to isomorphisms

$$E_\infty^{pq}(I) \to E_\infty^{pq}(J)$$

Now it is a general fact that a map of filtered complexes which induces isomorphisms on each graded piece must be an isomorphism of complexes. So the above implies that we get an isomorphism

$$H^n(\operatorname{tot} I^{\bullet,\bullet}) \xrightarrow{\simeq} H^n(\operatorname{tot} J^{\bullet,\bullet}).$$

□

26.2 Application: Derived Functors on the Category of Complexes

The above will be useful for us in the following situation. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories, where $\mathscr{A}$ has enough injectives. Let $K^\bullet$ be a complex in $\mathscr{A}$ with $K^p = 0$ for $p << 0$. We want to compute $R^iFK^\bullet \in \mathscr{B}$.

First resolve the complex $K^\bullet$ by injectives:

$$\begin{tikzcd}
\cdots \arrow[r] & K^{p-1} \arrow[r] \arrow[d] & K^p \arrow[r] \arrow[d] & K^{p+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & I^{p-1,\bullet} \arrow[r] & I^{p,\bullet} \arrow[r] & I^{p+1,\bullet} \arrow[r] & \cdots
\end{tikzcd}$$

Here the rows and columns are complexes, and the columns are also resolutions (i.e. exact). In particular, it's a double complex $I^{\bullet,\bullet}$. We can thus define

$$R^iFK^\bullet = H^i(\operatorname{tot} FI^{\bullet,\bullet})$$

Then of course we have to check that this does not depend on the resolution $I^{\bullet,\bullet}$. Suppose given two injective resolutions $I^{\bullet,\bullet}$ and $J^{\bullet,\bullet}$ of the complex $K^\bullet$. Then one argues as in the HW a few weeks ago that it is enough to consider the case when we have a morphism $I^{\bullet,\bullet} \to J^{\bullet,\bullet}$ which commutes with the inclusions of $K^\bullet$ into each double complex. Fixing p, we have injective resolutions $K^p \to I^{p,\bullet}$ and $K^p \to J^{p,\bullet}$. We know that the derived functors of F on objects do not depend on the resolution, i.e., the maps $I^{\bullet,\bullet} \to J^{\bullet,\bullet}$ induce isomorphisms $H^n(FI^{p,\bullet}) \simeq H^n(FJ^{p,\bullet})$, and hence the conditions of the corollary are met, showing that the derived functors R^nF are well-defined on the complex $K^\bullet$.

27 Wednesday, March 21st: Spectral Sequence of an Open Cover

Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathcal{F}$ an $\mathscr{O}_X$-module. Fix an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, and assume I is totally ordered. We use the notation

$$\mathbf{i} = (i_0, \ldots, i_k), \quad i_0 < i_1 < \ldots < i_k$$

for a multi-index in I, with $|\mathbf{i}| = k$ denoting the length (length minus one, really, since our multi-indices start with i_0). For such a multi-index, define $U_{\mathbf{i}} = U_{i_0} \cap \ldots \cap U_{i_k}$, and let

$$j_{\mathbf{i}} \colon U_{\mathbf{i}} \hookrightarrow X$$

be the inclusion.

We define a complex $\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$ of sheaves of abelian groups on X, the **Ĉech complex of $\mathcal{F}$ relative to the open cover $\mathfrak{U}$**, as follows. For each $k \geq 0$, set

$$\mathscr{C}^k(\mathfrak{U}, \mathcal{F}) = \prod_{|\mathbf{i}|=k} j_{\mathbf{i}*}\mathcal{F}\big|_{U_{\mathbf{i}}}.$$

The first few terms are thus

$$\mathscr{C}^0(\mathfrak{U}, \mathcal{F}) = \prod_{i \in I} j_{i*}\mathcal{F}\big|_{U_i}$$
$$\mathscr{C}^1(\mathfrak{U}, \mathcal{F}) = \prod_{i_1 < i_2} j_{(i_1, i_2)*}\mathcal{F}\big|_{U_{i_1} \cap U_{i_2}}$$

The differential on this complex is obtained by first defining, for each $0 \leq j \leq k+1$, a map

$$d_j \colon \mathscr{C}^k(\mathfrak{U}, \mathcal{F}) \to \mathscr{C}^{k+1}(\mathfrak{U}, \mathcal{F})$$

which sends an element

$$(a_{\mathbf{i}})_{\mathbf{i}=i_0 i_1 \cdots i_k} \mapsto (a_{s_0 s_1 \cdots \widehat{s_j} \cdots s_k})_{s_0 s_1 \cdots s_k}$$

That is, the component of $d_j((a_{\mathbf{i}}))$ on the $s_0 s_1 \cdots s_k$ factor is obtained by deleting the jth index s_j and taking the appropriate entry of $(a_{\mathbf{i}})$.

Then we define the differential on $\mathscr{C}^k(\mathfrak{U}, \mathcal{F})$ by

$$d = \sum_{j=0}^{k+1} d_j$$

One checks that this defines a complex of sheaves.

Example 27.1. Take $(X, \mathscr{O}_X)$ to be $\mathbb{P}^1_k$ with its structure sheaf, and $\mathfrak{U} = \{U_0, U_1\}$ the standard open cover of $\mathbb{P}^1$ by affines. Then

$$\mathscr{C}^0(\mathfrak{U}, \mathscr{O}_{\mathbb{P}^1}) = j_{0*}\mathscr{O}_{U_0} \oplus j_{1*}\mathscr{O}_{U_1}$$
$$\mathscr{C}^1(\mathfrak{U}, \mathscr{O}_{\mathbb{P}^1}) = j_{01*}\mathscr{O}_{U_{01}}$$
$$\mathscr{C}^k(\mathfrak{U}, \mathscr{O}_{\mathbb{P}^1}) = 0 \text{ for } k > 1$$

Then if (a_0, a_1) is a local section of $\mathscr{C}^1$, the two maps d_j for $j = 0, 1$ send it to a_0 and a_1, respectively, so the differential $d \colon \mathscr{C}^0 \to \mathscr{C}^1$ maps (a_0, a_1) to $a_1 - a_0$.

The reason we care about this Ĉech complex is the following:

Proposition 27.1. *There is a quasi-isomorphism of complexes* $\mathcal{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$.

Remark. Recall that when we speak of a sheaf $\mathcal{F}$ as a complex, we mean the complex which has $\mathcal{F}$ in degree zero and zeroes elsewhere, and with differential zero. Then to say that this complex is quasi-isomorphic to some other complex $K^\bullet$ (with $K^p = 0$ for $p \leq 0$) means that $K^\bullet$ must be exact except in degree zero, since the cohomology of the "complex" $\mathcal{F}$ is zero in higher degrees. Moreover, the kernel of $K^0 \to K^1$ must be isomorphic to $\mathcal{F}$ because of the isomorphism of cohomology in degree zero. In other words, "$\mathcal{F}$ quasi-isomorphic to $K^\bullet$" means "$K^\bullet$ is a resolution of $\mathcal{F}$".

Proof. First note that we have a map

$$\mathcal{F} \to \prod_{i \in I} j_{i*}\mathcal{F}\big|_{U_i}$$

Given by the product of the restriction maps. This map is actually an isomorphism onto the kernel of d^0 because of the sheaf axiom for $\mathcal{F}$. It remains to show that the higher cohomology sheaves $\mathscr{H}^q(\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F}))$ are zero for $q > 0$. The question is local since we're working with sheaves, so we can assume X is one of the U_is. In fact, after some fussing around with indices and signs, we may assume that $X = U_0$ for simplicity.

Now we define a homotopy operator

$$h_k \colon \mathscr{C}^{k+1}(\mathfrak{U}, \mathcal{F}) \to \mathscr{C}^k(\mathfrak{U}, \mathcal{F})$$

which sends a local section $(a_{i_0 \cdots i_{k+1}})$ to $(a_{0 i_0 \cdots i_k})_{i_0 \cdots i_k}$. One checks that these maps satisfy

$$dh_k + h_{k+1}d = \mathrm{Id}_{\mathscr{C}^\bullet},$$

that is, they define a homotopy between the identity map on $\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$ and the zero map. This shows that the identity map and the zero map induce the same map on cohomology, so it must be that $(\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F}), d)$ is exact (except, of course, in degree zero).

□

Next we want to apply our results on spectral sequences to Ĉech complexes. Take an injective resolution $\mathcal{F} \to \mathscr{I}^\bullet$ of $\mathcal{F}$. Then the Ĉech sheaves $\mathscr{C}(\mathfrak{U}, \mathscr{I})$ are all injective sheaves, and we obtain a double complex

$$\begin{tikzcd}
\vdots & \vdots & \\
\mathscr{C}^0(\mathfrak{U}, \mathscr{I}^1) \arrow[u] \arrow[r] & \mathscr{C}^1(\mathfrak{U}, \mathscr{I}^1) \arrow[u] \arrow[r] & \cdots \\
\mathscr{C}^0(\mathfrak{U}, \mathscr{I}^0) \arrow[u] \arrow[r] & \mathscr{C}^1(\mathfrak{U}, \mathscr{I}^0) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

The rows are (injective) resolutions of the sheaves $\mathscr{I}^k$, by the proposition. We can regard the complex $\mathscr{I}^\bullet$ as a double complex $\mathscr{I}^{\bullet,\bullet}$ with only one column. Since the maps $\mathscr{I}^k \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^k)$ are resolutions (i.e., quasi-isomorphisms), Corollary 26.1 from the previous lecture applies, and says that the map of total complexes $\operatorname{tot} \mathscr{I}^{\bullet,\bullet} \to \operatorname{tot} \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)$ is a quasi-isomorphism. But the total complex of $\mathscr{I}^{\bullet,\bullet}$ is just $\mathscr{I}^\bullet$. Thus we have a chain of quasi-isomorphisms

$$\mathcal{F} \simeq \mathscr{I}^\bullet \simeq \operatorname{tot}(\mathscr{C}^p(\mathfrak{U}, \mathscr{I}^q))$$

In other words, the total complex of the double complex above is a resolution of $\mathcal{F}$, also.

Now we take global sections all over the place. Let

$$C^q(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathscr{C}^q(\mathfrak{U}, \mathcal{F})) = \prod_{|\mathbf{i}|=q} \mathcal{F}(U_{\mathbf{i}}).$$

Then the quasi-isomorphisms $\mathscr{I}^k \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^k)$ give rise to quasi-isomorphisms

$$\Gamma(X, \mathscr{I}^k) \to C^\bullet(\mathfrak{U}, \mathscr{I}^k)$$

of complexes of abelian groups[35]. Then we have a double complex

$$\begin{tikzcd}
\vdots & \vdots & \\
C^0(\mathfrak{U}, \mathscr{I}^1) \arrow[u] \arrow[r] & C^1(\mathfrak{U}, \mathscr{I}^1) \arrow[u] \arrow[r] & \cdots \\
C^0(\mathfrak{U}, \mathscr{I}^0) \arrow[u] \arrow[r] & C^1(\mathfrak{U}, \mathscr{I}^0) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

where here the rows are resolutions of $\Gamma(X, \mathscr{I}^k)$. By the same argument as above, we find that the total complex $\operatorname{tot} C^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)$ is a resolution of $\Gamma(X, \mathscr{I}^\bullet)$, so we can use this total complex to compute the cohomology groups of $\mathcal{F}$ on X:

$$H^*(\operatorname{tot} C^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)) \simeq (H^*(X, \mathcal{F})).$$

On the other hand, our discussion of the spectral sequence of a double complex showed that there is a spectral sequence

$$E_1^{pq} = H^q(C^p(\mathfrak{U}, \mathscr{I}^\bullet)) \Rightarrow H^{p+q}(\operatorname{tot} C^\bullet(\mathfrak{U}, \mathscr{I}^\bullet))$$

But

$$H^q(C^p(\mathfrak{U}, \mathscr{I}^\bullet)) = H^q\big(\prod_{|\mathbf{i}|=p} \mathscr{I}^\bullet(U_{\mathbf{i}})\big) =^{36} \prod_{|\mathbf{i}|=p} H^q(U_{\mathbf{i}}, \mathscr{I}^\bullet) = \prod_{|\mathbf{i}|=p} H^q(U_{\mathbf{i}}, \mathcal{F})$$

[35] This is only true because we began with resolutions of *injective* objects $\mathscr{I}^k$. In general, if $K^\bullet$ and $K'^\bullet$ are quasi-isomorphic complexes of injectives, then application of a left exact functor F gives quasi-isomorphic complexes $FK^\bullet$ and $FK'^\bullet$, but the fact that both $K^\bullet$ and $K'^\bullet$ are injective is essential.

[36] Not sure under what circumstances we can interchange cohomology and products like this.

In conclusion, we have constructed a spectral sequence

$$E_1^{pq} = \prod_{|\mathbf{i}|=p} H^q(U_{\mathbf{i}}, \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

This is the spectral sequence of the open cover $\mathfrak{U}$.

28 Friday, March 23rd: Ĉech Cohomology Agrees with Derived Functor Cohomology in Good Cases

Last time, given an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of a topological space X, and a sheaf $\mathcal{F}$ of abelian groups on X, we constructed a spectral sequence

$$E_1^{pq} = \prod_{|\mathbf{i}|=p} H^q(U_{\mathbf{i}}, \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Recall that the "global" Ĉech complex consisted of groups

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{|\mathbf{i}|=p} \mathcal{F}(U_{\mathbf{i}}),$$

with differentials defined in the same way as for the sheafy version $\mathscr{C}^\bullet$ last time. We call the cohomology groups

$$H^p(C^\bullet(\mathfrak{U}, \mathcal{F}))$$

the **Ĉech cohomology of $\mathcal{F}$ relative to the open cover $\mathfrak{U}$**, and denote them by $\check{H}^p(\mathfrak{U}, \mathcal{F})$. Today we will use the above spectral sequence to prove that in good circmstances, the Ĉech cohomology groups agree with the cohomology groups defined by derived functors of $\Gamma(X, -)$.

Let us write out the first few pages of the "global" spectral sequence from last time. The E_1 page looks like

$$\begin{tikzcd}
\vdots & \vdots & \\
\prod_i H^1(U_i, \mathcal{F}) \arrow[r, "\gamma"] & \prod_{i<j} H^1(U_{ij}, \mathcal{F}) \arrow[r] & \cdots \\
C^0(\mathfrak{U}, \mathcal{F}) \arrow[r] & C^1(\mathfrak{U}, \mathcal{F}) \arrow[r] & \cdots \\
0 & 0 & \cdots
\end{tikzcd}$$

The second page looks as follows, letting $K = \ker \gamma$.

$$\begin{tikzcd}
\vdots & \vdots & & \\
K \arrow[drr, "\partial"] & \cdots \arrow[drr] & \cdots & \cdots \\
\check{H}^0(\mathfrak{U}, \mathcal{F}) \arrow[drr] & \check{H}^1(\mathfrak{U}, \mathcal{F}) \arrow[drr] & \check{H}^2(\mathfrak{U}, \mathcal{F}) & \cdots \\
0 & 0 & 0 & \cdots
\end{tikzcd}$$

The map in to $\check{H}^0(\mathfrak{U}, \mathcal{F})$ is also zero, so this shows that $\check{H}^0(\mathfrak{U}, \mathcal{F})$ is the only term in our filtration of $H^0(X, \mathcal{F})$, so

$$\check{H}^0(\mathfrak{U}, \mathcal{F}) \simeq H^0(X, \mathcal{F})$$

To study the filtrations on the higher cohomology groups $H^i(X, \mathcal{F})$, we look now at the third page:

$$\begin{tikzcd}
\vdots & \vdots & & \\
\ker \partial \arrow[dddrr] & \cdots & \cdots & \cdots \\
\check{H}^0(\mathfrak{U}, \mathcal{F}) & \check{H}^1(\mathfrak{U}, \mathcal{F}) & \cdots & \cdots \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Note that now the first antidiagonal $p + q = 1$ stabilizes, as the maps into and out of both $\ker \partial$ and $\check{H}^1(\mathfrak{U}, \mathcal{F})$ are zero hereafter. So we know that there is a filtration on $H^1(X, \mathcal{F})$:

$$0 = F^2 \subseteq F^1 \subseteq F^0 = H^1(X, \mathcal{F})$$

with

$$\begin{aligned}
F^0/F^1 &\simeq E_\infty^{0,1} = E_3^{0,1} \simeq \ker \partial \\
F^1/F^2 &\simeq E_\infty^{1,0} = E_3^{1,0} \simeq \check{H}^1(\mathfrak{U}, \mathcal{F})
\end{aligned}$$

Since $F^2 = 0$, we have

$$\ker \partial \simeq F^0/F^1 \simeq \big(F^0/F^2\big)/\big(F^1/F^2\big) \simeq H^1(X, \mathcal{F})/\check{H}^1(\mathfrak{U}, \mathcal{F})$$

So we get an exact sequence

$$0 \to \check{H}^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ker \partial \to 0.$$

It turns out that the map on the right, $H^1(X, \mathcal{F}) \to \ker \partial$, is compatible with the inclusions

$$\ker \partial \hookrightarrow K \hookrightarrow \prod_i H^1(U_i, \mathcal{F})$$

Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \to \check{H}^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \arrow[r] \arrow[ddr] & \ker \partial \arrow[r] \arrow[d] & 0 \\
& K \arrow[d] & \\
& \prod_i H^1(U_i, \mathcal{F}) &
\end{tikzcd}$$

This is good because it relates $H^1(X, \mathcal{F})$ to $\check{H}^1(\mathfrak{U}, \mathcal{F})$, but at the expense of the "smaller" groups $\prod_i H^1(U_i, \mathcal{F})$. Of course, what we want is a way to

get a handle on these $H^1(U_i, \mathcal{F})$. More generally, we would like to handle the groups $H^i(U_{\mathbf{i}}, \mathcal{F})$, where $i > 0$ and the multi-index $\mathbf{i}$ has arbitrary length. A first step in this direction is to take the cover $\mathfrak{U}$ to consist of open affines. The next theorem shows why this is useful. It is also the first of the two desired results mentioned at the beginning of our discussion of spectral sequences.

Theorem 28.1. *Let X be an affine scheme and $\mathcal{F} \in \mathrm{QCoh}(X)$. Then for all $q > 0$, $H^q(X, \mathcal{F}) = 0$.*

Proof. We proceed by induction on q. The case when $q = 1$ was established last semester: "the global sections functor is exact on affines". So now assume that for any affine scheme X and any quasicoherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ for $1 \leq i \leq q$. We fix an affine scheme X, with a cover[37] by open affines $X = U_0 \cup \ldots \cup U_r$. We can arrange it so that all possible intersections of the U_i are also affine (by taking them to be basic opens, say). We consider the spectral sequence from before, where by our inductive assumption, all $H^i(U_{\mathbf{i}}, \mathcal{F})$ are zero, where $\mathbf{i}$ is a multi-index and $0 < i \leq q$. The first page is thus

$$\begin{array}{ccccc}
\prod_i H^{q+1}(U_i, \mathcal{F}) & \to & \prod_{i<j} H^{q+1}(U_{ij}, \mathcal{F}) & \to & \cdots \\
0 & & 0 & & 0 \\
0 & & 0 & & 0 \\
C^0(X, \mathcal{F}) & \longrightarrow & C^1(X, \mathcal{F}) & \longrightarrow & \cdots
\end{array}$$

We include only two rows of zeroes for simplicity, but of course there will be more for larger q. These rows are zero by our inductive assumption.

We now proceed to page two. The first row will consist of the groups $H^p(C^\bullet(\mathfrak{U}, \mathcal{F}))$, which are the right derived functors of $\Gamma(X, -)$, applied to the "Čech sheaves" $\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$. These sheaves are quasicoherent, so since X is affine, $\Gamma(X, -)$ is exact and the higher right derived functors vanish. So the second page is

$$\begin{array}{ccc}
K & \cdots & \cdots \\
0 & 0 & 0 \\
0 & 0 & 0 \\
H^0(X, \mathcal{F}) & 0 & 0
\end{array}$$

Here K is just some subgroup of $\prod_i H^{q+1}(U_i, \mathcal{F})$. Since this spectral sequence converges to $H^{p+q}(X, \mathcal{F})$, we have that $H^{q+1}(X, \mathcal{F}) \simeq K \hookrightarrow \prod_i H^{q+1}(U_i, \mathcal{F})$. It turns out that, by the construction of the spectral sequence (which we did not go into in any detail), this map

$$H^{q+1}(X, \mathcal{F}) \hookrightarrow \prod_i H^{q+1}(U_i, \mathcal{F})$$

[37] Recall that an affine scheme is quasicompact, so we may take a finite cover here, although we will not use the finiteness.

is induced by the restriction maps along $U_i \hookrightarrow X$. We want to use this to show that $H^{q+1}(X, \mathcal{F}) = 0$, for which we need the following lemma.

Lemma 28.1. *IF X is any scheme, and $\alpha \in H^i(X, \mathcal{F})$ for $i > 0$. Then there is an open cover $X = \bigcup_j U_j$ such that the image of α in each $H^i(X, U_j)$ is zero.*

Proof. First choose an injective resolution $\mathcal{F} \to \mathscr{I}^\bullet$. Our element $\alpha \in H^i(X, \mathcal{F})$ can be represented by an element $\widetilde{\alpha} \in \ker(\Gamma(X, \mathscr{I}^i) \to \Gamma(X, \mathscr{I}^{i+1}))$. Since the copmlex of sheaves $\mathscr{I}^\bullet$ is exact, we can find a neighborhood U of any point in X such that $\widetilde{\alpha}\big|_U$ is in the image of $\mathscr{I}^{i-1}(U) \to \mathscr{I}^i(U)$. Now choose an open cover $\bigcup_j U_j$ of X by such U. Then we have $[\widetilde{\alpha}] = 0$ in $H^i(U_j, \mathcal{F})$, for each j, as desired. □

Using the Lemma, we may replace, i.e., refine, our original cover to get some new U_j such that $\alpha \in H^{q+1}(X, \mathcal{F})$ restricts to 0 in each $H^{q+1}(U_j, \mathcal{F})$. But the map $H^{q+1}(X, \mathcal{F}) \hookrightarrow \prod_i H^{q+1}(U_i, \mathcal{F})$ is injective, so α is zero, hence $H^{q+1}(X, \mathcal{F}) = 0$.

□

Corollary 28.1. *If $f\colon X \to Y$ is an affine morphism, and $\mathcal{F}$ a quasicoherent sheaf on X, then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.*

The proof follows immediately from the theorem and the following lemma.

Lemma 28.2. *If $f\colon X \to Y$ is a morphism of schemes, and $\mathcal{F}$ a sheaf on X, then the higher pushforward $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf on Y*

$$T^i_{\mathcal{F}}\colon U \mapsto H^i(f^{-1}(U), \mathcal{F}).$$

Proof. My notes for Professor Olsson's argument are unclear here. They include only the cryptic phrase "sheafification commutes with quotients". See Hartshorne III.8.1 for a different argument. □

Now we make good on the title of today's lecture

Theorem 28.2. *If X is a scheme and $\mathfrak{U} = \{U_i\}_{i=1}^r$ a cover by open affines with all possible intersections affine*[38]*, then for any quasicoherent sheaf $\mathcal{F}$ on X, we have*

$$H^n(X, \mathcal{F}) \simeq \check{H}^n(\mathfrak{U}, \mathcal{F})$$

Proof. Here the spectral sequence of the open cover, namely

$$E_1^{pq} = \prod_{|\mathbf{i}|=p} H^q(U_{\mathbf{i}}, \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

is especially simple. On the first page, all the rows except the first are zero, by Theorem 28.1. On the second page, the sequence has already converged, and the nonzero stable groups are the $\check{H}^n(\mathfrak{U}, \mathcal{F})$. By the theorem on this spectral sequence, these groups form a one term filtration of $H^n(X, \mathcal{F})$, which proves it.

□

[38] This happens, for instance, when X is separated.

Example 28.1. We'll use the above to compute $H^*(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k})$[39]. We have a cover of $\mathbb{P}^1$ by two affines $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k[y]$, whose intersection is $\operatorname{Spec} k[x^{\pm}]$, where y corresponds to $1/x$ on this intersection. The Ĉech complex is

$$C^0(\mathfrak{U}, \mathscr{O}) = k[x] \times k[y] \to k[x^{\pm}] = C^1(\mathfrak{U}, \mathscr{O})$$

where this map is given by $(p(x), q(y)) \mapsto p(x) - q(1/x)$. The kernel of this map consists of the constants $k \simeq \{(c, c)\} \subset k[x] \times k[y]$. It is surjective, so the cokernel is zero. Therefore

$$H^n(\mathbb{P}^1_k, \mathscr{O}) \simeq \begin{cases} k & i = 0 \\ 0 & \text{else} \end{cases}$$

29 Monday, April 2nd: Serre's Affineness Criterion

Today we prove a useful result of Serre that uses cohomology of ideal sheaves to test whether a Noetherian scheme is affine. The idea of the proof introduces the point of view that cohomology groups should be thought of as containing "obstructions" to various things.

We begin with a few more examples to review the concepts from last time. Recall that the hypotheses of Theorem 28.2 are automatically satisfied if X is a scheme over R which is *separated*. The reason is that for two open affines $U, V \subset X$, we have a diagram

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_R V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_R X \end{tikzcd}$$

and when X/R is separated, Δ is a closed immersion, hence so is the top arrow. But a closed immersion is an affine morphism and $U \times_R V$ is affine, hence so is $U \cap V$.

Example 29.1. Let X be the affine line with doubled origin, $\mathbb{A}^1_k \coprod_{\mathbb{G}_m} \mathbb{A}^1_k$. This scheme is not separated, but nonetheless we have a cover by two affines, with affine intersection, so the theorem applies.

Example 29.2. If X is affine, then we can take the open cover to consist of just X itself, so Theorem 28.2 applies and $\check{H}^i = H^i$. Moreover, the Ĉech groups are zero in positive degree since our open cover has only one element. This isn't really an application of the theorem, however, since we used this fact in the proof; but we see another point of view on why affine schemes have no higher cohomology.

Now to the main course:

Theorem 29.1 (Serre)**.** *If X is a Noetherian scheme, the following are equivalent:*

1. *X is affine.*

[39] Hereafter we'll drop the subscript $\mathbb{P}^1_k$ from the notation of the structure sheaf.

2. *$H^i(X, \mathcal{F}) = 0$ for all quasicoherent sheaves $\mathcal{F}$ on X and all $i > 0$.*
3. *$H^1(X, \mathscr{I}) = 0$ for all quasicoherent ideal sheaves $\mathscr{I}$ on X.*[40]

Proof. All that needs to be shown is that 3 implies 1. For this we use the following two facts

1. (Hartshorne II.2.17) X is affine if and only if there are global sections $f_1, \ldots, f_r$ of $\mathscr{O}_X$ such that for each i, X_{f_i} is affine and the ideal $(f_1, \ldots, f_r) = \Gamma(X, \mathscr{O}_X)$.
2. If X is a Noetherian scheme, and U an open subset which contains all the closed points of X, then $U = X$.

Now we produce the sections f_i "one point at a time." Specifically, pick a *closed* point $P \in X$, and an affine neighborhood U of P. Let $Y = X \setminus U$, and give it a scheme structure (the reduced one, say). We have the skyscraper sheaf $k(P)$ at P, and it fits into a short exact sequence

$$0 \to \mathscr{I}_{Y \cup P} \to \mathscr{I}_Y \to k(P) \to 0.$$

This comes from the fact that Y is a closed subscheme of $Y \cup P$, so we have an inclusion $\mathscr{I}_{Y \cup P} \hookrightarrow \mathscr{I}_Y$. This is an isomorphism away from P; the cokernel is then just the sheaf $k(P)$. Taking cohomology gives the long exact sequence

$$0 \to \Gamma(X, \mathscr{I}_{Y \cup P}) \to \Gamma(X, \mathscr{I}_Y) \to k(P) \to 0$$

where the rightmost term is zero because of our assumption (3) that H^1 vanishes for quasicoherent ideal sheaves. Thus the map $\Gamma(X, \mathscr{I}_Y) \to k(P)$ is surjective, so we can find a global section f of $\mathscr{I}_Y$ which *does not* vanish at P (namely a preimage of any nonzero element of $k(P)$). This means that $P \in X_f$, and moreover, since f vanishes on Y, we have that $X_f \subseteq U$. Therefore the points where f vanishes are exactly the points where the image of f in $\mathscr{O}_X / \mathscr{I}_Y \simeq \mathscr{O}_U$ vanishes, which is just U_f. So X_f is affine.

So we can cover all the closed points of X by these affine sets X_f. Therefore their union is all of X, by fact 2. Again using the Noetherian hypothesis, we see that finitely many such X_f suffice to cover X. Thus we have produced global sections $f_1, \ldots, f_r$ such that $X = \bigcup X_{f_i}$.

It remains to check that $(f_1, \ldots, f_r) = \Gamma(X, \mathscr{O}_X)$. The global sections f_i define a map

$$\mathscr{O}_X^r \to \mathscr{O}_X$$

by sending $(a_0, \ldots, a_r)$ to $\sum a_i f_i$. This map is surjective since we can check at stalks, and at each point of X, not all f_i vanish since the X_{f_i} cover X. So we obtain an exact sequence

$$0 \to \mathscr{K} \to \mathscr{O}_X^r \to \mathscr{O}_X \to 0$$

for some subsheaf $\mathscr{K}$ of $\mathscr{O}_X^r$. Looking at the induced long exact sequence in cohomology, we see that if we can show that $H^1(X, \mathscr{K}) = 0$, then we will have a surjection $\Gamma(X, \mathscr{O}_X)^r \to \Gamma(X, \mathscr{O}_X)$, which will finish the proof.

[40] (In reply to a question) An example of an ideal sheaf which is not quasicoherent: $j_! \mathscr{I} \subset \mathscr{O}_X$, where $j \colon U \hookrightarrow X$ is any proper open subset, and $\mathscr{I} \subset \mathscr{O}_U$ an ideal sheaf on U.

The idea is to filter $\mathscr{K}$ in such a way that we can apply our assumption 3 to the quotients of the filtration. The filtation

$$\mathscr{O}_X \hookrightarrow \dots \mathscr{O}_X^{r-2} \hookrightarrow \mathscr{O}_X^{r-1} \hookrightarrow \mathscr{O}_X^{r},$$

where $\mathcal{O}^{r-j}$ has local sections of the form $(*, \dots, *, 0, \dots, 0)$. This gives rise to a filtration of $\mathscr{K}$:

$$\mathscr{K}_r \hookrightarrow \dots \hookrightarrow \mathscr{K}_2 \hookrightarrow \mathscr{K}_1 \hookrightarrow \mathscr{K}$$

where $\mathscr{K}_i = \mathscr{O}_X^{r-i} \cap \mathscr{K}$. The local sections of $\mathscr{K}_i/\mathscr{K}_{i+1}$ have zeroes in all but one slot, so they are isomorphic to a subsheaf of $\mathscr{O}_X \simeq \mathscr{O}_X^{r-i}/\mathscr{O}_X^{r-i-1}$.

More precisely, we have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \ker\alpha \arrow[d] & \\
0 \arrow[r] & \mathscr{K}_{i+1} \arrow[r] \arrow[d] & \mathscr{K}_i \arrow[r] \arrow[d] & \mathscr{K}_i/\mathscr{K}_{i+1} \arrow[r] \arrow[d, dotted, "\alpha"'] & 0 \\
0 \arrow[r] & \mathscr{O}_X^{r-i-1} \arrow[r] \arrow[d] & \mathscr{O}_X^{r-1} \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_X^{r-i-1}/\mathscr{K}_{i+1} \arrow[r] \arrow[d] & \mathscr{O}_X^{r-1}/\mathscr{K}_i \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The map on the bottom is injective since the upper left square is cartesian. From the snake lemma, we obtain an exact sequence

$$0 \to \ker\alpha \to \mathscr{O}_X^{r-i-1}/\mathscr{K}_{i+1} \to \mathscr{O}_X^{r-1}/\mathscr{K}_i \to \dots$$

and since the map $\mathscr{O}_X^{r-i-1}/\mathscr{K}_{i+1} \to \mathscr{O}_X^{r-1}/\mathscr{K}_i$ is injective, $\ker\alpha = 0$. This shows that $\mathscr{K}_i/\mathscr{K}_{i+1}$ is a subsheaf of $\mathscr{O}_X$. We now use this to show that $H^1(X, \mathscr{K}_i) = 0$. The long exact sequence induced by $0 \to \mathscr{K}_{i+1} \to \mathscr{K}_i \to \mathscr{K}_i/\mathscr{K}_{i+1} \to 0$ gives, in degree 1,

$$\dots \to H^1(X, \mathscr{K}_{i+1}) \to H^1(X, \mathscr{K}_i) \to H^1(X, \mathscr{K}_i/\mathscr{K}_{i+1}) \to \dots$$

Now since $\mathscr{K}_i/\mathscr{K}_{i+1}$ is an ideal sheaf, $H^1(X, \mathscr{K}_i/\mathscr{K}_{i+1}) = 0$ by our assumption of (3). Assume inductively that $H^1(X, \mathscr{K}_{i+1}) = 0$; then it follows that $H^1(X, \mathscr{K}_i) = 0$. So we get that $H^1(X, \mathscr{K}) = 0$ by decreasing induction on i. As we mentioned above, this suffices to finish the proof that X is affine.

$\square$

30 Wednesday, April 4th: Cohomology of Projective Space

We begin with a quick warm-up, using the criterion of last time to show that $X = \mathbb{A}^2_k \setminus \{(0,0)\}$ is not affine. By 29.1 it will be enough to find a quasicoherent sheaf $\mathcal{F}$ on X for which $H^1(X, \mathcal{F}) \neq 0$. Using the open cover

$$\begin{aligned} U_1 &= \operatorname{Spec} k[x^{\pm}, y] \\ U_2 &= \operatorname{Spec} k[x, y^{\pm}] \\ U_1 \cap U_2 &= \operatorname{Spec} k[x^{\pm}, y^{\pm}], \end{aligned}$$

we have

$$H^1(X, \mathcal{F}) = \operatorname{coker}\left(\mathcal{F}(U_1) \times \mathcal{F}(U_2) \xrightarrow{(a,b)\mapsto a-b} \mathcal{F}(U_1 \cap U_2)\right)$$

Now we can take $\mathcal{F} = \mathscr{O}_X$. Then $x^{-1}y^{-1} \in \mathscr{O}_X(U_1 \cap U_2)$ is not in the image of the difference map, so $H^1(X, \mathscr{O}_X)$ is non-zero (it's not even finite-dimensional).

Now we turn to the calculation of the cohomology of the sheaves $\mathscr{O}_{\mathbb{P}^r}(n)$. Let A be a ring, $S = A[x_0, \ldots, x_r]$, and set $X = \mathbb{P}^r_A = \operatorname{Proj} S$. We use the notation $\Gamma_*(\mathscr{O}_X)$ for the graded ring $\bigoplus_{n \geq 0} \Gamma(X, \mathscr{O}_X(n))$.

Theorem 30.1. *(a) There is an isomorphism $S \xrightarrow{\simeq} \Gamma_*(\mathscr{O}_X)$ induced by the universal sections $s_0, \ldots, s_r \in \Gamma(X, \mathscr{O}_X(1))$.*

(b) $H^i(X, \mathscr{O}_X(n)) = 0$ for $0 < i < r$ and $i > r$.

(c) $H^r(X, \mathscr{O}_X(-r-1)) \simeq A$[41].

(d) For each n, the map

$$H^0(X, \mathscr{O}_X(n)) \times H^r(X, \mathscr{O}_X(-n-r-1)) \to H^r(X, \mathscr{O}_X(-r-1)) \simeq A$$

is a perfect pairing[42]

Remarks. 1. The theorem allows us to compute all cohomology groups of $\mathscr{O}_X(n)$ for any n.

2. The pairing of (d) is defined as follows. If $s \in H^0(X, \mathscr{O}_X(n))$, it defines a map of sheaves $\mathscr{O}_X \to \mathscr{O}_X(n)$, or equivalently, a map $\mathscr{O}_X(-n) \to \mathscr{O}_X$. Tensoring with $\mathscr{O}_X(-r-1)$ gives a map $\mathscr{O}_X(-n-r-1) \to \mathscr{O}_X(-r-1)$. Taking rth cohomology gives our map $H^r(\mathscr{O}_X(-n-r-1)) \to H^r(X, \mathscr{O}_X(-r-1))$.

Proof. Let $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathscr{O}_X(n)$[43]. We will compute $H^i(X, \mathcal{F})$. The standard open cover $\mathfrak{U}$ of $\mathbb{P}^r_A$ satisfies the hypotheses of Theorem 28.2, so we can use the Čech complex to compute this. Moreover, formation of the Čech complex commutes with direct sums, so we have

$$H^i(X, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathscr{O}_X(n)),$$

[41] Not canonically.

[42] This means it induces an isomorphism $H^0(X, \mathscr{O}_X(n)) \simeq H^r(X, \mathscr{O}_X(-n-r-1))^\vee$.

[43] This sheaf is actually the pushforward of the structure sheaf of $X = \mathbb{A}^{r+1}_A \setminus \{O\}$ along the map $X \to \mathbb{P}^r_A$.

so we will be able to recover the cohomology groups $H^i(X, \mathscr{O}_X(n))$ for each n from our calculation. More precisely, the Ĉech complex looks like

$$\bigoplus_{i=1}^{r} \mathcal{F}(U_i) \to \bigoplus_{i<j} \mathcal{F}(U_{ij}) \to \dots$$

Note that for each n, the sections of $\mathscr{O}_X(n)$ over the open set $U_i = D_+(x_i)$ are jsut the degree n part of S_{x_i}. Summing over all n gives $\mathcal{F}(U_i) \simeq S_{x_i}$, $\mathcal{F}(U_{ij}) \simeq S_{x_i x_j}$, etc., so we are interested in the following graded complex of graded S-modules

$$\bigoplus_{i=1}^{r} S_{x_i} \to \bigoplus_{i<j} S_{x_i x_j} \to \dots$$

Now the cohomology groups of this complex are by definition $H^i(C^\bullet(X, \mathcal{F})))$. But because the differential in the complex is just localization, it is compatible with the grading of the individual groups so the degree n piece of $H^i(C^\bullet(X, \mathcal{F})))$ is just $H^i(X, \mathscr{O}_X(n))$.

We now use the observations to prove the result, in the following order: (a),(c),(d),(b).

We've basically already seen (a) earlier. Here's an example in the case $r = 1$. then H^0 is the kernel of the map

$$A[x^\pm, y] \times A[x, y^\pm] \to A[x^\pm, y^\pm],$$

which is isomorphic to $A[x, y]$. For (c), since the Ĉech complex vanishes in degree $r + 1$ and larger, we have

$$H^r(X, \mathcal{F}) = \operatorname{coker}\Big(\prod_{k=0}^{r} S_{x_0 \cdots \widehat{x_k} \cdots x_r} \to S_{x_0 \cdots x_r}\Big).$$

But $S_{x_0 \cdots x_r}$ is a free A-module with basis $\{x_0^{l_0} \cdots x_r^{l_r} \mid l_i \in \mathbb{Z}\}$, so this cokernel is free with basis $\{x_0^{l_0} \cdots x_r^{l_r} \mid l_i < 0\}$. Looking at the degree n piece, we see that since each $l_i < 0$, $\sum l_i \leq -(r + 1)$, so

$$H^r(X, \mathscr{O}_X(n)) = 0$$

whenever $n > -r - 1$, and for $n = -r - 1$, the only monomial in the basis is $x_0^{-1} \cdots x_r^{-1}$, so $H^r(X, \mathscr{O}_X(-r-1))$ is a free A-module of rank one. This proves (c).

Now for (d). First, if $n < 0$, then $\mathscr{O}_X(n)$ has no global sections, and since $-n - r - 1 > -r - 1$, $H^r(X, \mathscr{O}_X(-n - r - 1)) = 0$, as was observed in the previous paragraph. So there is nothing to show. When $n = 0$, $H^0(X, \mathscr{O}_X) \simeq A$ and $H^r(X, \mathscr{O}_X(-r - 1)) \simeq A$ by (c). For each global section $s \in A$ we get a map

$$A \simeq H^r(X, \mathscr{O}_X(-r - 1)) \to H^r(X, \mathscr{O}_X(-r - 1)) \simeq A$$

and this association defines an isomorphism of $H^0(X, \mathscr{O}_X)$ with $H^r(X, \mathscr{O}_X(-r-1))^\vee$ since $H^r(X, \mathscr{O}_X(-r - 1))$ is free of rank one.

Now we consider the case when $n > 0$. Here $H^0(X, \mathscr{O}_X(n))$ is free with basis $\{x_0^{m_0} \cdots x_r^{m_r} \mid m_i \geq 0, \sum m_i = n\}$, while $H^r(X, \mathscr{O}_X(-n-r-1))$ has

basis $\{x_0^{l_0}\cdots x_r^{l_r} \mid l_i < 0, \sum l_i = -n-r-1\}$. The pairing acts on these bases by

$$\langle x_0^{m_0}\cdots x_r^{m_r}, x_0^{l_0}\cdots x_r^{l_r}\rangle = x_0^{m_0+l_0}\cdots x_r^{m_r+l_r}$$

But in $H^r(X, \mathscr{O}_X(-r-1))$, $x_0^{m_0+l_0}\cdots x_r^{m_r+l_r}$ is zero unless each $m_i + l_i$ is -1, in which case it is 1. So these form dual bases with respect to the pairing, which is therefore perfect. This proves (d). We will prove (b) next time.

31 Friday, April 6th: Conclusion of Proof and Applications

We will prove (b) by induction on r. Note that the statement for $i > r$ is true, regardless of r, simply because the Ĉech complex is zero after the rth degree. If $r = 1$, the claim is vacuous.

Note that $\mathbb{C}^\bullet(\mathfrak{U}, \mathcal{F})_{x_i}$ is the Ĉech complex of $\mathcal{F}\big|_{U_i}$ with respect to the open cover $\mathfrak{U}_i = \{U_i \cap U_j\}_{j=1}^r$. Therefore since U_i is affine, it is an exact complex except in degree 0. This means that for each j, an for all $i > 0$, $H^i(X, \mathcal{F})_{x_j} = 0$. But to say that this localization is zero is to say that the S-module $H^i(X, \mathcal{F})$ is annihilated by x_j, so to prove (b), it will be enough to show that

$$\cdot x_j \colon H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$$

is injective for each $0 < i < r$, $0 \le j \le r$. Now $x_j \in \Gamma(X, \mathscr{O}_X(1))$ gives a map $\mathscr{O}_X(n-1) \to \mathscr{O}_X$ for each n, so it defines a short exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{\cdot x_j} S \to S/(x_j) \to 0.$$

This gives rise to a short exact sequence of sheaves on $\mathbb{P}^r$

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

where $H = V_+(x_j) \simeq \mathbb{P}_A^{r-1} \subset \mathbb{P}_A^r$. Taking cohomology gives the long exact sequence

$$\begin{tikzcd}
H^0(X,\mathcal{F}) \arrow[r, "\cdot x_j"] & H^0(X,\mathcal{F}) \arrow[r] & H^0(H,\mathcal{F}_H) \arrow[dll] \\
H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(H,\mathcal{F}_H) \arrow[dll] \\
\vdots & \vdots & \vdots \\
H^{r-1}(X,\mathcal{F}) \arrow[r, "\cdot x_j"] & H^{r-1}(X,\mathcal{F}) & \\
& &
\end{tikzcd}$$

First, the map $H^0(X, \mathcal{F}) \simeq S \to H^0(H, \mathcal{F}_H) \simeq S/(x_j)$ is surjective, so the map

$$H^1(X, \mathcal{F}) \xrightarrow{\cdot x_j} H^1(X, \mathcal{F})$$

is injective. Next, we may assume by our induction hypothesis that $H^i(H, \mathcal{F}_H) = 0$ for $0 < i < r-1$, so the right hand column consists of zeroes except in the top row. Therefore each map

$$H^i(X, \mathcal{F}) \xrightarrow{\cdot x_j} H^i(X, \mathcal{F})$$

for $i = 2, 3, \ldots, r-1$ is injective. This finishes the proof.

□

Example 31.1. Let $F \in k[x_0, \ldots, x_r]$ be a homogeneous polynomial of degree d, and consider the hypersurface

$$X = V_+(F) \overset{i}{\hookrightarrow} \mathbb{P}^r_k.$$

We want to calculate $H^i(X, \mathscr{O}_X)$. The global section $F \in \Gamma(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r})$ defines an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^r}(-d) \overset{\cdot F}{\to} \mathscr{O}_{\mathbb{P}^r} \to i_* \mathscr{O}_X \to 0$$

Recall that since i is a closed immersion we have $H^i(X, \mathscr{O}_X) = H^i(\mathbb{P}^r, i_* \mathscr{O}_X)$. So the long exact sequence can be written as

$$\begin{tikzcd}
H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-d)) \arrow[r] & H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}) \arrow[r] & H^0(X, \mathscr{O}_X) \arrow[dll] \\
H^1(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-d)) \arrow[r] & H^1(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}) \arrow[r] & H^1(X, \mathscr{O}_X) \arrow[dll] \\
\vdots & \vdots & \vdots \\
H^r(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-d)) \arrow[r, "\alpha"] & H^r(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}) \arrow[r] & H^r(X, \mathscr{O}_X)
\end{tikzcd}$$

We have already computed the first two columns; the left column is zero except in degree r, the middle column is zero except in degrees 0 and r. Moreover, in degree zero, $H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}) \simeq k$. Also, we know $H^i(X, \mathscr{O}_X) = 0$ in degrees greater than $r-1$ for dimension reasons. Hence we have

$$\begin{aligned}
&H^0(X, \mathscr{O}_X) \simeq k \\
&H^i(X, \mathscr{O}_X) \simeq 0 \text{ for } r \neq 0, r-1
\end{aligned}$$

The only tricky part is to determine $H^{r-1}(X, \mathscr{O}_X)$, which is the kernel of the map $\alpha \colon H^r(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-d)) \to H^r(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r})$. Recall that the global section F defines a map

$$\cdot F \colon \mathscr{O}_{\mathbb{P}^r} \to \mathscr{O}_{\mathbb{P}^r}(d).$$

Twisting by $-r-1$ and taking global sections we have

$$H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-r-1)) \xrightarrow{F} H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(d-r-1))$$

Taking duals gives a map

$$H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(d-r-1))^\vee \xrightarrow{F} H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-r-1))^\vee.$$

Applying Serre duality, we get a map $H^r(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-d)) \to H^r(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r})$. Unwinding the definitions, it turns out that this map is in fact α. So the kernel of α is isomorphic to the kernel of $H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(-r-1)) \xrightarrow{F} H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(d-r-1))$.

Example 31.2. Specializing the previous example, let F be the polynomial $zy^2 - (x^3 - axz^2 + bz^3)$; this defines an elliptic curve $E \subset \mathbb{P}^2_k$. We have

$$h^0(E, \mathscr{O}_E) = 1$$
$$h^1(E, \mathscr{O}_E) = \binom{2}{2} = 1$$

Example 31.3. Recall that the canonical sheaf ω_X is the top exterior power of Ω^1_X; it's a line bundle on X. In the case of $\mathbb{P}^r$, we've calculated that $\omega_{\mathbb{P}^r}$ is the line bundle $\mathscr{O}_{\mathbb{P}^r}(-r-1)$. For $\mathbb{P}^1$, we have $\omega_{\mathbb{P}^1} = \Omega^1_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^2}(-2)$; so $H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2))$.

32 Monday, April 9th: Finite Generation of Cohomology

Today we discuss the question of whether the cohomology groups of a coherent sheaf are finitely generated modules over the base ring. The answer is yes in good circumstances.

Theorem 32.1. *If A is a Noetherian ring, X/A projective, $\mathscr{L}$ is a very ample sheaf on X, and $\mathcal{F}$ is any coherent sheaf on X, then*

(a) For all $i > 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module.

(b) There is an integer n_0 such that for all $i > 0$ and $n \geq n_0$,

$$H^i(X, \mathcal{F} \otimes \mathscr{L}^{\otimes n}) = 0.$$

Remark. Recall that $\mathscr{L}$ is very ample if there is an immersion $i\colon X \to \mathbb{P}^r_A$ such that $i^*\mathscr{O}_{\mathbb{P}^r}(1) \simeq \mathscr{L}$. In the following we will write $\mathscr{L}$ as $\mathscr{O}_X(1)$, in order to use the simpler notation $\mathcal{F}(n) = \mathcal{F} \otimes \mathscr{L}^{\otimes n}$.

The theorem also admits the following generalization.

Theorem 32.2. *1. If A is a Noetherian ring and X is proper over A, then for any coherent sheaf $\mathcal{F}$ on X and all $i > 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module.*

2. If $f\colon X \to Y$ is a proper morphism of locally Noetherian schemes, then for any coherent sheaf $\mathcal{F}$ on X and all $i > 0$, $R^i f_* \mathcal{F}$ is a coherent sheaf on Y.

Statement 2 is the most general; this is proved by reducing it to statement 1, and then reducing that to the proof of the previous theorem, which as we will now see, involves a reduction to the calculation of the cohomology of projective space.

We will need the following technical lemma

Lemma 32.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$ and let $s \in \Gamma(D_+(x_i))$. Then there is an integer m such that sx_i^m extends to a global section of $\Gamma(\mathbb{P}^r, \mathcal{F}(m))$, i.e. there is a section $\widetilde{s} \in \Gamma(\mathbb{P}^r, \mathcal{F}(m))$ such that $\widetilde{s}\big|_{D_+(x_i)} = sx_i^m \in \Gamma(D_+(x_i), \mathcal{F}(m))$.*

Proof. Here's a sketch of the idea. Given coherent $\mathcal{F}$, set $\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{F}(n)$. By the sheaf property, we have the following two equalizer diagrams, where the vertical arrows are the localizations maps

$$\begin{tikzcd}
\mathcal{G}(X) \arrow[r] \arrow[d] & \prod_j \mathcal{G}(D_+(x_j)) \arrow[r] \arrow[d] & \prod_{j<j'} \mathcal{G}(D_+(x_j x_{j'})) \arrow[d] \\
\mathcal{G}(D_+(x_i)) \arrow[r] & \prod_j \mathcal{G}(D_+(x_i x_j)) \arrow[r] & \prod_{j<j'} \mathcal{G}(D_+(x_i x_j x_{j'}))
\end{tikzcd}$$

One can lift sections up along the vertical maps by multiplying by a sufficiently high power of x_i. Use this and a diagram chase to lift $s \in \mathcal{G}(D_+(x_i))$ to a section of $\mathcal{G}(X)$. $\square$

Corollary 32.1. *If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^r_A$, then $\mathcal{F}$ admits a surjection $\mathscr{E} \to \mathcal{F}$, where $\mathscr{E}$ is a finite direct sum of sheaves of the form $\mathscr{O}_{\mathbb{P}^r}(q)$.*

Proof. If we have generators $s_1, \ldots, s_t$ for $\mathcal{F}$ over $D_+(x_i)$, multiply each one by an appropriate power of x_i to get a section of $\mathcal{F}(m)$. These sections define a map $\mathscr{O}^t_{\mathbb{P}^r} \to \mathcal{F}(m)$, where this m is the maximum of the twists involved in lifting the various s_j. This map is surjective over $D_+(x_i)$. Tensoring with $\mathscr{O}_{\mathbb{P}^r}(-m)$ gives a map of sheaves $\mathscr{O}_{\mathbb{P}^r} \to \mathcal{F}$ which is surjective over $D_+(x_i)$. Do this oover each $D_+(x_i)$, and then take the direct sum to prove the corollary.

$\square$

Proof of Theorem 32.1. First we reduce to the case $X = \mathbb{P}^r_A$. Let $i \colon X \to \mathbb{P}^r_A$ be the closed immersion. Since i_* is exact and has an exact left adjoint, it doesn't change the cohomology. Also, it preserves coherence, since this can be checked locally. If on some open affine, $\mathcal{F}$ corresponds to the B/I-module M, then $i_*\mathcal{F}$ corresponds to the module M_B obtained by base change along the ring map $B \to B/I$, which is still finitely generated since this is a surjective ring map. Thus $H^i(X, \mathcal{F}) \simeq H^i(\mathbb{P}^r, i_*\mathcal{F})$, so we may assume $X = \mathbb{P}^r$.

The result is true for $i > r$: in that case $H^i = 0$ since $\mathbb{P}^r$ can be covered by $r+1$ affines[44]. We also know the result in the case that $\mathcal{F}$ is $\mathscr{O}_X(n)$, since we have explicitly calculated $H^i(\mathbb{P}^r, \mathscr{O}_X(n))$; we also get

[44] Note - we cannot argue that $H^i = 0$ for $i > r$ because of dimension reasons, since A is arbitrary, so it is difficult to get our hands on the dimension of $\mathbb{P}^r$.

the result when $\mathcal{F}$ is a finite direct sum of copies of these since H^i is an additive functor.

Now for the inductive step: assume that for some fixed i_0, and for all coherent $\mathcal{G}$, $H^i(\mathbb{P}^r, \mathcal{G})$ is finitely generated whenever $i > i_0$. We will use this to show that $H^{i_0}(\mathbb{P}^r, \mathcal{F})$ is finitely generated. Now use the lemma above to find an $\mathscr{E} = \bigoplus_i \mathscr{O}_{\mathbb{P}^r}(q_i)$ and a morphism $\mathscr{E} \to \mathcal{F}$, and let $\mathscr{K}$ be the kernel. Then we have an exact sequence

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathcal{F} \to 0,$$

which gives a long exact sequence in cohomology in which the following terms appear:

$$\ldots \to H^{i_0}(\mathbb{P}^r, \mathscr{E}) \xrightarrow{\alpha} H^{i_0}(\mathbb{P}^r, \mathcal{F}) \xrightarrow{\beta} H^{i_0+1}(\mathbb{P}^r, \mathscr{K}) \to \ldots.$$

Note that $H^{i_0}(\mathbb{P}^r, \mathscr{E})$ is finitely generated, as we observed above, and that $H^{i_0+1}(\mathbb{P}^r, \mathscr{K})$ is finitely generated by induction. Since this sequence is exact, we can break off the following short exact sequence

$$0 \to \operatorname{im}\alpha \to H^i(\mathbb{P}^r, \mathcal{F}) \to \ker\beta \to 0,$$

and both $\operatorname{im}\alpha$ and $\ker\beta$ are finitely generated, since kernels and images of finitely generated modules over a Noetherian ring are also finitely generated. This proves (a).

For (b), we only provide a sketch; the crux of the argument is the same as (a). We will need the following result, which is useful enough to isolate as a separate lemma.

Lemma 32.2 (Projection Formula)**.** *If $i\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces, $\mathcal{F}$ an $\mathscr{O}_X$-module, and $\mathscr{E}$ a locally free $\mathscr{O}_Y$-module of finite rank, then*

$$i_*(\mathcal{F} \otimes_{\mathscr{O}_X} i^*\mathscr{E}) \simeq (i_*\mathcal{F}) \otimes_{\mathscr{O}_Y} \mathscr{E}$$

A similar argument was made for this statement in the special case when $\mathscr{E}$ is a line bundle in an earlier lecture. In our case, taking $Y = \mathbb{P}^r_A$, and $\mathscr{E} = \mathscr{O}_{\mathbb{P}^r}(1)$, we have

$$i_*\mathcal{F}(n) = i_*(\mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(1)^{\otimes n}) = (i_*\mathcal{F}) \otimes_{\mathscr{O}_{\mathbb{P}^r}} \mathscr{O}_{\mathbb{P}^r}(1)^{\otimes n} = (i_*\mathcal{F})(n),$$

which shows that we can assume that $X = \mathbb{P}^r_A$. Now, the statement of (b) is true when $i > r$, as in (a). We also know that for $\mathcal{F} = \mathscr{O}_X(n)$, $H^i(X, \mathscr{O}_X(n) = 0$ for $0 < i < r$ and $H^r(X, \mathscr{O}_X(n))$ is dual to $H^0(X, \mathscr{O}_X(-n-r-1))$, which shows that the statement holds when $\mathcal{F} = \mathscr{O}_X(n)$; it holds for finite direct sums of these again by additivity of H^i.

For the inductive step fix i_0, and assume that for any $\mathcal{G}$ and any $i > i_0$, there is an integer n_0 such that $H^i(X, \mathcal{G}(n)) = 0$ if $n \geq n_0$. Then one shows that there is an m_0 such that $H^{i_0}(X, \mathcal{F}(m)) = 0$ for $m \geq m_0$, by arguments similar to those of (a).

□

Note in conclusion that the integer n_0 of (b) in principle depends on all of X, A, $\mathscr{L}$, and $\mathcal{F}$, but the essential dependence is on $\mathcal{F}$. In particular, we can always twist $\mathcal{F}$ down to force larger values of n_0.

33 Wednesday, April 11th: The Hilbert Polynomial

Fix a projective scheme X over a Noetherian ring A. Let $\mathscr{O}_X(1)$ be a very ample sheaf. For a coherent sheaf $\mathcal{F}$ on X, we define the **Euler characteristic** of $\mathcal{F}$ to be the integer

$$\chi(\mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{F}).$$

Note that by the results of the previous lecture, each h^i is finite, and there are only finitely many nonzero terms, so the sum makes sense.

For such $\mathcal{F}$, we also define the **Hilbert polynomial** of $\mathcal{F}$ to be the function $p_{\mathcal{F}} \colon \mathbb{Z} \to \mathbb{Z}$ given by

$$p_{\mathcal{F}}(n) = \chi(\mathcal{F}(n))$$

Note that for sufficiently large n, we have $p_{\mathcal{F}}(n) = h^0(X, \mathcal{F}(n))$, again using the results of the previous lecture. To justify calling this function a polynomial, we prove the following

Theorem 33.1. *$p_{\mathcal{F}}$ is a polynomial function.*

Proof. First we reduce to the case where $X = \mathbb{P}^r_A$. Pick a closed immersion $i \colon X \hookrightarrow \mathbb{P}^r_A$ such that $i^*\mathscr{O}_{\mathbb{P}^r} \simeq \mathscr{O}_X(1)$. This can be done since $\mathscr{O}_X(1)$ is very ample. Then as we saw last time, the projection formula implies $(i_*\mathcal{F})(n) = i_*(\mathcal{F}(n))$, so $p_{i_*\mathcal{F}} = p_{\mathcal{F}}$, and we may replace X by $\mathbb{P}^r_A$.

Now pick a hyperplane $H \subset X = \mathbb{P}^r$; this gives us an exact sequence

$$0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X \to \mathscr{O}_H \to 0$$

where we are omitting the pushforward of $\mathscr{O}_H$ from the notation. Since tensoring with $\mathcal{F}$ is right exact we get another exact sequence

$$0 \to \mathscr{K} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}\big|_H \to 0,$$

where $\mathscr{K}$ is defined as the kernel of $\mathcal{F}(-1) \to \mathcal{F}$. Let $\mathscr{I}$ denote the image of the map $\mathcal{F}(-1) \to \mathcal{F}$. We can break this sequence into two short exact sequences

$$0 \to \mathscr{K} \to \mathcal{F}(-1) \to \mathscr{I} \to 0$$
$$0 \to \mathscr{I} \to \mathcal{F} \to \mathcal{F}\big|_H \to 0$$

Since χ is additive on short exact sequences (we proved this for curves a few weeks ago), we get two equations

$$p_{\mathscr{K}} + p_{\mathscr{I}} = p_{\mathcal{F}(-1)}$$
$$p_{\mathscr{I}} + p_{\mathcal{F}|_H} = p_{\mathcal{F}}.$$

The difference of these two equations is the equality

$$p_{\mathcal{F}|_H} - p_{\mathscr{K}} = p_{\mathcal{F}} - p_{\mathcal{F}(-1)}.$$

By induction, we may assume $p_{\mathcal{F}|_H}$ is a polynomial. Also, the sheaf $\mathscr{K}$ is supported on H, and a coherent sheaf supported on a closed subscheme

can be realized as the pushforward of a coherent sheaf on that closed subscheme (see lemma below for details). Thus we may also assume by induction that $p_{\mathscr{K}}$ is a polynomial. Finally, note that $p_{\mathcal{F}(-1)}(n) = p_{\mathcal{F}}(n-1)$. Thus the equation above shows that $p_{\mathcal{F}}(n) - p_{\mathcal{F}}(n-1)$ is a polynomial function. In fact (check) this implies that $p_{\mathcal{F}}$ itself is a polynomial function. $\square$

To explain our treatment of $\mathcal{F}\big|_H$ in the above, we have the following lemma and its corollary.

Lemma 33.1. *Let X be a Noetherian scheme, $\mathcal{F}$ a coherent sheaf on X with set-theoretic support $Z \subset X$. Give Z the reduced scheme structure, and let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal sheaf of Z. Then there exists r such that $\mathscr{I}^r$ annihilates $\mathcal{F}$.*

Corollary 33.1. *$\mathcal{F}$ is isomorphic to the pushforward of a coherent sheaf on $V(\mathscr{I}^r)$.*

Proof. The question of whether, for given r, $\mathscr{I}^r$ annihilates $\mathcal{F}$ can be checked locally. Having produced such r locally, we may take a finite cover of X by open affines and use the maximum of all the r obtained in our local constructions. Thus we may as well let $X = \operatorname{Spec} A$. For $I \subset A$ an ideal and F a finitely generated A-module, the condition that $\widetilde{F}$ is supported on $V(I)$ means that, if $g_1, \ldots, g_s$ are generators for I, then $\widetilde{F}$ is 0 on $D(g_1) \cup \ldots D(g_s)$, i.e., $F_{g_i} = 0$ for each i. Let $m_1, \ldots, m_t$ be generators for F. Since $F_{g_i} = 0$, there is an n_i such that $g_i^{n_i} m_j = 0$ for all j. Let n be the maximum of these n_i as i varies from 1 to s. Thus $g_i^n m_j = 0$ for all i, j.

We can choose r big enough so that, for any monomial $g_1^{a_1} \cdots g_s^{a_s}$ of degree r, at least one of the a_i is at least n. Then for this r, any monomial of degree r annihilates F, so $I^r F = 0$. This proves the lemma.

To see the corollary, observe that if $I^r F = 0$, then F has the structure of an A/I^r-module, so the sheaf $\widetilde{F}$ on $\operatorname{Spec} A$ can be viewed as the pushforward of a sheaf $\widetilde{F}'$ on $\operatorname{Spec} A/I^r$ along the ring map $A \to A/I^r$. Globalizing, we have a closd immersion $i\colon V(I^r) \hookrightarrow X$. The action of $i^{-1}\mathcal{O}_X$ on $i^{-1}\mathcal{F}$ factors through $\mathcal{O}_{V(I^r)}$, and $i^{-1}\mathcal{F}$ is still coherent as an $\mathscr{O}_{V(I^r)}$-module. Moreover, the map $\mathcal{F} \to i_* i^{-1}\mathcal{F}$ is an isomorphism of $\mathcal{O}_X$ modules, so we can view $\mathcal{F}$ as the pushforward of the coherent $\mathscr{O}_{V(I^r)}$-module $i^{-1}\mathcal{F}$.

$\square$