HYPOELLIPTICITY OF THE KOHN LAPLACIAN FOR THREE-DIMENSIONAL TUBULAR CR STRUCTURES

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By a tubular three-dimensional CR structure we will mean a CR structure defined in an open subset of $\mathbb{R}^3$, together with a coordinate system $(x, y, t) \in \mathbb{R}^3$, together with a Cauchy-Riemann operator of the form

$$\bar\partial_b = \partial_x + i(\partial_y - \phi'(x)\partial_t) \tag{1}$$

where $\phi \in C^\infty(\mathbb{R})$ is real-valued. Such CR structures may be realized as the boundaries of tube domains $\{z : \text{Im } z_2 > \phi(\text{Re } z_1)\}$ in $\mathbb{C}^2$. The Levi form may be identified with the function $\phi''(x)$. We always assume that ϕ is convex, so that the structure is pseudoconvex. By $\bar\partial_b^*$ we mean the adjoint of $\bar\partial_b$ with respect to $L^2(\mathbb{R}^3,\, dx\, dy\, dt)$; thus $\bar\partial_b^* = -\partial_x + i(\partial_y - \phi'(x)\partial_t)$.

The purpose of this note is to characterize hypoellipticity of the Kohn Laplacian $\bar\partial_b \bar\partial_b^*$ for this class of CR structures.

Main Theorem. *For any C^∞ pseudoconvex tubular three-dimensional CR structure, the following four conditions are equivalent in any open set.*

- (α) *$\bar\partial_b$ is C^∞ hypoelliptic, modulo its nullspace.*
- (β) *The CR structure is not exponentially degenerate.*
- (γ) *The pair $\{\bar\partial_b, \bar\partial_b^*\}$ satisfies a superlogarithmic estimate.*
- (δ) *There exists $s > 0$ such that $\bar\partial_b$ is H^s hypoelliptic, modulo its nullspace.*

The main new result here is the implication [exponential degeneracy] $\Rightarrow$ [nonhypoellipticity]. The implication [not exponentially degenerate] $\Rightarrow$ [hypoelliptic] is a sharpening of the known sufficient condition $x \log \phi''(x) \to 0$ as $|x| \to 0$.

This work is part of a broader investigation of related problems, concerning both hypoellipticity and global regularity in C^∞, C^ω, and Gevrey classes. See [3] and [15] for speculation on some of these matters in a wider context. The essential novelty in this paper is a characterization for a natural, though restricted, class of structures, as opposed to isolated examples; tube domains have long served as prototypical examples. The author had not anticipated obtaining such a characterization, because the behavior of smooth functions vanishing to infinite order can be so wild.

For arbitrary smooth, pseudoconvex three-dimensional CR structures, a superlogarithmic estimate for $\{\bar\partial_b, \bar\partial_b^*\}$ implies hypoellipticity [4], but the the converse is false in general [5].

The notions appearing in this characterization are defined as follows.

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Definition 1. $\bar\partial_b$ is C^∞ hypoelliptic modulo its nullspace in an open set U if for any open subset $V \subset U$ and every function $u \in L^2(V)$ such that $\bar\partial_b^* u \in L^2(V)$ and $\bar\partial_b \bar\partial_b^* u \in C^\infty(V)$, necessarily $\bar\partial_b^* u \in C^\infty(V)$.

For any parameter $s > 0$, $\bar\partial_b$ is H^s hypoelliptic modulo its nullspace in an open set U if for any open subsets $V \Subset \tilde V \subset U$ and every function $u \in L^2(\tilde V)$ such that $\bar\partial_b^* u \in L^2(\tilde V)$ and $\bar\partial_b \bar\partial_b^* u \in C^\infty(\tilde V)$, necessarily $\bar\partial_b^* u \in H^s(V)$.

Here H^s denotes the usual Sobolev space of functions having s derivatives in L^2; in the second part of the definition, V may be taken to be a ball, and $H^s(V)$ is then the space of all functions in $L^2(V)$ extendible to functions in $H^s(\mathbb{R}^3)$.

As is well known, C^∞ hypoellipticity, modulo the nullspace, holds whenever the structure is strictly pseudoconvex, or more generally of finite type, at every point of U, so the only issue here is the possible appearance of singularities at points where ϕ'' vanishes to infinite order. It is explained in [10, 11] why this notion of hypoellipticity is natural.

A variant would be merely to require $u, \bar\partial_b^* u$ to belong to $\mathcal{D}'(V)$; our results apply equally well to that variant, but we focus on the first formulation since it arises most directly in complex analysis.

Definition 2. A collection of (complex) vector fields $\{X_j\}$ is said to satisfy a superlogarithmic estimate in an open set $U \subset \mathbb{R}^d$ if for every relatively compact subset $V \subset U$ and every $\epsilon > 0$ there exists $C_\epsilon < \infty$ such that for every function $u \in C^1$ supported in V,

$$\int_{\mathbb{R}^d} (\log|\xi|)^2 \, |\hat u(\xi)|^2 \, d\xi \le \epsilon \sum_j \|X_j u\|_{L^2}^2 + C_\epsilon \|u\|_{L^2}^2, \tag{2}$$

where $\hat u$ denotes the Fourier transform of u.

An equivalent condition is that for each V there exists a function w, satisfying $w(r)/\log r \to +\infty$ as $r \to +\infty$, such that

$$\int_{\mathbb{R}^d} w^2(|\xi|) \, |\hat u(\xi)|^2 \, d\xi \le C \sum_j \|X_j u\|_{L^2}^2 + C\|u\|_{L^2}^2 \tag{3}$$

for all $u \in C^1$ supported in V. This formulation explains the terminology "superlogarithmic".

For a proof that the superlogarithmic estimate for the pair $\{\bar\partial_b, \bar\partial_b^*\}$ implies hypoellipticity modulo the nullspace for arbitrary smooth, pseudoconvex three-dimensional CR structures, see the proofs of Corollaries 3.2 and 3.3 of [4]. Hypoellipticity modulo the nullspace is not explicitly discussed in that reference, but follows from the arguments there together with a simple microlocalization as in [9, 10].

$|I|$ denotes the length of an interval $I \subset \mathbb{R}$. We denote the endpoints of a closed interval by $x_\pm$; thus $I = [x_-, x_+]$.

Definition 3. Let J be an open subinterval of $\mathbb{R}^1$. A CR structure satisfying the above conditions is said to be exponentially degenerate in $J \times \mathbb{R}^2$ if there exist $\delta > 0$

and a sequence of intervals $I_\nu \subset \mathbb{R}$, all contained in some compact subset of J, such that

$$|I_\nu| \to 0 \text{ as } \nu \to \infty \tag{4}$$

and

$$\int_{I_\nu} \phi''(x)\,dx \le e^{-\delta/|I_\nu|}\ . \tag{5}$$

For example, if $\phi''(x) \le \exp(-c/|x|)$ as $x \to 0$ for some $c > 0$ then the structure is exponentially degenerate; but the converse does not hold. Note that exponential degeneracy is defined only for tubular CR structures, whereas the other two notions appearing in the main theorem are defined in general.

For simplicity, we assume henceforth that ϕ is defined on all of $\mathbb{R}$, and that ϕ'' has a strictly positive lower bound outside some bounded interval. This can of course be arranged by restricting and then extending a given ϕ.

By separation of variables, everything reduces to an analysis of properties of certain ordinary differential operators.

Definition 4. For $\eta \in \mathbb{C}$ and $\tau \in \mathbb{R}$,

$$\mathcal{L}_{\eta,\tau} = (-\partial_x + (\eta - \tau\phi'(x))) \circ (\partial_x + (\eta - \tau\phi'(x)))\ . \tag{6}$$

Our first lemma is rather routine.

Lemma 1. *The following conditions are equivalent.*

- *The pair $\{\bar\partial_b, \bar\partial_b^*\}$ satisfies a superlogarithmic estimate.*
- *The lowest eigenvalue $\lambda(\tau,\eta)^2$ of $\mathcal{L}_{\eta,\tau}$ satisfies $\lambda(\tau,\eta)/\log\tau \to \infty$ as $\tau \to +\infty$, uniformly in $\eta \in \mathbb{R}$.*
- *The CR structure is not exponentially degenerate.*

Here $\lambda(\tau,\eta)^2$ is defined to be the infimum of $\langle \mathcal{L}_{\eta,\tau} f,\ f\rangle$ over all compactly supported C^2 functions f satisfying $\|f\|_{L^2} = 1$.

Proof. If the operators $\mathcal{L}_{\eta,\tau}$ satisfy the stated lower bounds, then the superlogarithmic estimate follows directly by application of a partial Fourier transform with respect to the coordinates (y,t).

To show that exponential degeneracy implies failure of the eigenvalue estimate, fix $\delta, \{I_\nu\}$ as in the definition of degeneracy. Set $\tau_\nu = \exp(\alpha/|I_\nu|)$, where $\alpha > 0$ is a constant to be specified below; thus $|I_\nu| = \alpha/\log\tau_\nu$. Fix $g \in C^\infty(\mathbb{R})$, supported in $(-1,1)$, not vanishing identically. Set $f = g \circ \ell_\nu$, where ℓ_ν is the unique linear transformation mapping the left endpoint of I_ν to -1, and the right endpoint to $+1$. Specify η_ν by

$$[\tau_\nu\phi'(x_-) - \eta_\nu] = -[\tau_\nu\phi'(x_+) - \eta_\nu]\ . \tag{7}$$

Since $\eta_\nu - \tau_\nu\phi'$ vanishes at some point of I_ν,

$$|\eta_\nu - \tau_\nu\phi'(x)| \le \tau_\nu \int_{I_\nu} \phi'' \tag{8}$$

for every $x \in I_\nu$.

Now dropping the subscripts ν to simplify notation,

$$\begin{aligned}\langle \mathcal{L}_{\eta,\tau} f,\ f\rangle &= \|f'\|^2 + \int_{\mathbb{R}} |f(x)|^2 |\eta - \tau\phi'(x)|^2\,dx + \tau\int_{\mathbb{R}} |f(x)|^2\phi''(x)\,dx \\ &\le C|I|^{-1} + C\max_I |\eta-\tau\phi'|^2\cdot|I| + C\tau\int_I \phi'' \\ &\le C|I|^{-1} + C|I|\tau^2\cdot(\int_I \phi'')^2 + C\tau\int_I\phi''\ .\end{aligned}$$

Hence by Cauchy-Schwarz,

$$\langle \mathcal{L}_{\eta,\tau} f,\ f\rangle/\|f\|^2 \le C|I|^{-2} + C\tau^2(\int_I \phi'')^2 \le C(\log\tau)^{-2} + C\tau^2\tau^{-2\delta/\alpha}. \tag{9}$$

Choosing any $\alpha < \delta$, the last term is bounded by a negative power of τ, so all together, the ratio is $\le C(\log\tau)^{-2}$, and the eigenvalue estimate fails.

Suppose next that the structure is not exponentially degenerate. To prove that the estimate for $\{\mathcal{L}_{\eta,\tau}\}$ does hold, let $\eta,\tau\in\mathbb{R}$ be given, with τ large and positive. Let A be a large parameter, and let $I = \{x : |\eta - \tau\phi'(x)| \le A\log\tau\}$. Then $\tau\int_I \phi'' = 2A\log\tau$, whence $|I| \le \delta/A\log\tau$. Now

$$\langle \mathcal{L}_{\eta,\tau} f,\ f\rangle \ge A^2\log^2\tau\int_{\mathbb{R}\setminus I} |f|^2 + \|f'\|^2. \tag{10}$$

By a Poincaré-type inequality, $\int_I |f|^2 \le C|I|^2\|f'\|^2 + C\int_{\mathbb{R}\setminus I}|f|^2$, so

$$\langle \mathcal{L}_{\eta,\tau} f,\ f\rangle \ge A^2\log^2\tau\int_{\mathbb{R}\setminus I} |f|^2 + c|I|^{-2}\int_I |f|^2 - C\|f\|^2. \tag{11}$$

Since $|I| \le \delta/A\log\tau$, $\min(A\log\tau, |I|^{-1}) \ge A\min(1,\delta^{-1})\log\tau$. Choosing A sufficiently large completes the proof of the eigenvalue bound.

It remains to show that exponential degeneracy precludes the superlogarithmic estimate. Fix an auxiliary function $h\in C^\infty(\mathbb{R})$, supported in a small neighborhood of the origin. For each interval I, let c_I be the center of I, and set $h_I(x) = h((x-c_I)/|I|)$. Consider functions $u_{I,\tau,\eta}(x,y,t) = h(y)h(t)e^{i\tau t + i\eta y}h_I(x)$. For such functions u, the superlogarithmic estimate is equivalent to

$$|I|\big(\log(2+\tau+|\eta|+|I|^{-1})\big)^2 \le \epsilon\|\bar\partial_b u\|^2 + \epsilon\|\bar\partial_b^* u\|^2 + C_\epsilon|I|, \tag{12}$$

for arbitrarily small $\epsilon > 0$. We have

$$\|\bar\partial_b u\|^2 + \|\bar\partial_b^* u\|^2 \le C|I|^{-1} + C|I|\max_{x\in I}|\eta-\tau\phi'(x)|^2, \tag{13}$$

so the superlogarithmic estimate becomes

$$\log(\tau+|\eta|) \le \epsilon|I|^{-1} + \epsilon\max_{x\in I}|\eta-\tau\phi'(x)| + C_\epsilon, \tag{14}$$

which is to hold uniformly in I, τ, η. Fixing I, τ, there exists η such that $\max_{x \in I} |\eta - \tau\phi'(x)| = \tau \int_I \phi''$; moreover, considering only intervals I of bounded length and located in a fixed neighborhood of the origin, we have $|\eta| < \tau$ and hence the superlogarithmic estimate becomes

$$\log \tau \leq \epsilon\big(|I|^{-1} + \tau \int_I \phi''\big) + C_\epsilon. \tag{15}$$

Suppose now that $\int_I \phi'' \leq \exp(-\delta/|I|)$, and set $\tau = 1/|I| \int_I \phi''$. Then $\log \tau \geq c \log |I|^{-1}$, so the superlogarithmic estimate implies that $c \log |I|^{-1} \leq \epsilon \log |I|^{-1} + C_\epsilon$ for arbitrarily small $\epsilon > 0$, a contradiction. □

We now begin the main step of the analysis, the proof that exponential degeneracy implies nonhypoellipticity. To each bounded closed interval $I = [x_-, x_+]$ associate

$$\theta_I = \tfrac{1}{2}(\phi'(x_-) + \phi'(x_+)). \tag{16}$$

An equivalent characterization of θ is that $\big[\phi'(x_-) - \theta_I\big] = -\big[\phi'(x_+) - \theta_I\big]$.

Lemma 2. *For each I, there exists a unique interval $I^* = [x_-^*, x_+^*]$ such that $I^* \supset I$, $|I^*| = 2|I|$, and $\big[\phi'(x_-^*) - \theta_I\big] = -\big[\phi'(x_+^*) - \theta_I\big]$.*

Proof. Consider $F(t) = \phi'(t) + \phi'(t + 2|I|)$ for $x_- - |I| \leq t \leq x_-$. F is strictly increasing, and $F(x_- - |I|) = \phi'(x_- - |I|) + \phi'(x_+) < \phi'(x_-) + \phi'(x_+) < \phi'(x_-) + \phi'(x_+ + |I|) = F(x_-)$, so there exists a unique $t \in \mathbb{R}$ with the desired property, and $t \in [x_- - |I|, x_-]$. Set $I^* = [t, t + 2|I|]$. □

Definition 5. For any interval $I \subset \mathbb{R}$,

$$\rho(I) = \frac{\int_{I^*} \phi''}{\int_I \ \phi''} \ . \tag{17}$$

$\rho(I)$ is bounded above, uniformly for all intervals, if and only if ϕ'' vanishes only to finite order at any point; that is, if and only if the CR structure is of finite type. Later, the key step of our analysis, Lemma 4, will exploit the unboundedness of ρ. The purpose of the next lemma is to show that there exist intervals for which $\int_I \phi''$ is small and $\rho(I)$ is large, simultaneously.

Lemma 3. *Suppose the CR structure is exponentially degenerate. Then there exist a positive constant δ and a sequence of intervals $I_\nu \subset \mathbb{R}$ such that $|I_\nu| \to 0$,*

$$\int_{I_\nu} \phi'' \leq e^{-\delta/|I_\nu|}, \tag{18}$$

and

$$\rho(I_\nu) \to \infty \ \text{as} \ \nu \to \infty. \tag{19}$$

Proof. The hypothesis gives a sequence of intervals whose lengths tend to zero, satisfying the first inequality. All such intervals necessarily lie in a bounded region, since ϕ'' is bounded below outside a compact set. Consider any large constant A, let $0 < \epsilon$ be a sufficiently small parameter to be chosen below, and consider the nonempty collection of all bounded closed intervals $J \subset \mathbb{R}$ such that $\int_J \phi'' \le \exp(-\delta/|J|)$ and $|J| \le \epsilon$.

We argue by contradiction; suppose that for all such intervals J, $\rho(J) \le A$. Then for any such J, consider the tower defined by $J_{n+1} = J_n^*$, with $J_0 = J$. Denote by N the smallest integer such that $\rho(J_N) > A$.

To see that such an N exists and that $|J_N|$ must itself be smaller than any preassigned quantity if ϵ is chosen to be sufficiently small, suppose instead that $|J_N| \ge 1$. Denote by M the unique integer such that $1 \le |J_M| < 2$. Then

$$\int_{J_M} \phi'' \le A^M \int_{J_0} \phi'' \le A^M e^{-\delta/|J|} \le e^{-C \log A \cdot \log |J|} e^{-\delta/|J|}; \tag{20}$$

since $|J| \le \epsilon$, the right-hand side can be made arbitrarily small by choosing ϵ sufficiently small. But since ϕ'' is uniformly bounded below outside a compact set and $|J_M| \ge 1$, $\int_{J_M} \phi''$ is bounded below by a strictly positive constant. Thus we have a contradiction; hence N must exist.

The same reasoning shows that if ϵ is sufficiently small, then $|J_N|$ must be smaller than any preassigned quantity α; just change the condition $1 \le |J_M| < 2$ to $\alpha \le |J_M| < 2\alpha$.

Now by the same reasoning,

$$\int_{J_N} \phi'' \le A^N e^{-\delta/|J|} \le e^{-\delta/|J_N|} C e^{C \log A \cdot [F(|J|) - F(|J_N|)]} \tag{21}$$

where $F(t) = \log(1/t) - \delta/t$. For any δ, $F'(t) > 0$ for all sufficiently small t; and we have already ensured that $|J|$ and $|J_N|$ are as small as may be desired. Thus $\int_{J_N} \phi'' \le e^{-\delta/|J_N|}$, J_N was chosen so that $\rho(J_N) > A$, and A is arbitrarily large. □

Definition 6.

$$\mathcal{N}(\zeta, \tau) = \int_{\mathbb{R}} e^{2(\zeta x - \tau \phi(x))} \, dx. \tag{22}$$

The following lemma is the core of our analysis. We write

$$\tau_\nu = \frac{1}{|I_\nu| \int_{I_\nu} \phi''} \; . \tag{23}$$

Lemma 4. *Suppose there exist a constant δ and sequence of intervals $\{I_\nu\}$ such that $\int_{I_\nu} \phi'' \le \exp(-\delta/|I_\nu|)$ for all ν, $|I_\nu| \to 0$, and $\rho(I_\nu) \to \infty$ as $\nu \to \infty$. Then there exist $C < \infty$ and a sequence $\{\nu_k\} \to \infty$ such that for each k there exists $\zeta_k \in \mathbb{C}$, satisfying $|\zeta_k| \le \tau_{\nu_k}$, such that*

$$\mathcal{N}(\zeta_k, \tau_{\nu_k}) = 0, \tag{24}$$

and

$$|\operatorname{Im} \zeta_k| \le C \log \tau_{\nu_k}. \tag{25}$$

Here Im denotes the imaginary part.

We have recently [5] given an example of a cylindrically symmetric CR structure, strictly pseudoconvex except on a single real curve transverse to the complex tangent spaces, for which hypoellipticity fails to hold. This was done by building in detail a CR structure for which analogues of $\mathcal{N}(\cdot, \tau_\nu)$, for a sequence $\tau_\nu \to \infty$, are small perturbations of an explicit function, which manifestly has zeroes. In the present article we take a different route, showing that the existence of appropriate zeroes is inevitable. It may well be possible to apply this technique to the cylindrically symmetric case, as well, to obtain a necessary and sufficient condition for hypoellipticity.

Before beginning the proof, we point out that unlike the real analytic case [2], there is no scaling symmetry which reduces matters to the case $\tau = 1$ modulo small perturbations. Perhaps paradoxically, the following argument actually exploits this lack of symmetry.

Proof. To simplify notation we drop the subscript ν for the first part of the proof. Let $I = [x_-, x_+]$ be one of the intervals I_ν. Set $\eta = \tau\theta$ where $\theta = \theta(I)$ was defined by the relation (16). Let ℓ be the unique linear endomorphism of $\mathbb{R}$ satisfying $\ell(\pm 1) = x_\pm$. Let $s^*_\pm = \ell^{-1}(x^*_\pm)$; then $s^*_- \le -1 < +1 \le s^*_+$, and $s^*_+ - s^*_- = 4$.

Set $\gamma = \min_{x \in I}(\tau\phi(x) - \eta x)$, and set

$$\psi = (\tau\phi - \eta x - \gamma) \circ \ell. \tag{26}$$

Then ψ is nonnegative, smooth and convex, and assumes its minimum value 0 at some point of $[-1, 1]$.

Crucial properties of ψ are:

$$0 \le \psi(\pm 1) \le 1. \tag{27}$$

$$|\psi'(s^*_\pm)| \ge \tfrac{1}{2}\rho(I). \tag{28}$$

Indeed, the maximum over $[-1, 1]$ of ψ equals the maximum over I of $\tau\phi(x) - \eta x - \gamma$. The derivative of the latter function is $\tau\phi' - \eta$, which vanishes at some point of I by the intermediate value theorem, since the choice of η means that $\big[\tau\phi'(x_-) - \eta x_-\big] = -\big[\tau\phi'(x_+) - \eta x_+\big]$. Hence since γ was chosen to make the minimum value equal 0, the maximum value is $\le \int_I |\tau\phi' - \eta| \le |I|\tau \int_I \phi'' = 1$, by the definition of τ.

The lower bounds for $|\psi'(s^*_\pm)|$ are obtained similarly. $I^* = [x^*_-, x^*_+]$ was constructed so that $\big[\tau\phi'(x^*_-) - \eta\big] = -\big[\tau\phi'(x^*_+) - \eta\big]$. Therefore

$$\tau\phi'(x^*_+) = \tfrac{1}{2}\big((\tau\phi'(x^*_+ - \eta) - (\tau\phi'(x^*_- - \eta)\big) = \tfrac{1}{2}\tau \int_{I^*} \phi'' = \tfrac{1}{2}\tau\rho(I) \int_I \phi''. \tag{29}$$

Thus

$$|\psi'(s^*_\pm)| = |I| \tfrac{1}{2}\rho(I)\tau \int_I \phi'' = \tfrac{1}{2}\rho(I). \tag{30}$$

Now by writing $\ell(s) = \frac{x_+ + x_-}{2} + \frac{|I|}{2}s$, we obtain

$$\begin{aligned} \mathcal{N}(\eta + \zeta, \tau) &= |I| \int_{\mathbb{R}} e^{2[\zeta\ell(s) - (\psi(s) + \gamma)]}\, ds \\ &= |I| e^{-2\gamma} \int_{\mathbb{R}} e^{\zeta(x_+ + x_-) + \zeta|I|s} e^{-2\psi(s)}\, ds = |I| e^{-2\gamma} e^{\zeta(x_+ + x_-)} \mathcal{M}_I(z), \end{aligned}$$

where

$$\mathcal{M}_I(z) = \int_{\mathbb{R}} e^{zs-2\psi(s)}\,ds \tag{31}$$

and

$$z = |I|\zeta\ . \tag{32}$$

$\mathcal{M}_I$ depends on I through ψ.

A simple normal families argument demonstrates that for all sufficiently large ν, there exists a zero $z \in \mathbb{C}$ of $\mathcal{M}_{I_\nu}$ satisfying $|z| \le C$, uniformly in ν. Indeed, since $-2 \le s_-^* < s_+^* \le +2$, $\exp(-2\psi_\nu(s)) \le \exp(-\rho(I_\nu)|s-2|)$ for every $|s| \ge 2$. Since $\rho(I_\nu) \to \infty$, and the functions $\exp(-2\psi_\nu)$ are everywhere ≤ 1, it follows that on any compact subset of $\mathbb{C}$, all but finitely many of the functions $\mathcal{M}_{I_\nu}(z)$ extend holomorphically, and the collection of all of these extensions forms a normal family.[1] Moreover, there exists a subsequence $\{\nu_k\}$ such that $\exp(-2\psi_{\nu_k})$ converges weakly to a bounded function f that is supported on $[-2,2]$. For any such subsequence, $f(s) \ge e^{-2}$ for all $|s| \le 1$, since $0 \le \psi(s) \le 1$. The limiting holomorphic function is the Fourier transform of f. Since f has compact support, $|\hat f(z)| \le C\exp(C|z|)$ for some $C < \infty$, for all $z \in \mathbb{C}$. Consequently $\hat f$ must have (an infinite discrete set of) zeroes, since the measure $f(s)\,ds$ is not a Dirac mass. Fixing one zero of f, any holomorphic function sufficiently close to $\hat f$ must have a nearby zero, so there exist a constant $C < \infty$, a subsequence $\{\nu_k\}$, and a sequence $\{z_k\}$ such that for each k, $\mathcal{M}_{I_{\nu_k}}$ has a zero satisfying $|z_k| \le C$.

Consequently $\mathcal{N}(\eta+\zeta,\tau) = 0$ and $|\operatorname{Im}(\eta+\zeta)| = |\operatorname{Im}(\zeta)| = |I|^{-1}|\operatorname{Im}(z)| \le C|I|^{-1}$. To conclude the proof, it suffices to show that

$$|I_\nu| \ge c/\log\tau_\nu. \tag{33}$$

Now the basic degeneracy condition $\int_I \phi'' \le \exp(-\delta/|I|)$, combined with the definition of τ, give $|I| \ge -\delta/\log\int_I \phi'' = \delta/\log(\tau|I|) = \delta(\log\tau - \log(|I|^{-1}))^{-1} \sim \delta/\log\tau$, since the definition $\tau = 1/|I|\int_I \phi''$ implies that $\tau \gg |I|^{-1}$ for all intervals I in a fixed compact set, and since only such intervals are among the degenerate intervals I_ν. $\square$

Rather than constructing singular solutions, we will show that $\bar\partial_b$ fails to be hypoelliptic by exhibiting functions that disprove certain *a priori* inequalities which are a consequence of hypoellipticity. Those functions could alternatively be used as building blocks in an infinite series whose sum is a singular solution.

If $\bar\partial_b$ is C^∞ hypoelliptic modulo its nullspace, then for any open subset V, any relatively compact subset V', and any α, there must exist $M, C < \infty$ such that for any $u \in C^\infty(V)$,

$$\|\partial^\alpha_{x,y,t}\bar\partial_b^* u\|_{C^0(V')} \le C\|u\|_{C^0(V)} + C\|\bar\partial_b^* u\|_{C^0(V)} + C\|\bar\partial_b\bar\partial_b^* u\|_{C^M(V)}; \tag{34}$$

this is a consequence of the Baire category theorem. Restricting attention to functions $u(x,y,t) = \exp(i\tau t + i\zeta y)f(x)$, where $\tau \in \mathbb{R}^+$ and $\zeta \in \mathbb{C}$ and f satisfies $\mathcal{L}_{\zeta,\tau}f \equiv 0$,

[1] It is here that the unboundedness of the numbers $\rho(I_\nu)$ is exploited.

setting $d_{\zeta,\tau} = \partial_x - (\zeta - \tau\phi'(x))$, and taking $V = \{|x| + |y| + |t| < \epsilon\}$ and $V' = \{|x| + |y| + |t| < \epsilon/2\}$, a consequence would be that for any $\epsilon > 0$,

$$\tau \sup_{|x|<\epsilon/2} |d_{\zeta,\tau} f(x)| \le C_\epsilon e^{\epsilon|\operatorname{Im}(\zeta)|} \sup_{|x|<\epsilon} \big(|f(x)| + |d_{\zeta,\tau} f(x)|\big), \tag{35}$$

uniformly for all f, ζ, τ; the crucial factor of τ on the left arises from taking one derivative of $\exp(i\tau t)$. In the same way, H^s hypoellipticity modulo the nullspace would imply that for any $\epsilon, \epsilon' > 0$,

$$\tau^s \|d_{\zeta,\tau} f\|_{L^2\{|x|<\epsilon'\}} \le C_{\epsilon,\epsilon'}\, e^{\epsilon|\operatorname{Im}(\zeta)|} \big(\|f\|_{L^2\{|x|<2\epsilon'\}} + \|d_{\zeta,\tau} f\|_{L^2\{|x|<2\epsilon'\}}\big). \tag{36}$$

Lemma 5. *Suppose there exist sequences of positive real numbers $\tau_\nu \to +\infty$, and complex numbers ζ_ν, such that $\mathcal{N}(\zeta_\nu, \tau_\nu) = 0$ for every ν, $|\zeta_\nu|/\tau_\nu \to 0$, and $|\operatorname{Im} \zeta_\nu| \le C \log \tau_\nu$. Then for any $s > 0$, $\bar\partial_b$ fails to be H^s hypoelliptic, modulo its nullspace.*

Proof. We will often drop the subscript ν in order to simplify notation. Set $\Phi(x) = \tau_\nu\phi(x) - \zeta_\nu x = \tau\phi(x) - \zeta x$ and set

$$f_\nu(x) = f(x) = e^{\Phi(x)} \int_{-\infty}^{x} e^{-2\Phi(s)}\, ds. \tag{37}$$

Then $\mathcal{L}_{\zeta,\tau} f \equiv 0$, and $d_{\zeta,\tau} f = e^{-\Phi}$.

The convex function $\operatorname{Re}\Phi$ has a unique critical point x_0, which tends to zero as $\nu \to \infty$ because $|\zeta_\nu|/\tau_\nu \to 0$. Moreover, for all but finitely many ν,

$$\|e^{-\Phi}\|_{L^2\{|x|>\epsilon'\}} \le C\|e^{-\Phi}\|_{L^2\{|x|<\epsilon'\}}\ . \tag{38}$$

Indeed, define $x_+ > x_0$ so that $\operatorname{Re}\Phi(x_+) = 1 + \operatorname{Re}\Phi(x_0)$. Clearly $x_+ \to 0$ as $\nu \to \infty$. Then $\operatorname{Re}\Phi'(x_+) \ge (x_+ - x_0)^{-1}$, and convexity of $\operatorname{Re}\Phi'$ and a simple comparison imply that

$$\int_{x_+}^{\infty} e^{-2\operatorname{Re}\Phi} \le (x_+ - x_0) e^{-2\operatorname{Re}\Phi(x_+)} \le (x_+ - x_0) e^{-2\operatorname{Re}\Phi(x_0)}. \tag{39}$$

On the other hand,

$$\int_{x_0}^{x_+} e^{-2\operatorname{Re}\Phi} \ge \int_{x_0}^{x_+} e^{-2} e^{-2\operatorname{Re}\Phi(x_0)} \ge e^{-2}(x_+ - x_0) e^{-2\operatorname{Re}\Phi(x_0)}, \tag{40}$$

so for large ν,

$$\int_{\epsilon'}^{\infty} e^{-2\operatorname{Re}\Phi} \le \int_{x_+}^{\infty} e^{-2\operatorname{Re}\Phi} \le C \int_{x_0}^{x_+} e^{-2\operatorname{Re}\Phi}. \tag{41}$$

The same reasoning applies on $(-\infty, x_0]$.

Thus (36) would imply that

$$\tau^s \|d_{\zeta,\tau} f\|_{L^2\{|x|<\epsilon'\}} \le C_{\epsilon,\epsilon'}\, \tau^\epsilon \big(\|f\|_{L^2\{|x|<\epsilon'\}} + \|d_{\zeta,\tau} f\|_{L^2\{|x|<\epsilon'\}}\big), \tag{42}$$

uniformly as $\nu \to \infty$ for all $\epsilon, \epsilon' > 0$. Choosing $\epsilon < s$, the second term on the right becomes much less than the left-hand side for large ν, so may be neglected. In order to obtain a contradiction, it suffices to verify that $\|f\|_{L^2\{|x|<\epsilon'\}} \le C\|d_{\zeta,\tau} f\|_{L^2\{|x|<\epsilon'\}}$; in fact there is a pointwise bound

$$|f(x)| \le C|d_{\zeta,\tau} f(x)| \qquad \text{for all } x. \tag{43}$$

To prove (43), consider first the case where $x \le x_0$. Since $-\Phi$ is increasing on $(-\infty, x_0)$,

$$|f(x)| \le \int_{-\infty}^{x} e^{-\text{Re } \Phi(s)}\, ds \ . \tag{44}$$

Let $J = J_\nu = \{s : |\text{Re } \Phi(s)| \le \text{Re } \Phi(x_0) + 1\} = [x_-, x_+]$. Then by convexity, $|\text{Re } \Phi'(x_\pm) \ge |J|^{-1}$, so $\tau|\int_{x_0}^{x_\pm} \phi''| = \ge |J|^{-1}$. The contribution of $J \cap (-\infty, x)$ to (44) is $\le |J|e^{-\text{Re } \Phi(x_0)} \le C|J|e^{-\text{Re } \Phi(x)}$ if $x \ge x_-$, and is zero if $x < x_-$.

Outside J, since Re Φ' is monotonic, we have as in the proof of (38) that the contribution of $(-\infty, x]\backslash J$ is

$$\le C|\text{Re } \Phi'(x)|]^{-1} e^{-\text{Re } \Phi(x)} \le |J|e^{-\text{Re } \Phi(x)}. \tag{45}$$

Thus

$$|f(x)| \le C|J||d_{\zeta,\tau} f(x)| \tag{46}$$

for all $x \le x_0$, uniformly in ν. Since $|J| \to 0$ as $\nu \to \infty$, this is stronger than (43) except for the restriction $x \ge x_0$.

In the analysis of the case $x \ge x_0$, the crucial condition $\mathcal{N}(\zeta, \tau) = 0$ finally comes into play. The vanishing of $\mathcal{N}(\zeta, \tau)$ means equivalently that f has the alternative representation

$$f(x) = -e^{\Phi(x)} \int_{x}^{\infty} e^{-2\Phi(s)}\, ds. \tag{47}$$

Hence for $x \ge x_0$, $|f(x)| \le \int_x^\infty \exp(-\Phi(s))\, ds$, and the reasoning of the preceding paragraph may be repeated to obtain the bound $|f(x) \le C|d_{\zeta,\tau} f(x)|$, uniformly in x, ν.

Combining the above steps, we conclude that for any positive s, (36) fails to hold, and hence that $\bar\partial_b$ fails to be H^s hypoelliptic, modulo its nullspace. □

Remark. It is natural to ask why Lemma 4 is needed. The simple counterexamples used in the proof of Lemma 1 to demonstrate the failure of the superlogarithmic estimate do not suffice to disprove hypoellipticity; they are not annihilated by $\bar\partial_b \bar\partial_b^*$, and the last term on the right-hand side of (34) turns out to be far larger than the left-hand side, so that no contradiction is reached.

A second explanation is that the related operators $-\partial_x^2 - (\phi'(x)\partial_t)^2$, in $\mathbb{R}^2$, are hypoelliptic so long as ϕ' vanishes only at one point; see [6] and the later works [4],[12], and for a more general analogue in the real analytic category see [7]. An argument like that used above to disprove the superlogarithmic estimate would apply equally to $-\partial_x^2 - (\phi'(x)\partial_t)^2$.

A third explanation is that if one runs the proof of Lemma 5 with $\zeta = \eta \in \mathbb{R}$, then $|f(x)|$ has size roughly $c \exp(c\epsilon\tau)$ for $x = \epsilon$, so the right-hand side of (35) is far larger than the left-hand side, and no contradiction results.

Remark. It is natural to ask about hypoellipticity for two closely related operators of the sum of squares type. These are

$$\begin{aligned} L &= -\partial_x^2 - \partial_y^2 - (\phi'(x)\partial_t)^2, \\ \tilde{L} &= -\partial_x^2 - (\partial_y - \phi'(x)\partial_t)^2. \end{aligned} \tag{48}$$

We believe that hypoellipticity is equivalent to the superlogarithmic estimate for both, under the supplementary hypothesis for $\tilde{L}$ that ϕ is convex. L is potentially easier to handle, since self-adjointness of the associated ordinary differential operators $-\partial_x^2 + \tau^2\phi'(x)^2$ translates into the existence of a genuine lowest eigenvalue $\lambda(\tau)^2$, and $\zeta(\tau)$ can be chosen to be $i\lambda(\tau)$; an eigenfunction of $-\partial_x^2 + \tau^2\phi'(x)^2$ then gives a solution of $-\partial_x^2 + \eta^2 + \tau^2\phi'(x)^2$. Convexity of ϕ is no longer a natural hypothesis, but it is elementary to show that for any ϕ, for large $\tau \in \mathbb{R}^+$, this lowest eigenvalue has order of magnitude

$$\lambda(\tau) \sim \inf_{I:\int_I |\phi'| = \tau^{-1}} |I|^{-1} \ , \tag{49}$$

where the infimum is taken over all bounded intervals I satisfying the stated equality.[2] If $\lambda(\tau)/\log\tau \to \infty$ as $\tau \to +\infty$, then L satisfies a superlogarithmic estimate, hence is C^∞ hypoelliptic [13],[4].

To show that L fails to be hypoelliptic when the superlogarithmic estimate fails to hold, would require an analogue of Lemma 5; we have not carried out that part of the analysis.

Remark. The second and more delicate cousin is $\tilde{L}$. We believe that for convex ϕ, the same criterion should be necessary and sufficient for its hypoellipticity as for $\bar\partial_b\bar\partial_b^*$. As in the analytic case [1], the appropriate analogue of $\mathcal{N}$ for these operators is the Wronskian of two properly normalized solutions of the associated ordinary differential equations, and everything should boil down to its having zeroes satisfying $|\operatorname{Im}(\zeta)| \le C\log\tau$. However, we have not carried out the analysis.

Remark. The main theorem is in accord with the general remarks in [3].

Remark. One could argue that this situation is parallel with that of constant-coefficient differential operators in $\mathbb{R}^n$, which are C^∞ hypoelliptic if and only if their (full) symbols have a sequence of zeros $\xi_\nu \in \mathbb{C}^n$ with $|\xi_\nu| \to \infty$ satisfying $|\operatorname{Im}(\xi_\nu)| \le C\log|\xi_\nu|$; see Theorem 11.1.3 of [8]. Of course, in the constant-coefficient setting, the formally weaker bound $|\operatorname{Im}(\xi_\nu)| \to \infty$ is equivalent [8] to the formally stronger bound $|\operatorname{Im}(\xi_\nu)| \ge c|\xi_\nu|^\delta$ for some $\delta > 0$, so there is a certain degree of fudging here.

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