SPIRALING AND NONHYPOELLIPTICITY FOR CR STRUCTURES DEGENERATE ALONG TRANSVERSE REAL CURVES

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1. Introduction

Consider a smooth, pseudoconvex CR structure on some small open set of real dimension three. A fundamental sufficient condition for an associated Cauchy-Riemann operator $\bar\partial_b$ to be C^∞ hypoelliptic, modulo its nullspace (this notion is defined below in §4) is that the CR structure should be of finite type, but this condition is far from necessary. While some sufficient conditions are known [2] (see also [16] for the higher-dimensional case), no useful complete characterization of hypoellipticity is known, or expected, in its absence.

Some sufficient conditions take a quantitative form. For example, one such condition is that the set of weakly pseudoconvex points should be contained in a manifold M of real dimension two that is everywhere transverse to the complex tangent space, and moreover the Levi form $\Lambda(z)$ should not tend to zero too rapidly as this degenerate manifold is approached; the critical threshold is distance $(z, M)|\log\Lambda(z)| \to \infty$ as $z \to M$. A more qualitative criterion is that points of infinite type should be isolated; then the rate of degeneration does not matter. It would be desirable to find other qualititative criteria. The main result of this paper is an example answering one specific question along these lines posed by J. J. Kohn.

Theorem 1.1. *There exists a smooth, pseudoconvex, embeddable three-dimensional CR structure for which the set of weakly pseudoconvex points consists of a single smooth real curve, which is everywhere transverse to the complex tangent space, yet such that $\bar\partial_b$ fails to be C^∞ hypoelliptic, modulo its nullspace.*

Such a result would be of modest interest, if the production of an example were simply a matter of making the Levi form degenerate sufficiently rapidly near the curve. It is not so; Theorem 1.2 states that for a certain class of such structures, hypoellipticity always holds provided that the size of the Levi form is a monotonic function of the distance to the curve, regardless of the rate at which it degenerates. And an extremely weak lower bound on the rate of degeneration suffices without monotonicity; see Theorem 1.3.

An established heuristic is that the formation and propagation of singularities should be closely related to complex disks located in $\mathbb{C}^2$ near the (embedded) CR

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manifold. The presence of a real curve γ on which the Levi form vanishes, *transverse* to the complex tangent space, might appear to have nothing to do with the presence of such disks; hence the hope for hypoellipticity. We prefer to work intrinsically on the CR manifold itself, rather than in a neighborhood of an embedded realization. One should then look for remnants of complex disks, consisting of sufficiently large pieces of manifolds of real dimension two, on which the Levi form is sufficiently small; bearing in mind that the meaning of "sufficiently" must be clarified. Our example has an infinite sequence of nearly Levi-flat open regions converging to γ; the "leaves" of these near-foliations wind around γ. An appropriate motion "tangent" to the "leaves" is microscopically transverse to γ, but is macroscopically parallel to it, as a result of repeated windings. The construction is based on an interplay between the rates of degeneration of the Levi form, and of the winding; the dangerous situation in which hypoellipticity fails is, in our example and according to the prediction of [1], when one loop around γ produces a comparatively large (though extremely small) degree of parallel motion, relative to the size of the Levi form. On the other hand, Theorem 1.4 states that if winding once around γ produces a parallel motion that is sufficiently small, relative to the size of the Levi form, then hypoellipticity does hold.

We emphasize that these results, both positive and negative, are predicted by a general but conjectural point of view formulated in [1]. Various works related to the construction in this paper, and likewise supporting those predictions, are listed in the bibliography. However, we have no general procedure for establishing the failure of hypoellipticity when predicted. Nor are those predictions entirely precise; they are intended only as a first-order guide to the truth. In the present paper, very special CR structures with certain symmetries are painstakingly constructed so as to support singular solutions, whose structure is not entirely obvious.

We conclude by indicating a few sufficient conditions for $\bar\partial_b$ to be C^∞ hypoelliptic, modulo its nullspace. With the exception of the second part of Theorem 1.3, these results, like the example in Theorem 1.1, concern a special class of CR structures, which we call *cylindrically symmetric*; in coordinates $(r,\theta,t) \in \mathbb{R}^+ \times [0,2\pi] \times \mathbb{R}$, the degenerate curve γ is the axis $r = 0$, and both translation with respect to t and rotation about γ are CR symmetries; the Levi form is thus a function $\Lambda(r)$ of r alone. See §2 for the precise definitions.

These theorems demonstrate (i) that the Levi form $\Lambda(r)$ can degenerate arbitrarily rapidly as $r \to 0$, without losing hypoellipticity, (ii) that (in the cylindrically symmetric case) hypoellipticity depends on an interplay between the sizes of $\Lambda(r)$ and a second quantity $\beta(r)$, rather than merely on the size of Λ alone, and (iii) that more rapid degeneration of Λ is needed near a real curve, than near a real hypersurface, to imperil hypoellipticity. These conditions are not exhaustive; more refined sufficient conditions can be obtained by the same method.

Theorem 1.2. *Consider any smooth, pseudoconvex, cylindrically symmetric three-dimensional CR structure, as described in §2. Suppose that the Levi form $\Lambda(r)$ vanishes only at $r = 0$, and is a nondecreasing function of $r > 0$. Then $\bar\partial_b$ is C^∞ hypoelliptic, modulo its nullspace.*

Theorem 1.3. *Consider any smooth, pseudoconvex, cylindrically symmetric three-dimensional CR structure. Suppose that there exists $M < \infty$ such that for all sufficiently small r, $\Lambda(r) \geq \exp(-r^{-M})$. Then $\bar\partial_b$ is C^∞ hypoelliptic, modulo its nullspace.*

More generally, consider any smooth, pseudoconvex three-dimensional CR structure for which the set of all points of infinite type forms a smooth real curve γ, everywhere transverse to the complex tangent space. Suppose that there exists $M < \infty$ such that for all points z sufficiently close to γ, $\Lambda(z) \geq \exp(-d(z)^{-M})$, where $d(z)$ denotes the distance from z to γ. Then $\bar\partial_b$ is C^∞ hypoelliptic, modulo its nullspace.

This should be contrasted with the corresponding result for the case where γ is a smooth real hypersurface; then the above condition suffices if $M < 1$, but not if $M = 1$ [11].

The next variant emphasizes the interplay between the magnitudes of Λ and a second quantity, which is related to the winding motion mentioned above.

Theorem 1.4. *Consider any smooth, pseudoconvex, cylindrically symmetric three-dimensional CR structure, as described in §2. Let*

$$\beta(r) = r^{-1} \int_0^r \rho\Lambda(\rho)\, d\rho \ .$$

If there exists $M < \infty$ such that

$$\Lambda(r) \geq e^{-M/\beta(r)}$$

for all sufficiently small $r > 0$, then $\bar\partial_b$ is C^∞ hypoelliptic, modulo its nullspace.

Remark. There are two different, though related, notions of hypoellipticity for $\bar\partial_b$. One, natural for a discussion of the canonical (global) solution of $\bar\partial_b u = f$ on a closed CR manifold without boundary, is hypoellipticity relative to its nullspace. The other notion is microhypoellipticity of the second-order operator $\Box_b = \bar\partial_b\bar\partial_b^*$, in the usual sense of wavefront sets and microlocal analysis. A close examination of our example reveals that $\Box_b$ also fails to be microhypoelliptic, on the nontrivial half of its characteristic variety (where the principal symbol of $[\bar\partial_b, \bar\partial_b^*]$ is nonnegative); failure of microhypoellipticity of the first-order operator $\bar\partial_b$ near the other half is quite easy to establish and requires none of the intricacy of our construction. Details are left to the reader.

2. Cylindrically symmetric CR structures

Consider a smooth CR structure in a neighborhood of the origin in $\mathbb{R}^3$, for which a Cauchy-Riemann operator takes the form $\bar\partial_b = X + \imath Y$ where, in coordinates (x, y, t),

$$X = \frac{\partial}{\partial x} + \frac{\partial a}{\partial y}\frac{\partial}{\partial t}$$
$$Y = \frac{\partial}{\partial y} - \frac{\partial a}{\partial x}\frac{\partial}{\partial t}$$

with $a = a(x, y)$ a real-valued C^∞ function independent of t. Thus translation with respect to t is one symmetry of the CR structure. Any such structure is embeddable; both $x + \imath y$ and $t + \imath a(x, y)$ are CR functions.

Since $[X,Y] = \Delta_{x,y}a \cdot \partial/\partial_t$, the Levi form is the function $\Delta_{x,y}a$. Assume that $\Delta_{x,y}a \geq 0$, and moreover that $\Delta_{x,y}a = 0$ if and only if $(x,y) = 0$. Then the CR structure is pseudoconvex, and the set of weakly pseudoconvex points consists of the single real curve $\gamma = \{(0,0,t) : t \in \mathbb{R}\}$, which is everywhere transverse to the complex tangent space $T^{1,0} \oplus T^{0,1}$. We will choose a to vanish to infinite order at the origin.

Introduce "polar" coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ with $r \geq 0$. Assume

$$a(x,y) = b(r), \tag{2.1}$$

and that $b \in C^\infty([0,1])$, and that b vanishes to infinite order at $r = 0$. Since $\Delta_{x,y}a = b''(r) + r^{-1}b'(r)$, the pseudoconvexity hypothesis becomes $b'' + r^{-1}b' > 0$ for all $r > 0$. Rotation about the axis $r = 0$ is a second CR symmetry.

The quantity $b'(r)$ was called $\beta(r)$ in the statement of Theorem 1.4, and can be expressed in terms of the Levi form:

$$b'(r) = r^{-1}\int_0^r \rho\Lambda(\rho)\,d\rho\ .$$

Alternatively, $\Lambda(r) = \beta'(r) + r^{-1}\beta(r)$.

Consider the vector field that, in coordinates (r,θ,t), equals $\cos(\theta)Y - \sin(\theta)X$. Its integral curves wind around γ; the radius r and hence the size of the Levi form Λ, the rate of change of θ, and the velocity $\beta(r) = b'(r)$ in the t direction all remain constant. This is the spiraling motion of our title.

3. Coordinates and separation of variables

Pass to coordinates (s,θ,t) where

$$s = \log r.$$

Set

$$b(r) = \Phi(s).$$

Since $b''(r) + r^{-1}b'(r) = e^{-2s}\Phi''(s)$, the pseudoconvexity hypothesis becomes

$$\Phi''(s) > 0 \qquad \text{for all } s \in (-\infty, 0]. \tag{3.1}$$

Our aim is to construct Φ such that $\bar\partial_b$ fails to be C^∞ hypoelliptic, in an appropriate sense to be specified below. Among the properties of the convex function Φ will be that $\Phi(s) > 0$ and $\Phi'(s) > 0$ for all s. The infinite order vanishing of b at $r = 0$ is equivalent to

$$\frac{d^k\Phi}{ds^k} = O(e^{Ns}) \qquad \text{as } s \to -\infty, \tag{3.2}$$

for all $k, N \geq 0$. Φ will be constructed to satisfy (3.2), thus guaranteeing that $a \in C^\infty$. This also ensures that $\Delta_{x,y}a(0) = 0$, so that the CR structure is merely weakly pseudoconvex on the axis $x = y = 0$.

Choose a Hermitian structure for which $\bar\partial_b^* = -(X - \imath Y)$. Then $-\Box_b = -\bar\partial_b\bar\partial_b^* = (X + \imath Y)\circ(X - \imath Y)$.

The two CR symmetries permit a separation of variables in solving $\Box_b u = 0$; we consider functions $u(x, y, t) = \exp(\imath\tau t + \imath k\theta) f(s)$. For any $k \in \mathbb{N}$ and $\tau \in \mathbb{C}$,

$$-e^{-\imath\tau t - \imath k\theta}\Box_b(e^{\imath\tau t + \imath k\theta} f(s)) = \mathcal{L}_{k,\tau} f(s) \tag{3.3}$$

where

$$\mathcal{L}_{k,\tau} = e^{-2s}\big(\partial_s - k + \tau\Phi'\big)\big(\partial_s + k - \tau\Phi'\big), \tag{3.4}$$

and the coordinates (s, θ, t) are related to (x, y, t) as above. Defining likewise

$$\bar{d}_b^* f(s) = e^{-\imath\tau t - \imath k\theta}\bar{\partial}_b^*(e^{\imath\tau t + \imath k\theta} f(s)), \tag{3.5}$$

one has

$$\bar{d}_b^* = e^{-s}(\partial_s + k - \tau\Phi')\ . \tag{3.6}$$

This operator also depends on the parameters k, τ, though this dependence is not made explicit in the notation.

4. Hypoellipticity and *a priori* inequalities

$\bar{\partial}_b$ is said to be hypoelliptic modulo its nullspace, in an open set V_0, if whenever u and $\bar{\partial}_b^* u$ belong to L^2 and $\bar{\partial}_b\bar{\partial}_b^* u \in C^\infty$ in some open subset $V \subset V_0$, then likewise $\bar{\partial}_b^* u \in C^\infty(V)$. The naturality of this notion is made apparent in [15] and [13]: on a compact, embeddable CR manifold without boundary, for any $f \in L^2$ belonging to the range of $\bar{\partial}_b$, there exists a solution $g \in L^2$ orthogonal to the nullspace of $\bar{\partial}_b$ of $\bar{\partial}_b g = f$; moreover, g may be expressed as $\bar{\partial}_b^* u$ for some $u \in L^2$.

It is convenient to disprove hypoellipticity by an argument by contradiction. If $\bar{\partial}_b$ is hypoelliptic modulo its nullspace in an open set, then the following *a priori* inequalities must hold: For any open subset V, any relatively compact subset V', and any α, there must exist $M, C < \infty$ such that for any $u \in C^\infty(V)$,

$$\|\partial_{x,y,t}^\alpha \bar{\partial}_b^* u\|_{C^0(V')} \le C\|u\|_{C^0(V)} + C\|\bar{\partial}_b^* u\|_{C^0(V)} + C\|\bar{\partial}_b\bar{\partial}_b^* u\|_{C^M(V)}. \tag{4.1}$$

It is a consequence of the Baire category theorem that hypoellipticity modulo the nullspace implies such families of inequalities.[1]

Restricting attention to functions $u(x, y, t) = \exp(\imath\tau t + \imath k\theta) f(s)$, a consequence would be that for any $\epsilon > 0$ there exist $M, C < \infty$ such that for any $\tau \in \mathbb{C},\ k \in \mathbb{N}$, and for any function $f(s)$ such that $\exp(\imath\tau t + \imath k\theta) f(s)$ is a C^∞ function of $(x, y, t) \in V$, for all $s \le -1$,

$$\begin{aligned} |\tau||\bar{d}_b^* f(s)| \le Ce^{\epsilon|\operatorname{Im}(\tau)|}\|f\|_{C^0} + Ce^{\epsilon|\operatorname{Im}(\tau)|}\|\bar{d}_b^* f\|_{C^0} \\ + Ce^{\epsilon|\operatorname{Im}(\tau)|}\sum_{|\beta| \le M}\|(e^{-s}\partial_s, e^{-s}k, |\tau|)^\beta \mathcal{L}_{k,\tau} f\|_{C^0}. \end{aligned} \tag{4.2}$$

[1]Alternatively, it is straightforward to verify that an appropriate infinite linear combination u of the sequence of trial functions constructed below satisfies $\Box_b u \in C^\infty$ and $u, \bar{\partial}_b^* u \in L^2$, but $\bar{\partial}_b^* u \notin C^\infty$.

This arises from (4.1) by considering a neighborhood V of the origin in which $|t| < \epsilon$. If we disprove (4.2), we will actually have proved that $\Box_b u \in C^\infty$ does not even imply[2] that $\bar\partial_b^* u \in C^1$, let alone C^∞.

We will always work with τ having $\mathrm{Re}\,(\tau) > 0$. It will be essential for the construction to have $\mathrm{Im}\,(\tau) \neq 0$. A certain tradeoff results, since the factors $\exp(\epsilon|\,\mathrm{Im}\,(\tau)|)$ on the right-hand side of (4.2) can then potentially become large. In order to have any reasonable hope of violating (4.2), we must thus ensure that $|\,\mathrm{Im}\,(\tau)| \le C \log|\tau|$.

5. Organization of the construction

Before embarking on the details, we outline the construction, oversimplifying some details in order to focus on the main line. Φ will take the form

$$\Phi = w + \sum_{j=1}^{\infty} \psi_j \tag{5.1}$$

where ψ_j, w are nonnegative, convex functions of s, all of whose derivatives tend to zero faster than any power of $\exp(s)$ as $s \to -\infty$, and the series will converge rapidly together with all the differentiated series, so that $\Phi(s)$ will define a C^∞, radial function $a(x,y)$. The term w will be strictly convex, and will tend to zero with enormous rapidity; its only purpose is to guarantee strict pseudoconvexity where $(x,y) \neq 0$, without significantly affecting either side of the *a priori* hypoellipticity inequalities (4.2) for the trial functions under consideration.

The functions ψ_j will be constructed by induction on j. For each $n \in \mathbb{N}$, parameters σ, k, τ and a function f will be chosen so that $\mathcal{L}_{k,\tau}^{\Phi_n} f \equiv 0$, where $\mathcal{L}_{k,\tau}^{\Phi_n}$ is defined by replacing Φ by $\Phi_n = \sum_{j=1}^n \psi_j$ in the definition of $\mathcal{L}_{k,\tau}$. Thus $\mathcal{L}_{k,\tau} f$ will not vanish identically, so the last of the three terms on the right-hand side of (4.2) will be nonzero. In order to make the contribution of $\psi_{n+1}, \psi_{n+2}, \dots$ to $\mathcal{L}_{k,\tau} f$ small, we will make those functions themselves sufficiently small, at future steps of the induction.

The most delicate role in the construction falls to (essentially but not exactly) $\mathrm{Im}\,(\tau)$, a tuning parameter governing the behavior of f. It will be chosen to be a zero of a certain holomorphic function, and is directly analogous to the nonlinear eigenvalues on which certain analyses of nonhypoellipticity in the real analytic category rest. See for example [3, 4, 5, 6, 9]. In those works, the existence of appropriate nonlinear eigenvalues is proved by indirect arguments. The smooth category is much less rigid than the real analytic category; in this paper we exploit that lack of rigidity by building in the nonlinear eigenvalues by hand.

6. Building blocks

In this section we introduce and analyze the basic building blocks of the construction. Choose a large parameter $\sigma > 0$, sufficiently large to satisfy various requirements to be imposed on it below. All the other parameters k, λ, ρ, ζ to be introduced below are functions of σ.

[2]In fact, the analysis shows that $\bar\partial_b^* u$ need not be Hölder continuous of any positive order. It is likely that arbitrarily weak types of regularity could be disproved, by refining the choices of parameters in §§6,7.

Define a very large positive integer k by[3]

$$\log\log k = \sigma\log\sigma\ . \tag{6.1}$$

Here we require σ to be chosen so that k is an integer; every other requirement imposed on σ will simply be that it is sufficiently large. Define a still larger positive real number λ by

$$\frac{\lambda}{\log\lambda} = k\sigma\ . \tag{6.2}$$

Let $\tau = \lambda + \imath\zeta$, where $\zeta \in \mathbb{C}$ is to be specified; ζ is not necessarily real.

Define

$$\psi(s) = \begin{cases} 0 & \text{if } s \le -5\sigma \\ k\lambda^{-1}(s+5\sigma) & \text{if } -5\sigma \le s \le -3\sigma \\ 2k\lambda^{-1}(s+4\sigma) & \text{if } -3\sigma \le s \le 0. \end{cases} \tag{6.3}$$

Then ψ is continuous, piecewise linear, nondecreasing, and convex.

We have remarked that any counterexample to hypoellipticity must have a non-monotonic Levi form; this is something of a red herring, and a more essential condition, phrased here very loosely, is simultaneous largeness of ψ' and smallness of ψ''. Here $\Lambda(r) = e^{-2s}\psi''$ is indeed highly non-monotonic; it is infinite at $s = -3\sigma, -5\sigma$ (though it is quite small in the sense of measures), and zero elsewhere. More essentially, for $s \in [-5\sigma, -3\sigma]$, $\psi'' \equiv 0$, while $\psi' > 0$. It will be smoothed out slightly in the next section, near -5σ and -3σ, but will retain this character.

Throughout the present section, let

$$\phi = \psi. \tag{6.4}$$

In later sections, ψ will be merely the main part of ϕ, which will have other constituents.

Define

$$h(s) = ks - \tau\phi(s), \tag{6.5}$$

$$f(s) = e^{-h(s)}\int_{-\infty}^{s} e^{2h(t)}\,dt\ . \tag{6.6}$$

Then

$$\bar{d}_b^* f = e^{h(s)}\cdot e^{-s},$$
$$\mathcal{L}_{k,\tau}^{\phi} f \equiv 0.$$

One consequence of this definition is that $f(s) = (2k)^{-1}e^{ks}$ for all $s \le -3\sigma$, whence $\exp(\imath\tau t + \imath k\theta)f(s) = (2k)^{-1}\exp(\imath\tau t)[re^{\imath\theta}]^k$ is a smooth function of (x,y,t).

[3]There is some arbitrariness in the choice of k. An absolute requirement is that $|\operatorname{Im}(\tau)| \le C\log|\tau|$. We began by aiming to have $|\operatorname{Im}(\tau)| \sim c\log|\tau|$, then worked backwards from the analysis of $\mathcal{N}$ to derive (6.2), then worked out a value of k that would satisfy various other requirements. Beginning with a different relation between the imaginary and real parts of τ would lead to different formulas for k, λ; so far as we know, the construction should still succeed with other such relations, so long as $|\operatorname{Im}(\tau)| \le C\log|\tau|$.

Define

$$\mathcal{N}(\zeta) = \int_{-\infty}^{0} e^{2[ks-(\lambda+\imath\zeta)\phi(s)]}\, ds \tag{6.7}$$

for all $\zeta \in \mathbb{C}$. If $\mathcal{N}(\zeta) = 0$, then f admits the alternative representation

$$f(s) = -e^{-h(s)} \int_{s}^{0} e^{2h(t)}\, dt, \tag{6.8}$$

which will be essential to estimating f for $s > -3\sigma$. If $\zeta = 0$, so that $\tau = \lambda > 0$ is real, then it is easily verified that f is an increasing function of s; when λ is large, $f(s)$ will be much larger at $s = 0$ than for s near $-\infty$, and this would thwart our efforts to contradict the *a priori* inequalities (4.2). Those inequalities do hold for s in any compact set, because $\bar\partial_b$ is indeed hypoelliptic modulo its nullspace, away from $r = 0$.

In order to make $\mathcal{N}(\zeta) = 0$, we wish to make $\mathrm{Re}\,(\zeta)$ large, to introduce some oscillation and hence cancellation into the integral. However, there is a price to be paid: the larger is $|\,\mathrm{Re}\,(\zeta)|$, the larger becomes the right-hand side of (4.2), threatening to thwart our purpose.

One computes

$$\mathcal{N}(\zeta) = e^{-10k\sigma} \cdot (I + II + III) \tag{6.9}$$

where, if $z = (2k\sigma\lambda^{-1}) \cdot \zeta = 2\zeta/\log\lambda$,

$$\begin{cases} I & = (2k)^{-1} \\ II & = 2\sigma z^{-1} e^{-\imath z} \sin(z) \\ III & = e^{-\imath 2z}(2k + \imath 2z\sigma^{-1})^{-1}(1 - e^{-6k\sigma - \imath 6z}). \end{cases} \tag{6.10}$$

If z is bounded, and bounded away from zero, then the principal term is II, since k, σ are large. Since $II = 0$ when $z = \pi$, we thus seek to prove the existence of a unique z satisfying $\mathcal{N}(\zeta) = 0$, with z in a small neighborhood of π.

Define

$$\rho = \exp(-\sigma \log\log\sigma); \tag{6.11}$$

this choice is not natural for the analysis of this section, but will be forced on us by the considerations of §7. Let B_ρ be the ball of radius ρ in $\mathbb{C}$, centered at π. ζ is a function of z; $\mathcal{N}(\zeta)$ is thereby a holomorphic function of $z \in B_\rho$. For $z \in \partial B_\rho$, $II = c_0\sigma(z - \pi + O(\rho^2))$, where c_0 is a nonzero constant, while $|I| = (2k)^{-1}$ and $|III| \le Ck^{-1}$, provided that σ is sufficiently large. Since $(k\sigma)^{-1} = \sigma^{-1}\exp(-\exp(\sigma\log\sigma))$ is much smaller than ρ, Rouché's theorem guarantees that there exists a unique $\zeta \in \mathbb{C}$ satisfying

$$\mathcal{N}(\zeta) = 0 \text{ and } |\zeta - \frac{\pi}{2}\log\lambda| \le \rho\log\lambda. \tag{6.12}$$

In particular, $|\zeta| \le 2\log\lambda$, provided σ is sufficiently large, and

$$|\,\mathrm{Im}\,(\zeta)| \le \rho\log\lambda \ll \log\lambda. \tag{6.13}$$

In order to analyze the validity of (4.2), we require bounds on the various terms appearing there. For all σ sufficiently large,

$$|\bar{d}_b^* f(-4\sigma)| \geq e^{-5k\sigma} \tag{6.14}$$

$$\sup_s |\bar{d}_b^* f(s)| \leq Ce^{-5k\sigma+5\sigma} \tag{6.15}$$

$$\sup_s |f(s)| \leq C\sigma e^{-5k\sigma} \tag{6.16}$$

where $C < \infty$ is independent of σ.

In justifying these, it will be useful to know that the factor $\exp(-\imath\zeta\phi(s))$ is bounded uniformly in s, σ; in fact, its absolute value approaches 1 as $\sigma \to \infty$. Indeed, $\phi(s) \equiv 0$ for $s \leq -5\sigma$, and for $-5\sigma \leq s \leq -3\sigma$, the real part of the exponent equals $k\lambda^{-1}(s+5\sigma)\,\mathrm{Im}\,(\zeta)$, whose absolute value is $\leq Ck\sigma\lambda^{-1} \cdot \rho \log \lambda = C\rho$, which tends rapidly to 0 as $\sigma \to \infty$. For $s \geq -3\sigma$, the exponent becomes $-\imath\zeta k\lambda^{-1} \cdot 2(s+4\sigma)$, to which the same analysis applies.

Thus $\bar{d}_b^* f = \exp(-s+ks-\lambda\phi(s)) \cdot \exp(-\imath\zeta\phi(s))$ has absolute value bounded above and below by constant multiples of $\exp(-s+ks-\lambda\phi(s))$. We have

$$e^{-s+ks-\lambda\phi(s)} = \begin{cases} e^{(k-1)s} & s \leq -5\sigma \\ e^{-5k\sigma-s} & -5\sigma \leq s \leq -3\sigma \\ e^{-8k\sigma-ks-s} & s \geq -3\sigma \ . \end{cases}$$

The maximum value is $\exp(-5k\sigma+5\sigma)$, assumed at $s = -5\sigma$. For $s = -4\sigma$, this last quantity equals $\exp(-5k\sigma+4\sigma)$. This establishes the first two of the above three bounds.

Lastly, consider f itself. By virtue of the uniform boundedness of $\exp(-\imath\zeta\phi(s))$,

$$|f(s)| \leq \tilde{f}(s) = e^{-ks+\lambda\phi(s)} \int_{-\infty}^{s} e^{2(kt-\lambda\phi(t))}\,dt \ . \tag{6.17}$$

Now $kt - \lambda\phi(t)$ is a nondecreasing function of t in $(-\infty, -3\sigma]$, so

$$\tilde{f}(s) \leq \int_{-\infty}^{s} e^{kt-\lambda\phi(t)}\,dt \qquad \text{for all } s \leq -3\sigma.$$

Thus for $s \leq -5\sigma$, $\tilde{f}(s) \equiv k^{-1}\exp(ks) \leq \exp(-5k\sigma)$. For $s \in [-5k\sigma, -3k\sigma]$,

$$\begin{aligned}\int_{-\infty}^{s} e^{kt-\lambda\phi(t)}\,dt &= k^{-1}e^{-5k\sigma} + \int_{-5\sigma}^{s} e^{-5k\sigma}\,dt \\ &= k^{-1}e^{-5k\sigma} + (s+5\sigma)e^{-5k\sigma} \leq (2\sigma+1)e^{-5k\sigma} \ ,\end{aligned}$$

whence $\tilde{f}(s) \leq (2\sigma+1)\exp(-5k\sigma)$. Lastly, for $s \geq -3\sigma$, we invoke the alternative representation (6.8) to majorize $|f(s)|$ in the same way by

$$Ce^{-ks+2k(s+4\sigma)} \int_{s}^{0} e^{2[kt-2k(t+4\sigma)]}\,dt \leq C \int_{s}^{0} e^{-kt-8k\sigma}\,dt \leq Ck^{-1}e^{-8k\sigma}e^{ks} \leq Ce^{-5k\sigma}.$$

This concludes the analysis of our basic building blocks.

7. Two modifications

The construction in §6 suffers from two defects. Firstly, ψ is not smooth. Secondly, we will be unable to make the final inductive scheme succeed, without taking into account in the construction of each building block ψ_j all preceding steps of the construction. In this section, we modify the construction to remedy both these defects.

Consider a mollification $\tilde\psi$ of ψ; define $\tilde\psi(s)$ to equal $\psi(s)$ unless[4] $|s+5\sigma| \le \rho$ or $|s+3\sigma| \le \rho$. In those two intervals, modify ψ so that $\tilde\psi$ is convex and smooth, $\tilde\psi \ge \psi$, $\tilde\psi \le \psi + 2k\lambda^{-1}|s+5\sigma|$ or $\le \psi + 2k\lambda^{-1}|s+3\sigma|$, respectively, $0 \le \tilde\psi'(s) \le 2k\lambda^{-1}$ for all s, and $\tilde\psi$ satisfies natural bounds in those intervals:

$$\Big|\frac{d^n\tilde\psi}{ds^n}\Big| \le C_n\rho^{1-n}k\lambda^{-1} \tag{7.1}$$

for every $n \ge 0$.

We claim that

$$|\frac{d^n\psi}{ds^n}(s)| \le C_{M,n}e^{-\sigma}e^{Ms} \text{ for every } M, n, \text{ for all } s \in (-\infty, 0], \tag{7.2}$$

uniformly for all sufficiently large σ. Indeed, since

$$\begin{aligned}\rho^{1-n}k\lambda^{-1} \le \rho^{-n}k\sigma\lambda^{-1} &= \exp[n\sigma\log\log\sigma - \log\log\lambda] \\ &\le \exp[n\sigma\log\log\sigma - \log\log k] \\ &= \exp[n\sigma\log\log\sigma - \sigma\log\sigma] \le C_n\exp[-\tfrac{1}{2}\sigma\log\sigma],\end{aligned}$$

(7.1) implies that $\tilde\psi$ satisfies (7.2) on the two intervals where it differs from ψ. Elsewhere, $\tilde\psi = \psi$ is linear. Its first derivative vanishes for $s \le -5\sigma$, and elsewhere equals either k/λ or $2k/\lambda$; $k/\lambda = 1/\sigma\log\lambda \le 1/\log k \le \exp(-\sigma\log\sigma)$ satisfies the required bound. The reasoning for $n = 0$ is the same.

We claim that all the bounds of §6 remain valid, if ψ is replaced by $\tilde\psi$, provided that ζ is such that $z = z(\zeta) \in B_\rho$. The crux of the analysis is $\exp(10k\sigma)\cdot\mathcal{N}$. Terms I, III are now strictly smaller than before, because τ has positive real part, and $\tilde\psi \ge \psi$. For any $z \in B_\rho$, the new principal term II differs from the old one by no more than $C\rho$, because both the new and the old integrands in the integral defining $\mathcal{N}$ are uniformly bounded multiples of $\exp(-10k\sigma)$. Thus $|II - c_0\sigma[z - \pi + O(\rho^2)]| \le C\rho$, with C independent of σ. Therefore for all sufficiently large σ, Rouché's theorem again applies on B_ρ.

We also require the three upper and lower bounds (6.14), (6.15), (6.16) on $f, \bar d_b^* f$. The upper bound on $\bar d_b^* f$ continues to hold, because $\exp(ks - \lambda\tilde\psi(s)) \le \exp(ks - \lambda\psi(s))$; the lower bound at $s = -4\sigma$ is unaffected because $\tilde\psi(-4\sigma) = \psi(-4\sigma)$. To analyze the upper bound for $f(s)$ for $s \le -3\sigma$, we pass to $\tilde f$ as in (6.17), and again majorize $\tilde f(s) \le \exp(ks - \lambda\tilde\psi(s))$ for all $s \le -3\sigma$. Since $\tilde\psi \ge \psi$, we obtain exactly the same bound as before.

[4] ρ here enjoys a role unrelated to its part in the preceding section; it is happenstance that the same quantity does both jobs.

Our second modification is to consider more general phases ϕ. Take

$$\phi = \psi + \varphi, \tag{7.3}$$

where ψ now denotes the function denoted above by $\tilde{\psi}$, and φ is some function which is considered to be known at the outset, before σ is chosen. Our aim is to show that if σ is chosen to be sufficiently large, then all bounds obtained previously remain valid, uniformly in σ, φ, provided only that $\sigma \geq C(\varphi)$, a constant which is permitted to depend on φ. Thus ψ will depend on φ.

We assume that φ is a nonnegative, C^∞ convex function of $s \in (-\infty, 0]$, having compact support. We impose on σ the additional requirement that it be sufficiently large that the support of φ is contained in $[-\sigma, 0]$. In the analysis of $\mathcal{N}(\zeta)$, only term III is changed. Since $\varphi \geq 0$, the exponent $-\tau k \lambda^{-1} \varphi$ has nonpositive real part, and hence the absolute value of the integrand in this region is no larger than it was before. Therefore term III satisfies the same upper bound, and hence the existence proof and resulting bounds for ζ are unchanged. The real part of $\imath \zeta k \lambda^{-1} \varphi(s)$ has absolute value $\leq C\rho \log \lambda \cdot k\lambda^{-1}$, where C depends on φ; this is $\leq C\rho\sigma^{-1}$, which tends to zero rapidly as $\sigma \to \infty$. So for all σ greater than some threshold depending on φ, $\exp(\imath\zeta k \lambda^{-1}\varphi(s))$ is bounded, uniformly in s, σ, φ. Retracing the analysis of the bounds for $\bar{d}_b^* f, f$, one finds that the same estimates remain valid, provided always that σ is sufficiently large relative to φ.

8. Conclusion of Proof

Construct Φ as follows. Set $\Phi = \Psi + w$ where $\Psi = \sum_{\jmath=1}^\infty \psi_\jmath$ and $w = \sum_{\jmath=1}^\infty w_\jmath$. The functions $\psi_\jmath, w_\jmath$ are constructed by induction; each $\psi_\jmath$ is constructed as in §6, with the two modifications of §7, and in particular is associated to a parameter $\sigma_\jmath$. We require that $\sigma_\jmath \geq 2^\jmath$, just to ensure that $\sigma_\jmath \to \infty$. We also require that $w_\jmath$ be nonnegative and convex, $w_{\jmath+1}(s) \equiv 0$ for all $s < -2\sigma_\jmath$, and $w''_{\jmath+1}(s) > 0$ for all $s \geq -\sigma_\jmath$.

Set $w_1 \equiv 0$. The inductive step is to construct first ψ_n, then w_{n+1}, given $\{\psi_\jmath,\ w_\imath\}$ for all $\jmath < n$ and $\imath \leq n$. Set $\varphi_n = \sum_{\jmath=1}^{n-1} \psi_\jmath + \sum_{\imath=1}^n w_\imath$. Choose a large parameter $\sigma_n \geq 2^n$, and in terms of it define $k_n, \lambda_n, \rho_n, \zeta_n$ and $\psi = \psi_n$ as in §§6,7. Setting $\phi = \phi_n = \psi_n + \varphi_n$, we have then a solution $f_n(s) = \exp(-h(s)) \int_{-\infty}^s \exp(2h(t))\,dt$ of $\mathcal{L}^{\phi_n}_{k_n,\tau_n} f_n \equiv 0$, where $h(s) = h_n(s) = k_n s - \tau_n \phi_n(s)$, and $\tau_n = \lambda_n + \imath\zeta_n$.

Choose σ_n sufficiently large that $|d^m\psi_n(s)/ds^m| \leq 2^{-n}e^{ns}$ for all s and all $0 \leq m \leq n$, and so that[5] $|\bar{d}_b^{*\phi_n} f_n(-4\sigma_n)| \geq \exp(-5k_n\sigma_n)$, $\|\bar{d}_b^{*\phi_n} f_n\|_{C^0} \leq C\exp(-5k_n\sigma_n + 5\sigma_n)$, and $\|f_n\|_{C^0} \leq C\sigma_n \exp(-5k_n\sigma_n)$, where C is a finite constant independent of n. Here $\bar{d}_b^{*\phi_n}$ is defined by replacing Φ by ϕ_n in the definition of $\bar{d}_b^*$, just as $\mathcal{L}^{\phi_n}$ is derived from $\mathcal{L}^\Phi$. That all the preceding inequalities hold for all sufficiently large σ_n, was proved in §§6,7.

Next, construct w_{n+1} to be a nonnegative, C^∞, convex function of s, satisfying $w_{n+1}(s) \equiv 0$ for all $s < -2\sigma_n$, and $w''_{n+1}(s) > 0$ for all $s \geq -\sigma_n$. In addition, choose $e^{-ns}w_{n+1}(s)$ to have C^n norm $\leq 2^{-n}$, so that the infinite series $\sum_\jmath w_\jmath$ converges in every C^m norm, as functions of the coordinate $r = \exp(s)$.

[5]Throughout this section, $\bar{d}_b^*$ is always defined relative to parameters k_n, τ_n; these have been suppressed in the notation.

Other conditions must also be imposed on ψ_n, w_{n+1}, in order that the last term on the right-hand side of (4.2) should not be too large. We wish to have

$$\sum_{|\beta|\le n} \|(e^{-s}\partial_s, e^{-s}k_n, |\tau_n|)^\beta \mathcal{L}^{\Phi}_{k_n,\tau_n} f_n\|_{C^0} \le e^{-5k_n\sigma_n}. \tag{8.1}$$

Indeed, we have $\mathcal{L}^{\phi_n}_{k_n,\tau_n} f_n \equiv 0$, but require control on $\mathcal{L}^{\Phi}_{k_n,\tau_n} f_n$ since it appears on the right-hand side of (4.2). We therefore impose on each ψ_n, w_{n+1} the supplemental condition

$$\sum_{|\beta|\le r} \|(e^{-s}\partial_s, e^{-s}k_r, |\tau_r|)^\beta \Big(\mathcal{L}^{\phi_n}_{k_r,\tau_r} - \mathcal{L}^{\phi_{n-1}}_{k_r,\tau_r}\Big) f_r\|_{C^0} \le 2^{-n} e^{-5k_r\sigma_r} \text{ for each } r < n.$$

(7.2) guarantees that this holds for $\big(\mathcal{L}^{\phi_{n-1}+\psi_n}_{k_r,\tau_r} - \mathcal{L}^{\phi_{n-1}}_{k_r,\tau_r}\big) f_r$, provided that σ_n is chosen to be sufficiently large. For the contribution $\big(\mathcal{L}^{\phi_n}_{k_r,\tau_r} - \mathcal{L}^{\phi_{n-1}+\psi_n}_{k_r,\tau_r}\big) f_r$ of w_{n+1}, it suffices to directly make w_{n+1} sufficiently small in C^{n+1} norm, since $f_r \in C^\infty$ and w_{n+1} is supported in a specified compact set. Now (8.1) follows by summing over all $n' > n$.

In the same way, we can and do choose ψ_n, w_{n+1} so that for each $r < n$,

$$\|(\bar{d}_b^{*\phi_n} - \bar{d}_b^{*\phi_{n-1}}) f_r\|_{C^0} \le 2^{-n-1} e^{-5k_r\sigma_r}\ .$$

Here $\bar{d}_b^{*\phi_n}$ is defined with respect to k_r, τ_r, not k_n, τ_n.

Putting this all together, we have the following estimates, uniformly in n:

$$|\bar{d}_b^* f_n(-4\sigma_n)| \ge \tfrac{1}{2} e^{-5k_n\sigma_n}$$

$$\|\bar{d}_b^* f_n\|_{C^0} \le C e^{5\sigma_n} e^{-5k_n\sigma_n}$$

$$\|f_n\|_{C^0} \le C\sigma_n e^{-5k_n\sigma_n}$$

$$\sum_{|\beta|\le n} \|(e^{-s}\partial_s, e^{-s}k_n, |\tau_n|)^\beta \mathcal{L}^{\Phi}_{k_n,\tau_n} f_n\|_{C^0} \le e^{-5k_n\sigma_n}.$$

For $f = f_n$, the left-hand side of (4.2), evaluated at $s = -4\sigma_n$, is therefore $\ge \frac{1}{2}|\tau_n| \exp(-5k_n\sigma_n)$. Since we have carefully ensured that $|\operatorname{Im}(\tau_n)| \le C \log|\tau_n|$, we may choose a fixed $\epsilon > 0$ such that $\exp(\epsilon|\operatorname{Im}(\tau_n)|) \le |\tau_n|^{1/2}$ for all n. The sum of the first two terms on the right is then $\le C|\tau_n|^{1/2}\exp(5\sigma_n)\exp(-5k_n\sigma_n)$. For any fixed M, the final term is $\le C|\tau_n|^{1/2}\exp(-5k_n\sigma_n)$. Thus (4.2) can only hold for large n if $|\tau_n| \le C|\tau_n|^{1/2}e^{5\sigma_n} + C|\tau_n|^{1/2}$. Since $|\tau_n| \sim \lambda_n \ge k_n = \exp(\exp(\sigma_n \log \sigma_n))$, this is impossible. Thus with our fixed choice of ϵ, no matter how large M is chosen to be, there exist a function f and parameters k, τ for which the *a priori* bound (4.2) does not hold. The proof is complete.

9. Proofs of positive results

We have the following general sufficient condition for hypoellipticity.

Theorem 9.1. *Consider a smooth, pseudoconvex three-dimensional CR structure, and let γ be a smooth real curve that is everywhere transverse to the complex tangent space. Suppose that the CR structure is strictly pseudoconvex at every point in the complement of γ. Suppose there exist a smooth real vector field T tangent to γ, and a* *real-valued function h defined in some tubular neighborhood of γ, such that $Th(z) > 0$ for all z, and such that for every $\epsilon > 0$ there exists $C < \infty$ such that for every $\tau \geq C$,*

$$\log\tau \cdot |\bar{\partial}_b(h)(z)| \leq \epsilon(1+\tau\Lambda(z))^{1/2}. \tag{9.1}$$

Then $\bar{\partial}_b$ is C^∞ hypoelliptic, modulo its nullspace.

The proof is based on the general results of [2], and in outline, goes as follows. Denote by $\Gamma \subset T^*\mathbb{R}^3\backslash(\mathbb{R}^3 \times \{0\})$ the characteristic variety associated to $\bar{\partial}_b$, that is, the set of all points at which the principal symbol of $\bar{\partial}_b$ vanishes. Γ is a line bundle over $\mathbb{R}^3$. Inside Γ is Γ_0, consisting of all fibers of Γ lying over the curve γ on which the Levi form vanishes; Γ_0 may equivalently be characterized as the subset of Γ on which the principal symbol of $[\bar{\partial}_b, \bar{\partial}_b^*]$ vanishes. Γ may be decomposed as a union of two ray bundles $\Gamma^\pm$, so that the principal symbol of $[\bar{\partial}_b, \bar{\partial}_b^*]$ is nonnegative in a conic neighborhood of Γ^+, and nonpositive in a conic neighborhood of Γ^-. Set $\Gamma_0^\pm = \Gamma_0 \cap \Gamma^\pm$.

Outside Γ, both $\bar{\partial}_b\bar{\partial}_b^*$ and $\bar{\partial}_b$ are microhypoelliptic, since they are elliptic. Moreover, as is well known, $\bar{\partial}_b\bar{\partial}_b^*$ is microhypoelliptic in any conic open subset of the cotangent bundle disjoint from Γ_0^-, while $\bar{\partial}_b$ is microhypoelliptic in any conic open subset of the cotangent bundle disjoint from Γ_0^+.

Let us fix coordinates $(p;q) = (x,y,t;\xi,\eta,\tau)$ in $T^*\mathbb{R}^3$, so that $\gamma = \{(x,y,t) : (x,y) = 0\}$, $\bar{\partial}_b$ belongs to the span of ∂_x, ∂_y at each point of γ, and (ξ,η,τ) is dual to (x,y,t). Then $\Gamma_0 = \{(0,0,t;0,0,\tau) : \tau \neq 0\}$. All functions discussed below are assumed to be supported in some small fixed neighborhood of γ, or of the cotangent bundle of such a neighborhood, as appropriate.

Under the hypotheses that the Levi form vanishes only on a curve γ transverse to the complex tangent space, there exists a (nonelliptic) symbol $w(q) \in S^1_{1,0}$ such that $w(q) \to +\infty$ as $|q| \to \infty$ and for any C^1 function u,

$$\|w(D)u\|^2 \leq C\|\bar{\partial}_b u\|^2 + C\|\bar{\partial}_b^* u\|^2 + C\|u\|^2, \tag{9.2}$$

where each norm is an L^2 norm. Here $w(D)$ denotes the pseudodifferential operator with symbol $w(q)$. This is a simple consequence of two facts. Firstly, for all u supported in the complement of any fixed neighborhood of γ, such an inequality holds, by subellipticity. Secondly, there is a Poincaré inequality: If U_δ is a tubular neighborhood of γ of width δ, then for any u,

$$\|u\|^2 \leq C\delta^2\|\bar{\partial}_b u\|^2 + C\delta^2\|\bar{\partial}_b^* u\|^2 + C\|u\|^2_{L^2(\mathbb{R}^3\backslash U_\delta)}.$$

It was shown in [2] that $\bar{\partial}_b$ is hypoelliptic modulo its nullspace, whenever it is sufficiently strong in the sense that an inequality (9.2) holds, with a symbol satisfying $w(q)/\log|q| \to \infty$ as $|q| \to \infty$. That condition holds if distance $(p,\gamma)|\log\Lambda(p)| \to 0$ as $p \to \gamma$; but we are now interested in establishing hypoellipticity in circumstances where Λ may vanish much more rapidly.

Combining this information with Theorems 2.3 and 2.4 of [2] and their proofs[6], plus the microlocalization procedure introduced by Kohn [13], plus Gårding's inequality,

[6]Those theorems are formulated for second-order operators, such as $\bar{\partial}_b\bar{\partial}_b^*$; here we also require the analogues for first-order operators, such as $\bar{\partial}_b$ (microlocally near Γ_0^-). The proofs of such analogues are parallel to the second-order case, with no additional complications.

we conclude that in order to prove that $\bar\partial_b$ is C^∞ hypoelliptic, modulo its nullspace, it establish the existence of certain auxiliary functions ψ. Given two distinct points $p_0, p_1 \in \gamma$, we seek an admissible C^∞ function ψ defined on $\mathbb{R}^3$, such that $\psi \equiv \jmath$ in a neighborhood of $p_\jmath$ for $\jmath = 0, 1$. Admissibility means that the pseudodifferential operator Ψ with symbol $\psi(p)\log(1+|q|^2)$ satisfies for every $\delta > 0$

$$\|[\Psi, \bar\partial_b]u\|^2 + \|[\Psi, \bar\partial_b^*]u\|^2 \le \delta\|\bar\partial_b u\|^2 + \delta\|\bar\partial_b u\|^2 + C_\delta\|u\|^2$$

for all $u \in C^1$ supported in a fixed open neighborhood of γ, and where all norms are L^2 norms. By Gårding's inequality and the microlocalized analysis in [13], this operator inequality would follow from the symbol condition

$$\begin{aligned}(9.3)\quad \log|\tau||\{\psi, \sigma(\bar\partial_b)\}(p,q)| + \log|\tau||\{\psi, \overline{\sigma(\bar\partial_b)}\}(p,q)| \\ \le \delta|\sigma(\bar\partial_b)(p,q)| + \delta|\sigma([\bar\partial_b, \bar\partial_b^*])(p,q)|^{1/2} + C_\delta \text{ for all } \delta > 0,\end{aligned}$$

where $\sigma(T)$ denotes the principal symbol of a pseudodifferential operator T. The notation $\{a, b\}$ denotes the Poisson bracket of two symbols $a(p,q), b(p,q)$. In certain proofs of hypoellipticity, a careful choice of ψ (or of a symbol $\psi(p,q)$) is essential, but in all the theorems stated above, any reasonable choice of ψ turns out to work and we will simply take ψ to be identically equal to a function of t alone in a small neighborhood of γ.

Now to complete the proof of Theorem 9.1 it suffices to take the auxiliary function ψ in the above discussion to have the form $g \circ h$, for an appropriate choice of g, where h is given in the hypotheses of Theorem 9.1.

Consider now the cylindrically symmetric case. Dropping the unhelpful term $\delta|\sigma(\bar\partial_b)|$ from the right-hand side of (9.3), our condition on $\psi(t)$ becomes simply

$$\log|\tau| \cdot b'(r) \le \delta|\tau|^{1/2}\Lambda(r)^{1/2} + C_\delta$$

for all $r > 0, \tau \ge 1$, for all small $\delta > 0$. Fixing a constant M depending on δ, this inequality holds automatically when $b'(r)\log|\tau| \le M$, and also when $|\tau|$ is less than any preassigned quantity. Thus what we need to show is that $\sup_{r,\tau} \frac{\log|\tau|}{|\tau|^{1/2}} \cdot \frac{b'(r)}{\Lambda(r)^{1/2}} \to 0$ as $A \to \infty$, where the supremum is taken over all pairs r, τ such that $|\tau| \ge A$ and $\log|\tau| b'(r) \ge M$. For fixed r, the supremum over all relevant τ occurs when $|\tau| = \exp(M/b'(r))$; then

$$\frac{\log|\tau|}{|\tau|^{1/2}} \cdot \frac{b'(r)}{\Lambda(r)^{1/2}} = \frac{M}{\Lambda(r)^{1/2}\, e^{M/2b'(r)}}\ .$$

Hence a sufficient condition for the existence of an auxiliary function ψ having the desired property is simply that there exist $M < \infty$ such that

$$\Lambda(r) \cdot \exp(M/b'(r)) \to \infty \text{ as } r \to 0\ .$$

This completes the proof of Theorem 1.4.

To deduce the first part of Theorem 1.3 from Theorem 1.4, it suffices to note that $b'(r) = O(r^n)$ for every $n < \infty$. As for Theorem 1.2, when $\Lambda(r) = r^{-1}(rb'(r))'$ is a monotonic function, we have $b'(r) = r^{-1}\int_0^r s\Lambda(s)\,dx \le r^{-1}\Lambda(r)\int_0^r s\,ds = \frac{1}{2}r\Lambda(r)$, so the hypothesis of Theorem 1.4 is satisfied.

The general case of Theorem 1.3 can be obtained from the same arguments. Choose coordinates (x,y,t) in which $\gamma = \{(0,0,t)\}$, and in which the real and imaginary parts X, Y of $\bar\partial_b$ lie in the span of ∂_x, ∂_y modulo $O(r^N)$ for every N, where still $r^2 = x^2+y^2$. This may be done by parametrizing γ as $\{\gamma(t)\}$, then introducing exponential coordinates $\exp(xX+yY)(\gamma(t))$. Then the above analysis applies, with no essential changes.

Remark. These arguments, based on the fact that the the quadratic form $Q(u,u) = \|\bar\partial_b u\|^2 + \|\bar\partial_b^* u\|^2 + \|u\|^2$ dominates the right-hand side of (9.1), have not exploited the full strength of Q. To go further, one could for instance exploit the uncertainty principle by arguing that because of the Poincaré-type inequality for tubular neighborhoods of γ used above, $Q^{1/2}$ dominates any pseudodifferential operator whose symbol belongs to $S^1_{1,0}$ and is $\le c\max_r \min\{r^{-1}, |\tau|^{1/2}\min_{\rho\ge r}\Lambda(\rho)^{1/2}\}$; thus it suffices to have $b'(r)\log|\tau|$ dominated by this quantity. Still sharper lower bounds can be obtained by pushing this reasoning to its natural conclusion, estimating a lower bound for $\partial_s + (k-\tau\phi'(s))$ in $L^2((-\infty,0],ds)$.

Remark. After this paper was completed, we discovered an additional idea which might potentially simplify the analysis and be applicable to a wide class of cylindrically symmetric structures, obviating the need for so much care in the construction. This approach has been worked out for tubular structures, degenerating along a single hyperplane, in [11]. As of this writing, its application to the cylindrically symmetric case is not yet complete.

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