LEBESGUE SPACE BOUNDS FOR ONE-DIMENSIONAL GENERALIZED RADON TRANSFORMS

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Abstract. An alternative proof is given of optimal mapping properties in L^p spaces, up to endpoints, for arbitrary Radon-like transforms involving integrals over one-dimensional manifolds.

1. Introduction

1.1. **Generalized Radon transforms.** Let $X, X^\star$ be open sets in smooth manifolds. Let $\mathcal{I} \subset X \times X^\star$ be a C^∞ submanifold of positive codimension, denote by $\pi, \pi^\star$ the restrictions to $\mathcal{I}$ of the natural projections of $X \times X^\star$ onto $X, X^\star$ respectively, and suppose that $\pi, \pi^\star$ are both submersions.

In this situation, in sufficiently small neighborhoods of any points $x_0 \in X$, $x_0^\star \in X^\star$ such that $(x_0, x_0^\star) \in \mathcal{I}$, the sets

$$\mathcal{M}_x = \{x^\star : (x, x^\star) \in \mathcal{I}\}, \quad \mathcal{M}^\star_{x^\star} = \{x : (x, x^\star) \in \mathcal{I}\} \tag{1.1}$$

are smooth manifolds of certain dimensions $k, k^\star$ respectively. There are associated integral operators

$$Tf(x) = \int_{\mathcal{M}_x} f \, d\sigma_x \tag{1.2}$$

where the measures σ_x are absolutely continuous with C^∞ densities with respect to k-dimensional measure on $\mathcal{M}_x$, and depend smoothly on x in the natural sense. Then for subsets $E, E^\star$ of $X, X^\star$ respectively, $\langle T(\chi_{E^\star}), \chi_E \rangle = \mu(\mathcal{I} \cap (E \times E^\star))$ for some smooth measure μ on $\mathcal{I}$.

Let $X, X^\star$ be equipped with positive measures having nonvanishing, smooth densities with respect to some local coordinate system. Thus we may speak of Lebesgue spaces $L^p(X), L^q(X^\star)$. T then has a transpose T^*, which is an operator of the same general form as T, associated to the dual family of submanifolds $\mathcal{M}^\star_{x^\star}$.

This paper presents an alternative proof, more accurately a partially alternative proof, of a theorem of Tao and Wright [37]. In terms of the geometry of $\mathcal{I}$, those authors characterized the interior of the set of all ordered pairs (p, q) for which an operator T maps L^p to L^q, *provided* that the manifolds $\mathcal{M}_x, \mathcal{M}^\star_{x^\star}$ are *one-dimensional* (equivalently, $X, X^\star$ have equal dimensions and the dimension of $\mathcal{I}$ is one greater). Our analysis revolves around a structural element of the problem which seems not to have been exploited in prior works. One motivation is to remove certain losses which prevent the attainment of endpoints, that is, pairs (p, q) of exponents on the boundary of the region for which an inequality might hold. Endpoint inequalities are not established in this paper, however; to obtain these will require additional independent refinements, on which work is underway. A second motivation, more fundamental though more speculative, is the hope that further insight into

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this one-dimensional case may eventually contribute to progress on the higher-dimensional problem. With this goal in mind we present alternative developments of some aspects of the theory.

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1.2. **Counting incidences.** Let $X, X^\star$ be sets, and suppose $\mathcal{I}$, the set of incidences, to be some prescribed subset of $X \times X^\star$. In various situations it is desired to find an upper bound on the cardinality of $\mathcal{I}$. Sometimes one is given infinite sets $X, X^\star, \mathcal{I}$, and one seeks to understand as a function of $N, N^\star$ the maximum number of incidences in $E \times E^\star$, where $E, E^\star$ are arbitrary subsets of $X, X^\star$ respectively of cardinalities $N, N^\star$. The Szemerédi-Trotter theorem [36], for instance, provides an upper bound in terms of $N, N^\star$ for the maximal number of incidences between N lines and $N^\star$ points in $\mathbb{R}^2$. Our problem is a continuum version of the general incidence counting problem, in which cardinality is replaced by Lebesgue measure, and the incidence relation $\mathcal{I}$ has the structure of a smooth manifold. The underlying problem is to relate differential geometric structure to extremal bounds for incidences.

1.3. **L^p-improving operators and curvature.** One should think of $X, X^\star$ as being sufficiently small neighborhoods of $x_0, x_0^\star$ respectively; all these structures exist in certain neighborhoods of $x_0, x_0^\star, (x_0, x_0^\star)$. Throughout the paper we will work in such a small neighborhood without explicitly saying so. Thus it is implicitly assumed that the measures σ_x are supported near x_0, that x is restricted to lie sufficiently close to x_0 that the sets $\mathcal{M}_x$ are indeed smooth manifolds, and so forth. In the case where $X, X^\star$ are compact manifolds without boundary, the type of question discussed in this paper reduces immediately to this local case, via a partition of unity.

Definition 1.1. T is said to be L^p-improving near $(x_0, x_0^\star)$ if there exist neighborhoods $U, U^\star$ of $x_0, x_0^\star$ respectively such that for any exponent $p \in (1, \infty)$ there exists $q > p$ such that for any family of measures σ_x having smooth densities, T maps $L^p(U^\star)$ boundedly to $L^q(U)$.

A geometric criterion for T to be L^p-improving was established (in the special case $k = k^\star$) in [11]. Moreover the following are mutually equivalent, with the same quantifiers on $U, U^\star, \sigma_x$:

(1) T is L^p-improving.
(2) There exists at least one pair of exponents $p < q$ for which T maps L^p to L^q.
(3) For all $p \in (1, \infty)$ there exists $\delta > 0$ such that T maps $L^p(U^\star)$ to the Sobolev space $L^p_\delta(U)$.
(4) For at least one $p \in (1, \infty)$ there exists $\delta > 0$ such that T maps $L^p(U^\star)$ to the Sobolev space $L^p_\delta(U)$.
(5) Several mutually equivalent curvature, or nonintegrability, conditions described in [11],[34]. See Definition 2.5 below.

It is a fundamental problem to determine the optimal exponents q, δ as functions of p; this problem includes both the local smoothing problem for the wave equation and (generalized to mixed norms) the Kakeya problem as special cases. The simplistic analysis in [11] gives poor estimates, even in comparatively simple examples.

1.4. **Some prior work.** One widely used method of analysis is based on L^2 orthogonality. In rough outline, T is decomposed into various pieces, precise microlocal L^2 estimates are derived for each summand, $L^p \mapsto L^q$ bounds are then obtained by interpolation of L^2

bounds with simple $L^1 \mapsto L^1$ and $L^1 \mapsto L^\infty$ estimates, real interpolation is invoked, and bounds are summed. See for instance [1], [12], [13], [14], [15], [18], [19], [20], [21], [29], [30], [31], [32], [33], [34], [28] for works in this vein.

The question of regularity in Sobolev spaces is still poorly understood. For instance, for the simple operator mapping functions in $\mathbb{R}^3$ to functions in $\mathbb{R}^3$ defined by convolution with arc length measure on a bounded portion of a helix or the curve $\{(t, t^2, t^3)\}$, it remains an open problem to determine for which p, δ the space L^p is mapped to L^p_δ; see [32], [27], and [28] for significant progress in this direction.

Oberlin [24],[25] had analyzed convolution with arc length measure on the curves $(t, t^2, \cdots, t^d)$ in $\mathbb{R}^d$ for $d = 3, 4$ by arguments which did not involve L^2 orthogonality, but relied on dimensional and numerological relations specific to those particular situations. Influential papers of Bourgain [4] and Wolff [38],[39],[40] on the Kakeya maximal function and related operators underscored the fundamentally combinatorial nature of this whole circle of questions.

The most direct precursors of the present paper were [6],[37],[9]. Christ [6] introduced a general method, and employed it in conjunction with more specialized arguments to analyze certain key particular operators. Tao and Wright [37] formulated in general terms the relation between $L^p \mapsto L^q$ estimates and the geometry of $\mathcal{I}$, in the case when $\text{dimension}\,(\mathcal{I}) = 1 + \text{dimension}\,(X) = 1 + \text{dimension}\,(X^\star)$. They introduced a notion of generic subsets, instrumental in overcoming the dependence of [6] on certain explicit formulae. They reinterpreted certain sets as two-parameter Carnot-Carathéodory balls in $\mathcal{I}$, and recognized the simplifying role of a technical condition which we call weak comparability. Christ and Erdoğan [9] analyzed mixed Lebesgue norm estimates, again only for a particular family of Radon-like transforms, combining [6] and the use of genericity with one more ingredient, certain spatial localizations.

The present paper synthesizes these works by reorganizing [6] to incorporate the localizations of [9] in an improved and more fundamental way, relying heavily on the notions of genericity and weak comparability from [37]. Although the main theorem is not new, we hope that the analysis both clarifies the structure of the problem, and establishes a foundation for further work, in particular for endpoint bounds in the real analytic case, which are now under investigation.

1.5. **Some notation.** $\mathbb{R}^+ = (0, \infty)$. $p, p^\star$ denote exponents in $[1, \infty]$ which are *not* necessarily conjugate to one another. $d, d^\star$ are respectively the dimensions of $X, X^\star$, and $k, k^\star$ are the dimensions of the submanifolds $\mathcal{M}, \mathcal{M}^\star$; our main result is restricted to the case where $k = k^\star = 1$ and $d = d^\star$, which will be assumed from the last paragraph of §2.2 on. $\gamma^\star$ denotes the mapping associated to the adjoint operator T^*. $\mathcal{I}$ denotes the incidence relation. $|S|$ denotes the measure of a set S contained in either X, $X^\star$, or $\mathcal{I}$, with respect to fixed measures which have continuous densities that are strictly positive near $x_0, x_0^\star, (x_0, x_0^\star)$, respectively. χ_E denotes the characteristic function of a set E. $A \lesssim B$ signifies an inequality of the form $A \le CB$ for some finite constant C, uniformly for all A, B in some classes of quantities; these classes will be clear from the context. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. Notation such as $\rho : X \times \mathbb{R} \to \mathbb{R}$ means that the domain of ρ is an appropriate open subset of $X \times \mathbb{R}$. $q' = q/(q-1)$ denotes the exponent conjugate to an exponent q.

2. Preliminaries and formulation of the main theorem

2.1. **Weak comparability.** As motivation consider the function $F(x,y) = x^4y^2 + x^2\exp(-1/y^2)$ for $0 \le x, y \le 1$. For any small $\delta > 0$ and any $C < \infty$ there exists $C' < \infty$ such that $1 \le \frac{F(x,y)}{x^4y^2} \le C'$ whenever $y \le Cx^\delta$. Thus F is comparable in every such region $y \le Cx^\delta$ to a polynomial which is independent of C, δ, even though the constants bounding the ratio do depend on C, δ.

Definition 2.1. Let N, c be positive real numbers. Two positive real numbers $s, s^\star \lesssim 1$ are said to be (N,c)–weakly comparable if $s \ge cs^{\star N}$ and $s^\star \ge cs^N$.

In practice, N will be large and c will be small. We will sometimes simply write "weakly comparable" without specifying N, c. A statement of the form "If $s, s^\star$ are weakly comparable then P" will always mean that for any N, c, P holds for any parameters $s, s^\star$ which are (N,c)– weakly comparable. If proposition P is an inequality, then it is to be understood to hold with an implicit constant depending on N, c but not on $s, s^\star$.

2.2. **Parametrized form.** There exist C^∞ functions $\gamma : X \times \mathbb{R}^k \to X^\star$ and $\gamma^\star : X^\star \times \mathbb{R}^{k^\star} \to X$ such that $\mathcal{M}_x = \{\gamma(x,t) : t \in \mathbb{R}^k\}$ and $\mathcal{M}^\star_{x^\star} = \{\gamma^\star(x^\star,t) : t \in \mathbb{R}^{k^\star}\}$, these are injective functions of t for each $x, x^\star$ respectively, and $\partial\gamma/\partial t, \partial\gamma^\star/\partial t$ are everywhere of maximal ranks $k, k^\star$ respectively. Here t is restricted to sufficiently small open sets in $\mathbb{R}^k, \mathbb{R}^{k^\star}$. The incidence relation takes the form

$$\mathcal{I} = \{(x,x^\star) \in X \times X^\star : \exists t : x^\star = \gamma(x,t)\} = \{(x,x^\star) \in X \times X^\star : \exists t : x = \gamma^\star(x^\star,t)\}.$$

In these terms our generalized Radon transform may be expressed in suitable local coordinates as

$$Tf(x) = \int_{\mathbb{R}^k} f(\gamma(x,t))\eta(x,t)\,dt. \tag{2.1}$$

Here $x, \gamma(x,t)$ belong to Euclidean spaces $\mathbb{R}^d$. The adjoint operator takes the same form

$$T^*g(x^\star) = \int_{\mathbb{R}^{k^\star}} g(\gamma^\star(x^\star,s))\eta^\star(x^\star,s)\,ds. \tag{2.2}$$

Here $\eta, \eta^\star$ are smooth, compactly supported functions. Since we are interested in upper bounds in L^p spaces, we may always assume that $\eta, \eta^\star$ are nonnegative.

We will always take f, g to be characteristic functions of Borel sets $E, E^\star$. By real interpolation theory, a restricted weak type bound of the form $|\langle T(\chi_E), \chi_{E^\star}\rangle| \le A|E|^{1/p}|E^\star|^{1/p^\star}$ for some finite constant A implies that T is bounded from L^{q_1} to L^{q_2} whenever $q_1 > p$ and $q_2 < \frac{p^\star}{p^\star - 1}$. Thus we will work primarily with

$$\mathcal{T}(E,E^\star) = \langle T(\chi_{E^\star}), \chi_E\rangle = \langle \chi_{E^\star}, T^*(\chi_E)\rangle. \tag{2.3}$$

We may assume without loss of generality that $\mathcal{T}(E,E^\star)$, and hence $|E|, |E^\star|$, are strictly positive.

A fundamental role [6] is played by two parameters.

Definition 2.2. The average numbers of incidences $\alpha, \alpha^\star$ per point of $E, E^\star$ respectively are defined to be

$$\alpha = \frac{\mathcal{T}(E,E^\star)}{|E|}, \qquad \alpha^\star = \frac{\mathcal{T}(E,E^\star)}{|E^\star|}. \tag{2.4}$$

The Radon-like transform T is bounded from L^p to L^p for any $p \in [1,\infty]$, and therefore since $E, E^\star$ are subsets of bounded regions, $\mathcal{T}(E,E^\star) \lesssim |E|^{1/p}|E^\star|^{1/p^\star}$ whenever $\frac{1}{p}+\frac{1}{p^\star} \le 1$. Thus we assume throughout the paper that

$$\frac{1}{p}+\frac{1}{p^\star} > 1. \tag{2.5}$$

2.3. **Iterated mappings and Jacobians.** Associated to γ are iterated mappings Γ_n which map $X \times \mathbb{R}^n$ to $X^\star$ for odd[1] n and to X for even n, defined by $\Gamma_1(x,t) = \gamma(x,t)$ and

$$\Gamma_n(x,t_1,\cdots,t_n) = \begin{cases} \gamma(\Gamma_{n-1}(x,t_1,\cdots,t_{n-1}),t_n) \text{ if } n>1 \text{ is odd,} \\ \gamma^\star(\Gamma_{n-1}(x,t_1,\cdots,t_{n-1}),t_n) \text{ if } n \text{ is even.} \end{cases} \tag{2.6}$$

These have of course certain domains, a point which will be consistently slurred over. There are corresponding mappings $\Gamma_n^\star$ from $X^\star \times \mathbb{R}^n$ to one of $X, X^\star$, obtained by interchanging $\gamma, \gamma^\star$ everywhere in the definition. This construction applies for any values of $k, k^\star$, the only change beginning that $\mathbb{R}^n$ should be replaced by Euclidean space of dimension $k + k^\star + k + k^\star + \cdots$ or $k^\star + k + k^\star + k + \cdots$, where the sum extends over n terms. We will often denote by τ the vector $(t_1,\cdots,t_n)$.

Γ_n parametrizes chains of points $(x_0,x_1,\cdots,x_n) \in X \times X^\star \times X \times X^\star \times \cdots$ such that $(x_i,x_{i+1}) \in \mathcal{I}$ whenever i is even and $(x_{i+1},x_i) \in \mathcal{I}$ whenever i is odd, and $x_0 = x$. For $n=2$ such chains arise in the computation of the composition of T with its adjoint. For $n>2$ they were used in the analysis of generalized Radon transforms in [5]. A closely related construction has been exploited by Baouendi, Ebenfelt, and Rothschild [2],[3] in connection with complex analysis in several variables.

In the special case when the domain of Γ_n has the same dimension as its codomain, associated to these are the Jacobian determinant

$$J(x,\tau) = \det\left(\frac{\partial \Gamma_d(x,\tau)}{\partial \tau}\right). \tag{2.7}$$

This has the analogue $J^\star(x^\star,\tau) = \det\left(\frac{\partial \Gamma_n^\star(x^\star,\tau)}{\partial \tau}\right)$, provided again that the domain and codomain of $\Gamma_n^\star$ have the same dimensions. This is in particular the case when $n=d$ and $k=k^\star=1$. When the dimension of the domain exceeds the dimension of the codomain then determinants of subJacobians are similarly defined.

Besides its smoothness, J has several fundamental properties. For the sake of simplicity, and because our main theorem concerns only this case, we assume here and throughout the remainder of the paper that $k=k^\star=1$. Choose $n=d$ or $n=d^\star$ so that the dimension of the codomain X or $X^\star$ of Γ_n is n. Then firstly, $J(x,0) \equiv 0$. Secondly, T is L^p-improving in some neighborhood of x if and only if there exists β such that $\frac{\partial^\beta J}{\partial \tau^\beta}(x,0) \ne 0$; in general there is an analogous statement involving determinants of appropriate subJacobians. Thirdly, $J(x,\tau)$ vanishes identically on certain canonical lower-dimensional sets; see Remark 11.7. This last property will not be exploited in this paper, but will be essential in an analysis of sharper endpoint estimates.

2.4. **Canonical towers.** We change notation slightly for the iterated mappings Γ_k originally defined in (2.6). For $k=1$, for each $z=(x,x^\star) \in \mathcal{I}$, define $\Gamma_1((x,x^\star),t_1) = \gamma(x,t_1)$

[1] The necessity to distinguish between even and odd dimensions will plague the entire exposition.

if d is odd, and $\Gamma_1((x,x^\star),t_1)=\gamma^\star(x^\star,t_1)$ if d is even. Then inductively define

$$\Gamma_{k+1}((x,x^\star),t_1,\cdots,t_{k+1})=\begin{cases}\gamma(\Gamma_k((x,x^\star),t_1,\cdots,t_k),t_{k+1}) & \text{if } d-k \text{ is even,}\\ \gamma^\star(\Gamma_k((x,x^\star),t_1,\cdots,t_k),t_{k+1}) & \text{if } d-k \text{ is odd.}\end{cases} \tag{2.8}$$

Define $s_1\in\mathbb{R}$ so that $\Gamma_1((x,x^\star),s_1)=x^\star$ if d is odd, and $\Gamma_1((x,x^\star),s_1)=x$ if d is even. Define $\Omega_1^\dagger=\{t_1\in\mathbb{R}^1:|t_1-s_1|<\alpha\}$ if d is odd, and $\Omega_1^\dagger=\{t_1\in\mathbb{R}^1:|t_1-s_1|<\alpha^\star\}$ if d is even. Inductively define sets $\Omega_k^\dagger\subset\mathbb{R}^k$ and functions $s_k:\Omega_{k-1}^\dagger\to\mathbb{R}$ by the equations

$$\Gamma_k((x,x^\star),t_1,\cdots,t_{k-1},s_k)=\begin{cases}x^\star & \text{if } d-k \text{ is even}\\ x & \text{if } d-k \text{ is odd,}\end{cases} \tag{2.9}$$

and

$$\Omega_k^\dagger=\Big\{(t_1,\cdots,t_k)\in\mathbb{R}^k:|t_k-s_k(t_1,\cdots,t_{k-1})|<\begin{cases}\alpha \text{ if } d-k \text{ is even,}\\ \alpha^\star \text{ if } d-k \text{ is odd.}\end{cases}\Big\} \tag{2.10}$$

Definition 2.3 (Canonical towers)**.** Let $z=(x,x^\star)\in\mathcal{I}$. The canonical tower $\Omega^\dagger(z,\alpha,\alpha^\star)\subset\mathbb{R}^d$ is defined to be

$$\Omega^\dagger(z,\alpha,\alpha^\star)=\Omega_d^\dagger.$$

These sets are special cases of *towers*, which will be discussed below.

In other words, we iterate $\gamma,\gamma^\star$ beginning at either x or $x^\star$, depending on parity, but consider only multi-times $(t_1,t_2,\cdots)$ for which each of these iterates returns either within distance $c\alpha$ of $x^\star$, or within distance $c\alpha^\star$ of x, as dictated by parity. This definition of $\Omega^\dagger$ is coordinate-free, but awkward to work with. In §3.5 below we will introduce parametrizations of $\gamma,\gamma^\star$ under which $\Omega^\dagger$ becomes a Cartesian product of intervals $[-\alpha,\alpha]$ and $[-\alpha^\star,\alpha^\star]$.

Definition 2.4. For any $z=(x,x^\star)\in\mathcal{I}$ and $0<\alpha,\alpha^\star\lesssim 1$,

$$\lambda(z,\alpha,\alpha^\star)=\int_{\Omega^\dagger(z,\alpha,\alpha^\star)}|\tilde J(\tilde x,\tau)|\,d\tau \tag{2.11}$$

$$\lambda(\alpha,\alpha^\star)=\inf_{z\in\mathcal{I}}\lambda(z,\alpha,\alpha^\star) \tag{2.12}$$

where $\tilde J,\tilde x=J,x$ if d is odd, and $=J^\star,x^\star$ if d is even.

Lemma 2.1. *Under the L^p-improving hypothesis, if d is odd then there exists a finite set S of multi-indices such that for any parameters N,c, for any $\alpha,\alpha^\star$ that are (N,c)–weakly comparable, for all $z=(x,x^\star)\in\mathcal{I}$ sufficiently close to any point $(x_0,x_0^\star)$,*

$$\lambda(z,\alpha,\alpha^\star)\sim\sum_{\beta\in S}c_\beta(z)\alpha^{(d+1)/2}\alpha^{\star(d-1)/2}\alpha^{\sum_{j\ odd}\beta_j}\alpha^{\star\sum_{j\ even}\beta_j}. \tag{2.13}$$

Here β_j are the components of β. The constants implicit in the inequalities expressed by $\sim$ depend on N,c, but S does not. For even d there is a corresponding formula, with $\alpha^{(d+1)/2}\alpha^{\star(d-1)/2}$ replaced by $\alpha^{d/2}\alpha^{\star d/2}$ and J,x replaced by $J^\star,x^\star$.

This is proved by expanding J in Taylor series with respect to $(r,r^\star)$ about $(z,0,0)$, approximating it by a Taylor polynomial of a very high degree depending on N, and integrating that polynomial in $r,r^\star$ with $(x,x^\star)$-dependent coefficients over $\Omega,\Omega^\dagger$. Using the fact that at least one coefficient in this expansion about $(x_0,x_0^\star,0,0)$ is nonzero, one finds that only a finite set S of monomials, independent of N,c,x, can make a nonnegligible contribution, provided that $r,r^\star$ are sufficiently small. Further details are left to the reader.

As a consequence, if $z \in \mathcal{I}$ varies over a sufficiently small neighborhood of a point z_0 then $\lambda(z,\alpha,\alpha^\star) \gtrsim \lambda(z_0,\alpha,\alpha^\star)$, whence $\lambda(\alpha,\alpha^\star) \sim \lambda(z_0,\alpha,\alpha^\star)$.

2.5. **Vector fields and curvature.** On $\mathcal{I}$ there exist pairs of smooth real vector fields $V, V^\star$, which are everywhere linearly independent, such that the integral curves of V are the sets $\pi^{-1}(\{x\})$ for points $x \in X$, and the integral curves of $V^\star$ are the sets $\pi^{\star -1}(\{x^\star\})$ for $x^\star \in X^\star$. Fix one such pair $V, V^\star$ for the remainder of the paper.

Definition 2.5. $\mathcal{I}$ possesses rotational curvature if and only if the Lie algebra generated by $V, V^\star$ spans the tangent space to $\mathcal{I}$ at each of its points.

This condition has many equivalent formulations [11], and in particular is equivalent [11] to T being L^p-improving. We assume it throughout this paper.

2.6. **Balls.** Two different kinds of "balls" enter naturally into the analysis. The first of these are subsets of $\mathcal{I}$ and will be denoted $\mathcal{B}(z,r,r^\star)$, where $z \in \mathcal{I}$ and $r, r^\star$ are small positive numbers. $\mathcal{B}(z,r,r^\star)$ is the Carnot-Carathéodory ball of radius 1 associated to the vector fields $rV, r^\star V^\star$, that is, $\mathcal{B}(z,r,r^\star)$ is the set of all points $\varphi(1)$ where $\varphi : [0,1] \to \mathcal{I}$ varies over all Lipschitz continuous curves whose tangent vectors $\varphi'(t)$ take the form $[a(t)rV + b(t)r^\star V^\star](\varphi(t))$ for almost every $t \in [0,1]$, with $a^2+b^2 \le 1$. When $z = (x,x^\star)$ we will also write $\mathcal{B}(x,x^\star,r,r^\star)$.

The second families of "balls" are subsets of the two ambient manifolds $X, X^\star$. To any $(x,x^\star) \in \mathcal{I}$, and to any sufficiently small $r, r^\star \in \mathbb{R}^+$, are associated

$$B^\star(x,x^\star,r,r^\star) = \pi^\star(\mathcal{B}(x,x^\star,r,r^\star)) \subset X^\star \text{ and } B(x,x^\star,r,r^\star) = \pi(\mathcal{B}(x,x^\star,r,r^\star)) \subset X.$$

These are Carnot-Carathéodory balls in a generalized sense. They can be described as all points accessible via travel along certain vector fields, which are not finite in number but depend smoothly on a one-dimensional control parameter; the traveler is allowed to flow along one vector field, then change the parameter, then flow again, and so forth; but not only is the total time of travel restricted by r, but also the total variation of the control parameter is restricted by a second quantity, $r^\star$.

Sets equivalent to $B, B^\star$, described in different terms, underlie the discussion in [6].

Lemma 2.2. *Let $\mathcal{I}$ be an incidence manifold possessing rotational curvature. Let N, c be finite positive numbers. Then uniformly for every $(x,x^\star) \in \mathcal{I}$ and every $r, r^\star \lesssim 1$ which are (N,c)–weakly comparable, the ball $\mathcal{B} = \mathcal{B}(x,x^\star,r,r^\star)$ satisfies*

$$r|\pi(\mathcal{B})| \sim r^\star|\pi^\star(\mathcal{B})| \sim |\mathcal{B}|, \tag{2.14}$$

$$|\mathcal{B}(x,x^\star,r,r^\star)| \sim \Theta(x,x^\star,r,r^\star), \tag{2.15}$$

where $\Theta(x,x^\star,r,r^\star) = \sum_\beta c_\beta(x,x^\star) r^{\beta_1} r^{\star\beta_2}$ is a certain polynomial in $(r,r^\star)$ whose coefficients c_β are nonnegative C^∞ functions of $x, x^\star$. Θ is independent of N, c, but the constants implicit in these inequalities may depend on N, c. If $r = 0$ or $r^\star = 0$ then $\Theta(x,x^\star,r,r^\star) = 0$. For any $(x,x^\star) \in \mathcal{I}$ there exists β such that $c_\beta(x,x^\star) \neq 0$.

The coefficients c_β can be expressed in terms of determinants of $(d+1)$-tuples of specific elements of the Lie algebra generated by $V, V^\star$. Lemma 2.2 is proved in §9.

2.7. **Geometric ratios and inequalities.** By Lemma 2.2,

$$\frac{|\mathcal{B}|}{|\pi(\mathcal{B})|^{1/p}|\pi^\star(\mathcal{B})|^{1/p^\star}} \sim r^{1/p} r^{\star 1/p^\star} |\mathcal{B}|^{1-\frac{1}{p}-\frac{1}{p^\star}}. \tag{2.16}$$

Assuming always that $\frac{1}{p}+\frac{1}{p^\star}>1$, or equivalently that $p+p^\star>pp^\star$, define positive, finite exponents $a(p,p^\star), a_\star(p,p^\star)$ by

$$a=\frac{p^\star}{p^\star+p-pp^\star},\qquad a_\star=\frac{p}{p+p^\star-pp^\star}. \tag{2.17}$$

The mapping $(p^{-1},p^{\star-1})\mapsto(a,a_\star)$ happens to be an involution.

Lemma 2.3. *Suppose that $\mathcal{I}$ possesses rotational curvature, and that $\frac{1}{p}+\frac{1}{p^\star}>1$. Let $a,a_\star$ be defined in terms of $p,p^\star$ by (2.17). Then the following are equivalent:*

(1) *For every (N,c), for all (N,c)–weakly comparable radii $0<r,r^\star\lesssim 1$ and for all $z\in\mathcal{I}$, the balls $\mathcal{B}=\mathcal{B}(z,r,r^\star)$ satisfy*

$$\sup_{\mathcal{B}}\frac{|\mathcal{B}|}{|\pi(\mathcal{B})|^{1/p}|\pi^\star(\mathcal{B})|^{1/p^\star}}<\infty. \tag{2.18}$$

(2) *For every (N,c) there exists $c'>0$ such that*

$$|\mathcal{B}(x,x^\star,r,r_\star)|\ge c' r^a r_\star^{a_\star} \tag{2.19}$$

for all (N,c)–weakly comparable pairs of positive numbers $r,r_\star\lesssim 1$.

(3) *For any (N,c) there exists $c'>0$ such that*

$$\lambda(\beta,\beta_\star)\ge c'\beta^a\beta_\star^{a_\star-1} \tag{2.20}$$

for all (N,c)–weakly comparable $\beta,\beta_\star\lesssim 1$.

The equivalence of the first two conditions follows directly from the definitions and (2.16) by algebra. The proof of their equivalence with the third condition will be completed in §9.

Definition 2.6. We say that $\Lambda(p,p^\star)<\infty$ if (2.18) holds for each N,c.

$\Lambda(p,p^\star)$ is not defined to be an actual number.

2.8. **Formulation of the main theorem.**

Theorem 2.4. *Suppose that $\mathcal{I}$ possesses rotational curvature. Suppose that $1<\tilde p<\tilde q<\infty$. If $\Lambda(\tilde p,\tilde q')<\infty$ for all N,c then T maps L^p boundedly to L^q whenever $p>\tilde p$ and $q<\tilde q$. Conversely if $1<p<q<\infty$ and T is of restricted weak type (p,q), then $\Lambda(p,q')<\infty$.*

This theorem is due to Tao and Wright [37]. Recall that $r'=r/(r-1)$ denotes the exponent conjugate to r.

In a sufficiently small neighborhood of z_0, $\lambda(\alpha,\alpha^\star)=\min_z\lambda(z,r,r^\star)$ is comparable to $\lambda(z_0,\alpha,\alpha^\star)$, as follows from (2.13). Therefore it suffices to consider only balls $\mathcal{B}(z,r,r^\star)$ centered at $z=z_0$ in defining $\Lambda(p,p^\star)$. Thus if $\mathcal{I}\subset X\times X^\star$ is expressed as $\{z:F(z)=0\}$ for a smooth $\mathbb{R}^{d-1}$–valued function F, then Theorem 2.4 gives in principle a criterion for an $L^p\to L^q$ inequality for T in terms of the Taylor expansion of F about z_0.

An equivalent formulation of this upper bound for T is that whenever $(p,p^\star)$ belongs to the interior of the set of all pairs of exponents for which Λ is finite,

$$\mathcal{T}(E,E^\star)\le C|E|^{\frac{1}{p}}|E^\star|^{\frac{1}{p^\star}}. \tag{2.21}$$

Rather than (2.21), the analysis leads most directly to a bound of the form

$$|E^\star|\ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon\lambda(\alpha,\alpha^\star) \tag{2.22}$$

for any Borel sets $E,E^\star$ of finite, positive measures satisfying $\mathcal{T}(E,E^\star)>0$. (2.22) implies (2.21) via Lemma 2.3 and real interpolation theory. The balls $\pi(\mathcal{B}),\pi^\star(\mathcal{B})$ provide pairs

$(E, E^\star)$ leading to lower bounds for $\mathcal{T}(E, E^\star)/|E|^{1/p}|E^\star|^{1/p^\star}$, but do not enter explicitly into the proof of the main bound (2.21).

3. Towers

3.1. **The role of towers.** Rather than seeking an upper bound on $\mathcal{T}(E, E^\star)$, we imagine a lower bound to be given, and seek a corresponding lower bound on $|E|^{1/p}|E^\star|^{1/p^\star}$. It is shown in [6] how a measurable set Ω and a smooth mapping $\Phi : \Omega \to E^\star$ are naturally associated with this situation, where Ω has a certain structure including lower bounds on its size in terms of $|E|, |E^\star|, \mathcal{T}(E, E^\star)$. Thus $|E^\star| \ge |\Phi(\Omega)|$, for which it remains to establish a suitable lower bound.

Here one faces a continuum version of the usual combinatorial issue of overcounting due to the failure of Φ to be injective. This problem becomes acute when Ω has larger dimension than does the target space for Φ. A more serious problem is that even for the basic example of convolution with arc length measure on the curve $(t, t^2, \cdots, t^d)$ in $\mathbb{R}^d$ for $d \ge 4$, the desired lower bound on $|\Phi(\Omega)|$ is simply false, if Ω has the natural dimension d. Those particular examples were treated in an *ad hoc* way in [6], which replaced a natural space Ω of dimension d with an analogous set of dimension $2d-2$, and analyzed the resulting image set by exploiting explicit formulae unavailable in the general situation.

Tao and Wright [37] modified this construction so as to endow Ω with certain additional structure, using sets Ω of arbitrarily high dimensions along the way. In the present paper a version of this additional structure is obtained in a different way, and it is shown how the required estimates still follow. One advantage is that only towers of dimension d are used.

3.2. **Towers defined.**

Definition 3.1. A tower Ω of height D and multi-dimension $(k_1, k_2, \cdots, k_D)$ is a D-tuple $\Omega = (\Omega_1, \cdots, \Omega_D)$ of Lebesgue measurable sets $\Omega_n \subset \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$ such that for any $(t_1, \cdots, t_{n-1}, t_n) \in \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$,

$$(t_1, \cdots, t_{n-1}, t_n) \in \Omega_n \Rightarrow (t_1, \cdots, t_{n-1}) \in \Omega_{n-1}.$$

If $1 < n \le D$ then to each $\tau \in \Omega_{n-1}$ is associated the fiber

$$F_n(\tau) = \{t \in \mathbb{R}^{k_n} : (\tau, t) \in \Omega_n\}.$$

To unify notation set $F_1 = \Omega_1$.

Definition 3.2. Ω has multisize $\ge (\alpha_1, \cdots, \alpha_D)$ if each $\alpha_n \in (0, \infty)$, $|F_1| \ge \alpha_1$, and for all $n > 1$ and almost every $\tau \in \Omega_{n-1}$,

$$|F_n(\tau)| \ge \alpha_n. \tag{3.1}$$

By deleting sets of measure zero we may ensure that this condition holds for every point τ, and henceforth will make this type of reduction without further comment.

In this paper only parameter space towers having one-dimensional fibers arise; $k_j = 1$ for all j for the remainder of the discussion.

Definition 3.3. Let $\ell \in (0, \infty)^D$. A tower Ω with one-dimensional fibers has multilength $\ell = (\ell_1, \cdots, \ell_D)$ if each of its fibers $F_j(\tau)$ is contained in some interval of length $\le \ell_j$. Its multilength is said to be monotonic if $\ell_n \le \ell_{n+2}$ for all $n \le D - 2$.

Here C is any sufficiently large fixed constant. If Ω has multilength ℓ then it also has multilength $\tilde{\ell}$ for every $\tilde{\ell}$ satisfying $\tilde{\ell}_j \ge \ell_j$ for all j.

Definition 3.4. A mapping tower which generates subsets of $E, E^\star$ is a tower Ω together with mappings $\{\Phi_n : 1 \le n \le D\}$ with domains Ω_n such that

$$\Phi_n(\Omega_n) \subset E \text{ whenever } d-n \text{ is odd, and } \Phi_n(\Omega_n) \subset E^\star \text{ whenever } d-n \text{ is even.} \tag{3.2}$$

In this paper, the mappings Φ_n will always take the form $\Phi_n(\tau) = \Gamma_n(x,\tau)$ for some $x \in E$ if d is odd, and $\Phi_n(\tau) = \Gamma^\star_n(x^\star,\tau)$ for some $x^\star \in E^\star$ if d is even, where $\Gamma, \Gamma^\star$ are the iterated mappings associated to $\mathcal{I}$ as described above. The points $x, x^\star$ will not depend on n.

3.3. **Generic subsets.** Let $\varepsilon > 0$ be a constant, sufficiently small for later purposes. Let $I \subset \mathbb{R}^1$ be a bounded interval, and let $E \subset I$ be a Lebesgue measurable set having positive measure.

Definition 3.5. $E \subset I$ is an (ε,δ)-generic subset of I if for any subinterval $J \subset I$ of measure $|J| \le \delta |I|$,

$$|E \cap J| \le \varepsilon |E|. \tag{3.3}$$

Lemma 3.1. [37] *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite $\rho > 0$ and any Lebesgue measurable set $E \subset \mathbb{R}^1$ having positive measure and having diameter $\le \rho$, there exists an interval I such that $E \cap I$ is an (ε,δ)-generic subset of I and*

$$|E \cap I| \ge |E| \cdot (|E|/\rho)^\varepsilon. \tag{3.4}$$

We will sometimes speak simply of generic subsets, without specifying the parameters ε, δ. For related alternative notions of genericity see [6],[9],[8],[10].

For the sake of completeness we indicate a proof.

Proof. Let $\delta < \varepsilon$ be a small positive number to be specified. Let E be given and fix an interval I_0 of length ρ containing E. If there exists no subinterval $J \subset I_0$ of length $\delta |I_0|$ satisfying $|E \cap J| \ge \varepsilon |E|$ then E is generic in I_0. Otherwise choose some such interval J, and call it I_1. Repeat this process, generating a sequence of intervals $I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$ satisfying $|I_n| = \delta^n |I_0|$, and $|E \cap I_n| \ge \varepsilon^n |E|$.

This process cannot continue indefinitely; $\varepsilon^n |E| \le |E \cap I_n| \le |I_n| = \delta^n |I_0|$, so $n \le \frac{\ln(|I_0|/|E|)}{\ln(\varepsilon/\delta)}$. Define $I = I_n$ for the first integer n for which no subinterval J with the required property exists. It remains only to verify that $|E \cap I_n| \ge |E| \cdot (|E|/\rho)^\varepsilon$. We have $n \le \ln(\rho/|E|)/\ln(\varepsilon/\delta)$, so

$$|E \cap I_n| \ge |E| \varepsilon^{\ln(\rho/|E|)/\ln(\varepsilon/\delta))} = |E| \cdot (|E|/\rho)^\gamma$$

where $\gamma = \frac{\ln(1/\varepsilon)}{\ln(\varepsilon/\delta)}$. Choosing $\delta < \varepsilon$ sufficiently small yields (3.4). □

We'll need a version with parameters. Given a measurable set $\mathcal{E} \subset \mathbb{R}^K \times \mathbb{R}^1$, set $E_y = \{t \in \mathbb{R}^1 : (y,t) \in \mathcal{E}\}$. Suppose that E_y has diameter $\le \rho$, for all $y \in \mathbb{R}^K$. Then there exist measurable functions $y \mapsto a(y), b(y)$ such that for almost every $y \in \mathbb{R}^K$, the interval $I_y = [a(y), b(y)]$ satisfies $|E_y \cap I_y| \ge c|E_y| \cdot (|E_y|/\rho)^\varepsilon$, and $E_y \cap I_y$ is an (ε,δ)-generic subset of I_y. This can be achieved by making arbitrary choices in the above stopping-time construction in any reasonable manner.

Definition 3.6. A tower Ω with multilength $(\ell_1, \cdots, \ell_D)$ is *generically arranged* if for all j, each fiber $F_j(\tau)$ is a generic subset of some interval of length $\sim \ell_j$.

More precisely, F_j is required to be an (ε_0,δ)-generic subset of an appropriate interval, where the exponent ε_0 is to be specified later in the proof, and $\delta=\delta(\varepsilon_0)$ is then chosen as in Lemma 3.1. The intervals in question are required to depend measurably on τ. By an interval of length $\sim \ell_j$ we mean an interval whose length lies in the interval $(C^{-1}\ell_j,\ell_j)$, where C is some sufficiently large but fixed constant.

3.4. **Centrality.** There are unique C^∞ mappings $\rho: X\times\mathbb{R}\to\mathbb{R}$ and $\rho^\star: X^\star\times\mathbb{R}\to\mathbb{R}$ satisfying the relations

$$\gamma^\star(\gamma(x,t),\rho(x,t))\equiv x, \qquad \gamma(\gamma^\star(x^\star,t),\rho^\star(x^\star,t))\equiv x^\star. \tag{3.5}$$

Definition 3.7. A mapping tower Ω of multilength $\ell=(\ell_1,\cdots,\ell_D)$ with one-dimensional fibers is *centered* if both of the following conditions hold. (i) For any $1\le j\le D-2$, for almost every $\mathbb{R}^j\times\mathbb{R}^1\ni(\tau,t)\in\Omega_{j+1}$, the fiber $F_{j+2}(\tau,t)$ is contained in the interval of length ℓ_{j+2} centered at either $\rho(\Phi_j(\tau),t)$ or $\rho^\star(\Phi_j(\tau),t)$, as dictated by parity. (ii) If d is odd there exists $x\in X$ such that for every $\tau\in\Omega_1$, the fiber $F_2(\tau)$ is contained in the interval of length ℓ_2 centered at $\rho(x,\tau)$. If d is even there exists $x^\star\in X^\star$ such that for every $\tau\in\Omega_1$, the fiber $F_2(\tau)$ is contained in the interval of length ℓ_2 centered at $\rho^\star(x^\star,\tau)$.

This notion of centrality is essentially that of Tao and Wright [37]. As defined here, centrality by itself means practically nothing, since it can be achieved simply by choosing each ℓ_j to be sufficiently large. However we will work with towers which are also generically arranged, thus imposing a reasonably tight upper bound on ℓ.

Here and elsewhere, the phrase "as dictated by parity" refers to the alternating character of the tower and mappings; $\Phi_n(\Omega_n)\subset X$ for all odd[2] $d-n$ and $\Phi_n(\Omega_n)\subset X^\star$ for all even $d-n$.

3.5. **A special parametrization.** Choose C^∞ functions $\Pi: X\to\mathbb{R}^1$ and $\Pi^\star: X^\star\to\mathbb{R}^1$ such that for all $x,x^\star,t$, $\frac{d}{dt}\Pi\circ\gamma^\star(x^\star,t)\ne 0$ and $\frac{d}{dt}\Pi^\star\circ\gamma(x,t)\ne 0$. More precisely, these are defined in sufficiently small neighborhoods of a point $(x_0,x_0^\star)\in\mathcal{I}$ and for t sufficiently close to some $t_0,t_0^\star$, respectively.

Because the maps $(x,t)\mapsto(x,\Pi^\star(\gamma(x,t)))$ and $(x^\star,t)\mapsto(x^\star,\Pi(\gamma^\star(x^\star,t)))$ are local diffeomorphisms, the mappings $\gamma,\gamma^\star$ can be smoothly reparametrized so that

$$\Pi^\star(\gamma(x,t))\equiv t \ \text{ and } \ \Pi(\gamma^\star(x^\star,t))\equiv t \tag{3.6}$$

for all $x,x^\star,t$. We henceforth assume (3.6).

For any x,t_1,t_2, $\rho(\Gamma_1(x,t_1),t_2)=t_3$ is the unique solution of $\Gamma_3(x,t_1,t_2,t_3)=\gamma(x,t_1)$. Given (3.6), this equation is equivalent to $t_3=t_1$. Indeed,

$$t_3=\Pi^\star(\gamma(\Gamma_2(x,t_1,t_2),t_3))=\Pi^\star(\Gamma_3(x,t_1,t_2,t_3))=\Pi^\star(\gamma(x,t_1))=t_1.$$

This statement, together with the dual statement in which the roles of $\gamma,\gamma^\star$ are reversed, means that with (3.6) in force, a mapping tower Ω is centered if and only if there exists t_0 such that for any $\tau=(t_1,\cdots,t_D)\in\Omega_D$,

$$|t_{j-2}-t_j|\lesssim \ell_j \text{ for all } 2\le j\le D. \tag{3.7}$$

Under this parametrization, the canonical towers $\Omega^\dagger(z,\alpha,\alpha^\star)$ introduced in Definition 2.3 take the simple form

$$\Omega^\dagger(z,\alpha,\alpha^\star)=\cdots\times[-\alpha^\star,\alpha^\star]\times[-\alpha,\alpha]\times[-\alpha^\star,\alpha^\star]\times[-\alpha,\alpha] \tag{3.8}$$

[2]The reverse parity convention would be equally reasonable.

with d factors in the Cartesian product; the leftmost factor is $[-\alpha^\star,\alpha^\star]$ if d is even, and is $\times[-\alpha,\alpha]$ if d is odd.

4. Spatial localization and orthogonality

After some preparations, we discuss in §4.3 a form of L^p orthogonality which will be at the core of the analysis.

4.1. **A crude localization.**

Our purpose here is to reduce matters to the case where $\alpha,\alpha^\star$ are weakly comparable. A loss of an arbitrarily small power of $\alpha\alpha^\star$ will be incurred in the process.

Let sets $E,E^\star$, and consequently parameters $\alpha,\alpha^\star$, be given; we may always assume that $\mathcal{T}(E,E^\star)>0$. Let $0<\delta_1\le(4d)^{-1}$ be a small parameter, which eventually will be taken to depend on the exponent ε in the main theorems; δ_1 will tend to zero as $\varepsilon\to 0$.

Via a preliminary partition of unity we may restrict attention to an arbitrarily small neighborhood of a single point in $\mathcal{I}$. Since the derivatives of $\gamma(x,t)$, $\gamma^\star(x^\star,t)$ with respect to t are nonzero, both $\alpha,\alpha^\star$ are bounded by a constant multiple of the diameter of such a neighborhood. Thus we may, and will, assume from the outset that $\alpha,\alpha^\star$ are smaller than any fixed constant.

Partition $X,X^\star$ into cubes $Q,Q^\star$ of sidelength $(\alpha\alpha^\star)^{\delta_1}$. Since $\mathcal{T}(E,E^\star)=\sum_{Q,Q^\star}\mathcal{T}(E\cap Q,E^\star\cap Q^\star)$ and there are $O((\alpha\alpha^\star)^{-d\delta_1})$ cubes Q and $O((\alpha\alpha^\star)^{-d\delta_1})$ cubes $Q^\star$, there exist $Q_0,Q_0^\star$ such that $\mathcal{T}(E\cap Q_0,E^\star\cap Q_0^\star)\gtrsim(\alpha\alpha^\star)^{2d\delta_1}\mathcal{T}(E,E^\star)$. Fix such $Q_0,Q_0^\star$ and set $\tilde E=E\cap Q_0$ and $\tilde E^\star=E^\star\cap Q_0^\star$. Thus

$$\mathcal{T}(\tilde E,\tilde E^\star)\ge c(\alpha\alpha^\star)^{2d\delta_1}\mathcal{T}(E,E^\star). \tag{4.1}$$

To $\tilde E,\tilde E^\star$ are associated new parameters $\tilde\alpha=\mathcal{T}(\tilde E,\tilde E^\star)/|\tilde E|$ and $\tilde\alpha^\star=\mathcal{T}(\tilde E,\tilde E^\star)/|\tilde E^\star|$.

Lemma 4.1. *Let $E,E^\star$ be Borel sets such that $\mathcal{T}(E,E^\star)>0$. Then the associated subsets $\tilde E,\tilde E^\star$ satisfy*

$$(\alpha\alpha^\star)^2\lesssim\min(\tilde\alpha,\tilde\alpha^\star)\le\max(\tilde\alpha,\tilde\alpha^\star)\lesssim(\alpha\alpha^\star)^{\delta_1} \tag{4.2}$$

In particular, there exist N,c, depending only on δ_1, such that $\tilde\alpha,\tilde\alpha^\star$ are (N,c)–weakly comparable.

Proof. For any $x\in X$, $T(\chi_{Q_0^\star})(x)\le C|\{t:\gamma(x,t)\in Q_0^\star\}|\le C(\alpha\alpha^\star)^{\delta_1}$ because $\partial\gamma(x,t)/\partial t$ never vanishes. Therefore $T(\chi_{E^\star\cap Q_0^\star})(x)\le T(\chi_{Q_0^\star})(x)\le C(\alpha\alpha^\star)^{\delta_1}$, and consequently

$$\tilde\alpha=\mathcal{T}(E\cap Q_0,E^\star\cap Q_0^\star)/|E\cap Q_0|\le|E\cap Q_0|^{-1}\int_{E\cap Q_0}T(\chi_{Q_0^\star})\le C(\alpha\alpha^\star)^{\delta_1}.$$

The same reasoning applies to $\tilde\alpha^\star$, so $\max(\tilde\alpha,\tilde\alpha^\star)\lesssim(\alpha\alpha^\star)^{\delta_1}$.

On the other hand,

$$\tilde\alpha\gtrsim(\alpha\alpha^\star)^{2d\delta_1}\mathcal{T}(E,E^\star)/|\tilde E|\ge(\alpha\alpha^\star)^{2d\delta_1}\mathcal{T}(E,E^\star)/|E|=(\alpha\alpha^\star)^{2d\delta_1}\alpha\ge(\alpha\alpha^\star)^{1+2d\delta_1}\ge(\alpha\alpha^\star)^2$$

since $\alpha^\star\le 1$. The same applies to $\tilde\alpha^\star$, so $\min(\tilde\alpha,\tilde\alpha^\star)\gtrsim(\alpha\alpha^\star)^2$. □

4.2. **Finer localizations.** Given $n \in \mathbb{Z}$, $\mathbb{R}^1$ may be partitioned (up to a discrete set which will be ignored) as the union of all dyadic intervals of lengths 2^n. This induces a partition of X, up to null sets, as the union of the inverse images of these dyadic intervals under Π. The same goes for $X^\star, \Pi^\star$. Denote these partitions by $X^\star = \cup_i X^\star_{n,i}$, $X = \cup_i X_{n,i}$.

For any $x \in X$ and any interval $I \subset \mathbb{R}$ define $\gamma(x, I) = \{\gamma(x,t) : t \in I\} \subset X^\star$, and define $\gamma^\star(x^\star, I) \subset X$ analogously for $x^\star \in X^\star$.

Let $C \geq 1$ be arbitrary. For any interval $I \subset \mathbb{R}$ satisfying $|I| \leq C2^n$ and any $x \in X$, there are at most C' indices i for which $\gamma(x, I) \cap X^\star_{n,i}$ is nonempty; C' is independent of n. The corresponding assertion holds for $X, \Pi, \gamma^\star$.

We will work with localized operators

$$\tilde{T} f(x) = \int_{I_x} f(\gamma(x,t))\, dt$$

where each interval I_x has length Between 2^{n-1} and 2^n, and $\{(x,y) \in \mathcal{I} : y \in \gamma(x, I_x)\}$ is Borel measurable. Particular families of intervals I_x will be constructed in §5. Let Borel measurable subsets $E \subset X, E^\star \subset X^\star$ be given. Partition $E^\star \subset X^\star$ as $\cup_i E^\star_i$, where

$$E^\star_i = E^\star \cap X^\star_{n,i}. \tag{4.3}$$

Define associated subsets $E_i \subset E$ by

$$E_i = \{x \in E : \exists t \in I_x : \gamma(x,t) \in E^\star_i\}; \tag{4.4}$$

the definitions of $E_i, E^\star_i$ are not symmetric, but we will also use the analogous definitions with the roles of $X, X^\star$ interchanged.

Each point $x \in E$ satisfying $T(chi_{E^\star})(x) > 0$ belongs to at least one of these sets E_i, and no point belongs to more than three. Thus uniformly in $n, E, E^\star$, the bilinear form $\tilde{\mathcal{T}}$ associated to the localized operator $\tilde{T}$ satisfies

$$\tilde{\mathcal{T}}(E, E^\star) \sim \sum_i \tilde{\mathcal{T}}(E_i, E^\star_i). \tag{4.5}$$

4.3. **An orthogonal decomposition due to localization.** By a bilinear form we mean an expression $\mathcal{T}(E, E^\star)$, acting on pairs of Borel sets $E \subset X$, $E^\star \subset X^\star$, associated to some linear operator $T : L^\infty(X^\star) \to L^1(X)$ by $\mathcal{T}(E, E^\star) = \int_E T(\chi_{E^\star})$. Here, as elsewhere, $\chi_{E^\star}$ denotes the characteristic function of $E^\star$. It is assumed that $f \geq 0 \Rightarrow T(f) \geq 0$.

Proposition 4.2. *Let $p, p^\star \in (1, \infty)$, and suppose that $\frac{1}{p} + \frac{1}{p^\star} > 1$, with strict inequality. Let $\tilde{\mathcal{T}}$ be a bilinear form for which*

$$A_{\tilde{\mathcal{T}}} = \sup_{E, E^\star} |\tilde{\mathcal{T}}(E, E^\star)| \cdot |E|^{-1/p} |E^\star|^{-1/p^\star} \tag{4.6}$$

is finite, where the supremum is taken over all measurable sets having finite, positive measures. Then for any families $\{E_i\}, \{E^\star_i\}$ of subsets of $X, X^\star$ respectively, such that no point of E belongs to E_i for more than 3 indices i, and likewise no point of $E^\star$ belongs to $E^\star_i$ for more than 3 indices i,

$$\sum_i \tilde{\mathcal{T}}(E_i, E^\star_i) \leq C \max_i \Big(\frac{|E_i|}{|E|} \cdot \frac{|E^\star_i|}{|E^\star|} \Big)^\delta \cdot A_{\tilde{\mathcal{T}}} |E|^{1/p} |E^\star|^{1/p^\star} \tag{4.7}$$

for certain $C, \delta \in \mathbb{R}^+$ which depend only on $p, p^\star$.

Consequently if $\tilde{\mathcal{T}}$ is localized in the sense (4.5), then a pair of sets $E, E^\star$ can realize the extremal situation for the inequality (4.6), up to a bounded factor, only when $E, E^\star$ are nearly equal to single components $E_i, E_i^\star$.

Proof. Choose a pair of exponents $r, r^\star \in (1,\infty)$ satisfying $\frac{1}{r} + \frac{1}{r^\star} = 1$, $\frac{1}{r} < \frac{1}{p}$ and $\frac{1}{r^\star} < \frac{1}{p^\star}$. Set $\delta = \frac{1}{p} - \frac{1}{r}$, $\delta' = \frac{1}{p^\star} - \frac{1}{r^\star}$, both of which are strictly positive. Then

$$\begin{aligned}
\sum_i \tilde{\mathcal{T}}(E_i, E_i^\star) &\le \sum_i A_{\tilde{\mathcal{T}}} |E_i|^{1/p} |E_i^\star|^{1/p^\star} \\
&\le A_{\tilde{\mathcal{T}}} \max_i \big(|E_i|^{\delta} |E_i^\star|^{\delta'} \big) \sum_i |E_i|^{1/r} |E_i^\star|^{1/r^\star} \\
&\le A_{\tilde{\mathcal{T}}} \max_i \big(|E_i|^{\delta} |E_i^\star|^{\delta'} \big) \big(\sum_i |E_i| \big)^{1/r} \big(\sum_i |E_i^\star| \big)^{1/r^\star} \\
&\le C A_{\tilde{\mathcal{T}}} \max_i \big(|E_i|^{\delta} |E_i^\star|^{\delta'} \big) |E|^{1/r} |E^\star|^{1/r^\star} \\
&= C A_{\tilde{\mathcal{T}}} \max_i \left(\frac{|E_i|}{|E|} \right)^{\delta} \left(\frac{|E_i^\star|}{|E^\star|} \right)^{\delta'} |E|^{1/p} |E^\star|^{1/p^\star}.
\end{aligned}$$

$\square$

5. Construction of a mapping tower

5.1. The setup.

Define

$$A = A(p, p^\star) = \sup_{E, E^\star} \mathcal{T}(E, E^\star) |E|^{-1/p} |E^\star|^{-1/p^\star}, \tag{5.1}$$

the supremum being taken over Borel sets having strictly positive measures. $A(p, p^\star)$ is an element of $(0, +\infty]$.

We aim to show that $A(p, p^\star)$ is finite, under the hypothesis of Theorem 2.4 that there exist $q < p, q^\star < p^\star$ such that $\Lambda(q, q^\star) < \infty$. The main step will be to show that there exists a finite constant $A^\dagger = A^\dagger(p, p^\star)$ such that *if* $A(p, p^\star)$ is finite, then for any pair of measurable sets $E, E^\star$,

$$\mathcal{T}(E, E^\star) > \tfrac{1}{2} A |E|^{1/p} |E^\star|^{1/p^\star} \;\Rightarrow\; \mathcal{T}(E, E^\star) \le A^\dagger |E|^{1/p} |E^\star|^{1/p^\star}. \tag{5.2}$$

Moreover, $A^\dagger$ will depend only on geometric information encoded by the incidence manifold $\mathcal{I}$, not in any way on A. It follows that $A \le A^\dagger$, provided that A is finite. The finiteness assumption can be removed in at least two distinct ways; see §8.

5.2. The algorithm's output.

We will specify an algorithm whose inputs are the incidence manifold $\mathcal{I}$ together with arbitrary Borel subsets $E \subset X$ and $E^\star \subset X^\star$ satisfying $\mathcal{T}(E, E^\star) > \frac{1}{2} A(p, p^\star) |E|^{1/p} |E^\star|^{1/p^\star}$, and whose output consists of sets with the following properties.

Proposition 5.1. *Let $p, p^\star \in (1, \infty)$. Suppose that $A(p, p^\star) < \infty$. Then for any $\varepsilon_0, \varepsilon > 0$ there exist $c, N, \delta > 0$ with the following property. Let $E \subset X$, $E^\star \subset X^\star$ be arbitrary Borel sets satisfying $\mathcal{T}(E, E^\star) > \frac{1}{2} A(p, p^\star) |E|^{1/p} |E^\star|^{1/p^\star}$. Let $\alpha, \alpha^\star$ be the average numbers of incidences* (2.4) *per point of $E, E^\star$, respectively. Then there exists a nested sequence of measurable sets*

$$E \times E^\star \supset E^{(d)} \times E^{\star(d)} \supset E^{(d-1)} \times E^{\star(d-1)} \supset \cdots \supset E^{(1)} \times E^{\star(1)} := E_\flat \times E_\flat^\star \tag{5.3}$$

such that $E^{(d)}, E^{\star(d)}$ are contained in cubes of sidelengths $\le (\alpha\alpha^\star)^{C\varepsilon}$, and

$$\mathcal{T}(E_\flat, E_\flat^\star) \ge c(\alpha\alpha^\star)^\varepsilon \mathcal{T}(E, E^\star). \tag{5.4}$$

The average numbers of incidences $\alpha_\flat, \alpha_\flat^\star$ associated to $E_\flat, E_\flat^\star$ are (N,c)–weakly comparable.

Furthermore there exist sets $F_n(x) \subset \mathbb{R}^1$ whenever $d-n$ is even and $x \in E^{(n)}$, and $F_n(x^\star) \subset \mathbb{R}^1$ whenever $d-n$ is odd and $x^\star \in E^{\star(n)}$, such that

$$d-n \text{ even} \Rightarrow \gamma(x,t) \in E^{\star(n)} \quad \text{for all } x \in E^{(n)},\ t \in F_n(x), \tag{5.5}$$

$$d-n \text{ odd} \Rightarrow \gamma^\star(x^\star,t) \in E^{(n)} \text{ for all } x^\star \in E^{\star(n)},\ t \in F_n(x^\star), \tag{5.6}$$

and

$$|F_n(x)| \ge c(\alpha\alpha^\star)^\varepsilon \alpha \text{ for even } d-n, \text{ and } |F_n(x^\star)| \ge c(\alpha\alpha^\star)^\varepsilon \alpha^\star \text{ for odd } d-n. \tag{5.7}$$

Moreover for each n there exists a single bounded interval I_n such that $F_n(y)$ is an (ε_0,δ)–generic subset of I_n for each $y \in E^{(n)}$ when $d-n$ is even, and for each $y \in E^{\star(n)}$ when $d-n$ is odd. Each I_n has length 2^{m_n} where the exponents m_n satisfy

$$m_{n-2} \le m_n \text{ for all } n \ge 3. \tag{5.8}$$

The constants c, C depend on $\mathcal{I}$, but neither on $A(p,p^\star)$ nor on $E, E^\star$.

Since $E_\flat \subset E$ and $E_\flat^\star \subset E^\star$, (5.4) together with the hypothesis $\mathcal{T}(E,E^\star) > \frac{1}{2}A|E|^{1/p}|E^\star|^{1/p^\star}$ force

$$\begin{aligned} |E_\flat| \ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon|E| &\text{ and } |E_\flat^\star| \ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon|E^\star| \\ \alpha_\flat \ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon\alpha \quad &\text{ and } \alpha_\flat^\star \quad \ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon\alpha^\star. \end{aligned}$$

Propositioin 5.1 is proved in §§5.3–5.4.

5.3. **Preparation for the first stage.** Let $E, E^\star$ be any two Borel sets satisfying $\mathcal{T}(E,E^\star) > \frac{1}{2}A(p,p^\star)|E|^{1/p}|E^\star|^{1/p^\star}$. Let $\delta_0, \delta_1, \varepsilon_0 > 0$ be small constants, chosen sufficiently small to satisfy constraints which will be specified, in principle, later in the proof. In particular, we will require that δ_1/δ_0 is sufficiently small, and that $\delta_0 \le \epsilon_0$.

Localize $E, E^\star$ to cubes $Q, Q^\star$ of sidelengths $\le (\alpha\alpha^\star)^{\delta_1}$ as detailed in §4.1 to obtain subsets $E_\sharp = E \cap Q$ and $E_\sharp^\star = E^\star \cap Q^\star$ for which

$$\mathcal{T}(E_\sharp, E_\sharp^\star) \ge c(\alpha\alpha^\star)^{2d\delta_1}\mathcal{T}(E,E^\star). \tag{5.9}$$

Since $\mathcal{T}(E,E^\star) > \frac{1}{2}A|E|^{1/p}|E^\star|^{1/p^\star} \ge \frac{1}{2}A|E_\sharp|^{1/p}|E_\sharp^\star|^{1/p^\star}$, (5.9) forces $|E_\sharp| \ge (\alpha\alpha^\star)^{2dp\delta_1}|E|$ and $|E_\sharp^\star| \ge (\alpha\alpha^\star)^{2dp^\star\delta_1}|E^\star|$. The associated average numbers of incidences

$$\alpha_\sharp = |E_\sharp|^{-1}\mathcal{T}(E_\sharp, E_\sharp^\star), \qquad \alpha_\sharp^\star = |E_\sharp^\star|^{-1}\mathcal{T}(E_\sharp, E_\sharp^\star)$$

thus satisfy

$$c\alpha(\alpha\alpha^\star)^{C\delta_1} \le \alpha_\sharp \le c'\alpha(\alpha\alpha^\star)^{-C\delta_1} \text{ and } c\alpha^\star(\alpha\alpha^\star)^{C\delta_1} \le \alpha_\sharp^\star \le c'\alpha^\star(\alpha\alpha^\star)^{-C\delta_1} \tag{5.10}$$

for certain constants c, c' which depend on δ_1, and a constant C which depends only on $d, p, p^\star$. Moreover $\alpha_\sharp, \alpha_\sharp^\star$ are (N,c) weakly comparable by Lemma 4.1, where N, c depend only on the choice of δ_1. The sets $E \setminus E_\sharp$ and $E^\star \setminus E_\sharp^\star$ are now discarded; the sets $E^{(d)}, E^{\star(d)}$ constructed below will be subsets of $E_\sharp, E_\sharp^\star$ and hence subsets of $Q, Q^\star$, respectively.

5.4. **The first stage.** The first stage of the construction takes as input $(E_\sharp, E^\star_\sharp)$, and outputs a pair $(E^{(d)} \subset E_\sharp,\ E^{\star(d)} \subset E^\star_\sharp)$. Define

$$\tilde E^{(d)} = \{x \in E_\sharp : T(\chi_{E^\star_\sharp})(x) \ge \tfrac12 \alpha_\sharp\}.$$

Then $\mathcal{T}(\tilde E^{(d)}, E^\star_\sharp) \ge \frac12 \mathcal{T}(E_\sharp, E^\star_\sharp)$. $E^{(d)}$ will eventually be defined to be a certain subset of $\tilde E^{(d)}$.

For $x \in \tilde E^{(d)}$, define $\tilde F_d(x) = \{t : \gamma(x,t) \in E^\star_\sharp\}$; thus $|\tilde F_d(x)| \ge c\alpha_\sharp$. For each $x \in \tilde E^{(d)}$, apply Lemma 3.1 to obtain a set $F_d(x) \subset \tilde F_d(x)$ and an interval $I_x \subset \mathbb{R}^1$ satisfying $|I_x| \lesssim (\alpha\alpha^\star)^{\delta_1}$, such that

$$|F_d(x)| \gtrsim |\tilde F_d(x)|^{1+\delta_0} \ge c\alpha_\sharp^{\delta_0} |\tilde F_d(x)|, \tag{5.11}$$

and $F_d(x)$ is an (δ_0, δ)-generic subset of I_x. Here $\delta = \delta(\delta_0)$ is the quantity appearing in Lemma 3.1. Since $\delta_0 \le \varepsilon_0$, $F_d(x)$ is (ε_0, δ)-generic. All such choices are to be made measurably. Here, as elsewhere, the value of c changes from one occurrence to the next. Partition $\tilde E^{(d)} = \cup_{i\in\mathbb{Z}} E_i^{(d)}$, where

$$E_i^{(d)} = \{x \in \tilde E^{(d)} : 2^{i-1} \le |I_x| < 2^i\}.$$

Since $|I_x| \ge |F_d(x)| \gtrsim \alpha_\sharp^{1+\delta_0}$, the pigeonhole principle guarantees that there exists $m_d \in \mathbb{Z}$ satisfying $c\alpha_\sharp^{1+\delta_0} \le 2^{m_d}$, such that

$$\mathcal{T}(E^{(d)}_{m_d}, E^\star_\sharp) \gtrsim \frac{\mathcal{T}(E_\sharp, E^\star_\sharp)}{\log(1/\alpha_\sharp)} \gtrsim \alpha_\sharp^{\delta_0} \mathcal{T}(E_\sharp, E^\star_\sharp). \tag{5.12}$$

We work henceforth only with $E_{\sharp\sharp} = E^{(d)}_{m_d}$, discarding the rest of $E_\sharp$.

Partition $\mathbb{R}^1$ as a union of dyadic intervals $J_i = [i2^{m_d}, (i+1)2^{m_d})$, and set $E_i^\star = \{x^\star \in E^\star : \Pi^\star(x^\star) \subset J_{i-1} \cup J_i \cup J_{i+1}\}$. Then $E^\star_\sharp = \cup_i E_i^\star$, and no point of $E^\star_\sharp$ belongs to more than three sets $E_i^\star$. Let $\tilde T, \tilde{\mathcal{T}}$ be respectively the localized operator and bilinear form associated to the family of intervals $\{I_x\}$ as in §4.2, related to one another by $\tilde{\mathcal{T}}(\mathcal{E}, \mathcal{E}^\star) = \int_{\mathcal{E}} \tilde T(\chi_{\mathcal{E}^\star})$. Since

$$\tilde T(\chi_{\mathcal{E}^\star})(x) \sim |F_d(x)| \gtrsim \alpha_\sharp^{\delta_0} |\tilde F_d(x)| \sim \alpha_\sharp^{\delta_0} T(\chi_{E^\star_\sharp})(x) \text{ for all } x \in E_{\sharp\sharp} \subset E_\sharp,$$

(5.12) with the substitution of the definition $E_{\sharp\sharp} = E^{(d)}_{m_d}$ implies

$$\tilde{\mathcal{T}}(E_{\sharp\sharp}, E^\star_\sharp) \gtrsim (\alpha_\sharp \alpha^\star_\sharp)^{2\delta_0} \mathcal{T}(E_\sharp, E^\star_\sharp) \tag{5.13}$$

Express $E_{\sharp\sharp} = \cup_i E_i$ as in (4.4): $E_i = \{x \in E : \exists t \in I_x : \gamma(x,t) \in E_i^\star\}$. No point belongs to more than 3 of the sets E_i. Then $\tilde{\mathcal{T}}(E_{\sharp\sharp}, E_i^\star) = \tilde{\mathcal{T}}(E_i, E_i^\star)$ for all i and consequently

$$\mathcal{T}(E_{\sharp\sharp}, E^\star_\sharp) \le \sum_i \tilde{\mathcal{T}}(E_i, E_i^\star). \tag{5.14}$$

Now the orthogonality discussed in §4.3 comes into play. Let c_0 be a small positive constant, to be specified momentarily. If constants C_1, c_1 are chosen to be sufficiently large and small respectively, then by (4.7) and (5.13), either

$$\mathcal{T}(E_{\sharp\sharp}, E^\star_\sharp) \le c_0 (\alpha_\sharp \alpha^\star_\sharp)^{3\delta_0} A(p, p^\star) |E_\sharp|^{1/p} |E^\star_\sharp|^{1/p^\star} \tag{5.15}$$

or there exists at least one index i for which

$$|E_i| \ge c_1 (\alpha_\sharp \alpha^\star_\sharp)^{C_1\delta_0} |E_\sharp| \text{ and } |E_i^\star| \ge c_1 (\alpha_\sharp \alpha^\star_\sharp)^{C_1\delta_0} |E^\star_\sharp|. \tag{5.16}$$

We will show momentarily that (5.15) cannot hold, so (5.16) must.

Now for any $x \in E_i$,

$$T(\chi_{E_i^\star})(x) \ge \tilde{T}(\chi_{E_i^\star})(x) \gtrsim \alpha^{\delta_0} T(E_\sharp^\star)(x) \gtrsim \alpha_\sharp^{1+\delta_0}.$$

Therefore $\mathcal{T}(E_i, E_i^\star) \gtrsim \alpha_\sharp^{1+\delta_0}|E_i|$ and consequently

$$\mathcal{T}(E_i, E_i^\star) \ge c(\alpha_\sharp \alpha_\sharp^\star)^{C\delta_0} \mathcal{T}(E_\sharp, E_\sharp^\star). \tag{5.17}$$

Choose any one such index i, and define

$$E^{(d)} = E_i \text{ and } E^{\star(d)} = E_i^\star. \tag{5.18}$$

Stage one is then complete.

If (5.15) were to hold then by definition of $A(p, p^\star)$,

$$\begin{aligned}\mathcal{T}(E, E^\star) &\le C(\alpha\alpha^\star)^{-C\delta_1}(\alpha_\sharp\alpha_\sharp^\star)^{-2\delta_0}\mathcal{T}(E_{\sharp\sharp}, E_\sharp^\star)\\ &\le c_1 C(\alpha\alpha^\star)^{-C\delta_1}(\alpha_\sharp\alpha_\sharp^\star)^{(3-2)\delta_0}A(p,p^\star)|E_\sharp|^{1/p}|E_\sharp^\star|^{1/p^\star}\\ &\le c_1 C(\alpha\alpha^\star)^{-C\delta_1}(\alpha\alpha^\star)^{(1-C\delta_1)\delta_0}A(p,p^\star)|E|^{1/p}|E^\star|^{1/p^\star}\\ &< \tfrac{1}{2}A(p,p^\star)|E|^{1/p}|E^\star|^{1/p^\star}\end{aligned}$$

since $\alpha, \alpha^\star \lesssim 1$, provided that δ_1/δ_0 and c_1 are sufficiently small. (5.15) thus contradicts the hypothesis that $\mathcal{T}(E, E^\star) > \frac{1}{2}A(p, p^\star)|E|^{1/p}|E^\star|^{1/p^\star}$.

Note that the factors $(\alpha_\sharp\alpha_\sharp^\star)^{C\delta_0}$ in (5.16) and (5.17) can be replaced by $(\alpha\alpha^\star)^{C'\delta_0}$ for some constant C', by (5.10). Thus

$$\begin{gathered}|E^{(d)}| \ge c(\alpha\alpha^\star)^{C\delta_0}|E_\sharp| \text{ and } |E^{(d)\star}| \ge c(\alpha\alpha^\star)^{C\delta_0}|E_\sharp^\star|\\ \mathcal{T}(E_i, E_i^\star) \ge c(\alpha\alpha^\star)^{C\delta_0}\mathcal{T}(E_\sharp, E_\sharp^\star).\end{gathered}$$

This concludes the first stage of the construction.

What has been gained by all this is not only that for each $x \in E^{(d)}$, a large subset of $\{t \in \mathbb{R} : \gamma(x,t) \in E^{\star(d)}\}$ is a *generic* subset of an interval I, but also, crucially for our analysis, that $X^\star$ has been replaced by $\pi^{\star-1}(J_{i-1} \cup J_i \cup J_{i+1})$, and that I is linked to this subset in a specific way.

5.5. **Subsequent stages of the construction.** Stage k takes as input a pair $(E^{(d-k+2)}, E^{\star(d-k+2)})$, and outputs a pair $(E^{(d-k+1)}, E^{\star(d-k+1)})$. The roles of $X, X^\star$ and all associated quantities alternate with each stage. New constants c_k, C_k appear, and are chosen to be sufficiently small and large, respectively, relative to quantities chosen at earlier stages, so that sets $E^{(d-k+1)}, E^{\star(d-k+1)}$ are constructed. This procedure can be repeated an arbitrary finite number of times, but we need do so only d times.

Built into the construction are the relation

$$m_{k-2} \le m_k \quad \text{for all } 3 \le k \le d \tag{5.19}$$

and the nesting property $E^{n+1} \times E^{\star(n+1)} \supset E^{(n)} \times E^{\star(n)}$.

5.6. **Definition of Ω.** The sets constructed in Proposition 5.1 supply the data for a mapping tower, which we next describe.

Proposition 5.2. *Suppose that $A(p, p^\star) < \infty$. Let $E, E^\star$ be Borel sets with $\mathcal{T}(E, E^\star) > \frac{1}{2}A(p,p^\star)|E|^{1/p}|E^\star|^{1/p^\star}$. Let $\alpha, \alpha^\star$ be the average numbers of incidences per point of $E, E^\star$, respectively. Then for any $\varepsilon > 0$ there exist N, c and a mapping tower Ω which generates subsets of $E, E^\star$, is central, is generically arranged, has monotonic multilength, and has* *multisize* $(\alpha_1, \cdots, \alpha_d)$ *where* $\alpha_n \ge c(\alpha\alpha^\star)^\varepsilon\alpha$ *if* $d-n$ *is even, and* $\alpha_n \ge c(\alpha\alpha^\star)^\varepsilon\alpha^\star$ *if* $d-n$ *is odd. Moreover* α_m, α_n *are* (N,c)*–weakly comparable for all* m,n*.*

To begin the construction of Ω, define $\alpha_n = \mathcal{T}(E^{(n)}, E^{\star(n)})/|E^{(n)}|$ and $\alpha_n^\star = \mathcal{T}(E^{(n)}, E^{\star(n)})/|E^{(n)\star}|$. The sets $E^{(1)}, E^{\star(1)}$ are nonempty, since $\mathcal{T}(E^{(1)}, E^{\star(1)}) \gtrsim (\alpha\alpha^\star)^{C\delta_0}\mathcal{T}(E,E^\star) > 0$. Choose a point $y \in E^{(1)}$ if d is odd, and $y \in\in E^{\star(1)}$ if d is even. y will remain fixed for the rest of the proof.

Consider the case where the ambient dimension d is even. Then $y \in E^{\star(1)}$. There exist a Borel set $F_1(y) \subset \mathbb{R}^1$, and an interval I_y of length $\sim 2^{m_1}$ which contains $F_1(y)$, such that $|F_1(y)| \gtrsim \alpha_1$, $F_1(y)$ is a generic subset of I_y, and $\gamma^\star(y,t) \in E^{(2)}$ for all $t \in F_1(y)$. Define $\Omega_1 = F_1(y)$. Set $\Phi_1(t) = \gamma^\star(y,t)$; $\Phi_1(\Omega_1) \subset E^{(2)} \subset E$.

Each point $\gamma^\star(y,\tau)$ with $\tau \in \Omega_1$ belongs to $E^{(2)}$, and consequently there are a Borel set $F_2(\Phi_1(\tau)) \subset \mathbb{R}^1$, and an interval I of length $\sim 2^{m_2}$ which contains $F_2(\Phi_1(\tau))$, such that $|F_2(\Phi_1(\tau))| \gtrsim \alpha_2$, $F_2(\Phi_1(\tau))$ is a generic subset of I, and $\gamma(\Phi_1(\tau),t) \in E^{\star(3)}$ for all $t \in F_2(\Phi_1(\tau))$. Define

$$\begin{aligned} \Omega_2 &= \{(\tau,t) \in \Omega_1 \times \mathbb{R}^1 : t \in F_2(\Phi_1(\tau))\}, \\ \Phi_2(\tau,t) &= \gamma(\Phi_1(\tau),t) \in E^{\star(3)} \subset E^\star \text{ for } (\tau,t) \in \Omega_2. \end{aligned} \tag{5.20}$$

Continue in this way, constructing Ω_n by ascending induction on n for $1 \le n \le d$. Ω has multisize $\gtrsim (\alpha_1, \cdots, \alpha_d)$. Define its multilength to be $\ell = (2^{m_1}, \cdots, 2^{m_d})$. Since $m_n \le m_{n+2}$ for all n, this multilength is monotonic.

When d is odd, the only change is that the roles of $E, E^\star$ are reversed at the first stage $n = 1$, and hence also at all subsequent stages.

Certainly $\Phi_n(\Omega_n)$ is contained in E if $d-n$ is odd and in $E^\star$ if $d-n$ is even, so Ω generates subsets of $E, E^\star$. By construction, Ω is generically arranged and has monotonic multilength. Proposition 5.1 guarantees that Ω has large multisize, in the sense that for all n

$$\begin{aligned} (\alpha\alpha^\star)^{C\delta_0}\alpha \lesssim \alpha_n \lesssim (\alpha\alpha^\star)^{-C\delta_0}\alpha \quad &\text{if } d-n \text{ is even} \\ (\alpha\alpha^\star)^{C\delta_0}\alpha^\star \lesssim \alpha_n^\star \lesssim (\alpha\alpha^\star)^{-C\delta_0}\alpha^\star \quad &\text{if } d-n \text{ is odd.} \end{aligned} \tag{5.21}$$

Lemma 5.3. *The tower* Ω *is centered.*

Proof. This is merely a matter of unraveling notation. Consider the case where d is even. We will first show that if $(t_1,t_2,t_3) \in \Omega_3$ then $|t_1 - t_3| \lesssim \ell_3 = 2^{m_3}$.

There is a certain interval $J_3 \subset \mathbb{R}^1$ of length $\sim \ell_3$ such that $\Pi(E^{(3)}) \subset J_3$. Likewise there is an interval J_1 of length $\sim \ell_1 \le \ell_3$ such that $\Pi(E^{(1)}) \subset J_1$. Since $E^{(1)} \subset E^{(3)}$, J_1 intersects the longer interval J_3.

Since $(t_1,t_2,t_3) \in \Omega_3$, $\gamma^\star(y,t_1) \in E^{(1)}$; since $\Pi(\gamma^\star(y,t_1)) \equiv t_1$ we have $t_1 \in J_1$. Likewise $\Gamma_3^\star(y,t_1,t_2,t_3) \in E^{(3)}$, and since $\Gamma_3^\star(y,t_1,t_2,t_3) = \gamma^\star(\Gamma_2^\star(y,t_1,t_2),t_3))$, necessarily $t_3 \in J_3$. This forces $|t_1 - t_3| \lesssim \ell_3$.

The corresponding inequalities $|t_{j-2} - t_j| \lesssim \ell_j$ for $j = 4, \cdots, d$ and the corresponding restriction on t_2, as well as the case where d is odd, are treated in the same way except for notational changes. □

6. Lower bounds for polynomials restricted to towers

We seek a lower bound on $|E^\star|$, in order to establish (2.22), and now have $\Phi_d(\Omega_d) \subset E^\star$. In §7 we will establish a lower bound for $|\Phi_d(\Omega_d)|$ in terms of $\int_{\Omega_d} |\det(D\Phi_d)|$. The present section focuses on such integrals, relating them to $\int_{\Omega^\dagger(z,\beta,\beta^\star)} |\det(D\Phi_d))|$ for appropriate $z, \beta, \beta^\star$; $\Omega^\dagger(z,\beta,\beta^\star)$ was defined in Definition 2.3. Centrality, genericity, and monotonicity properties of Ω_d all come into play.

6.1. **A property of polynomials.**

Lemma 6.1. *Let $D \in \mathbb{Z}^+$. There exists $\varepsilon > 0$ such that for any polynomial Q of degree $\le D$ in one real variable, any $\delta > 0$, and any (ε,δ)-generic subset S of any bounded interval I,*

$$\int_S |Q| \gtrsim \frac{|S|}{|I|} \int_I |Q| \sim |S| \max_{x\in I} |Q(x)|. \tag{6.1}$$

Moreover under the same hypotheses, if $J \subset I$ are nested intervals, and if Q vanishes at some point in J, then

$$\frac{|I|}{|J|} \max_J |Q| \lesssim \max_I |Q| \lesssim (|I|/|J|)^D \max_J |Q|. \tag{6.2}$$

The constants in these inequalities depend only on D, δ.

The routine proof is left to the reader.

6.2. **Polynomials on towers.**

Lemma 6.2. *For any $N < \infty$ there exists $\varepsilon > 0$ with the following property. Let $\delta > 0$ and let Ω be any centered parameter space tower of height D. Suppose moreover that Ω has monotonic multilength $(\ell_1, \cdots, \ell_D)$, and that for each n, for each $\tau \in \Omega_{n-1}$, the fiber $F_n(\tau) = \{t \in \mathbb{R}^1 : (\tau, t) \in \Omega_n\}$ is an (ε,δ)-generic subset of an interval of length comparable to ℓ_n. Then there exists $c > 0$ such that for any polynomial $P : \mathbb{R}^D \to \mathbb{R}$ of degree N,*

$$|\{\tau \in \Omega_D : |P(\tau)| \ge c \sup_{t\in\Omega_D} |P(t)|\}| \ge c|\Omega_D|. \tag{6.3}$$

c depends on D, N, δ but not on otherwise on P, Ω.

Proof. If P has degree $\le N$ and I is any bounded interval, then for any $\delta > 0$ it is possible to choose N or fewer subintervals $J_i \subset I$, each of length $\delta|I|$, such that $|P(s)| \ge c(\delta) \sup_I |P|$ for all $s \in I \setminus \cup_i J_i$. To do this, express P as a product of linear factors, discard any factors corresponding to real or complex zeros lying at distance $\gtrsim |I|$ from I, and choose each J_i to be centered at the real part of one of the remaining zeros. Details are left to the reader.

Suppose further that E is an (ε,δ)-generic subset of such an interval I. Then $|E \cap J_i| \le \varepsilon|E|$ for each subinterval J_i. Therefore $|E \cap (I \setminus \cup_i J_i)| \ge \frac{1}{2}|E|$, provided that $\varepsilon \le 1/2N$. Consequently $|P(s)| \gtrsim \sup_I |P|$ on a subset of E having measure comparable to the measure of E. This is the case $D = 1$ of the lemma.

It suffices to prove the conclusion of the lemma for P^2. There exist $s, \tilde{s} \in \mathbb{R}$ such that for any n, for any $\tau \in \Omega_{n-1}$, $F_n(\tau)$ is an (ε,δ)-generic subset of an interval $I(\tau)$ of length ℓ_n, which is centered at a point within distance $C\ell_n$ of s if n is odd, and of $\tilde{s}$ if n is even. Let J_n denote the interval of length $C'\ell_n$ centered at s if n is odd, and at $\tilde{s}$ if n is even; choose C' sufficiently large relative to C to guarantee that $I(\tau) \subset J_n$ for all $\tau \in \Omega_{n-1}$.

Suppose that D is even; the case of odd D will follow from the same reasoning with small changes of notation. For any $\tau \in \Omega_{D-1}$, $P^2(\tau, t) \gtrsim \ell_D^{-1} \int_{J_D} P^2(\tau, s)\, ds$ for all t in a subset of $F_D(\tau)$ having measure comparable to that of $F_D(\tau)$, as shown two paragraphs above. The right-hand side is comparable to $\sup_{s\in I(\tau)} P^2(\tau, s)$, which we have already shown to be comparable to $\sup_{s\in F_D(\tau)} P^2(\tau, s)$. Moreover, $\int_{J_D} P^2(\tau, s)\, ds$ is a polynomial in τ, of degree less than or equal to the degree of P^2. Therefore to conclude the proof it suffices to reason by induction on D, applying the induction hypothesis to the tower $(\Omega_1, \cdots, \Omega_{D-1})$ of height $D-1$, and the polynomial $\tau \mapsto \ell_D^{-1} \int_{J_D} P^2(\tau, s)\, ds$ in one fewer variable. $\square$

Continue to assume that $\gamma, \gamma^\star$ have been parametrized so that (3.6) holds: $\Pi(\gamma(x,t)) \equiv t$ and $\Pi^\star(\gamma^\star(x^\star, t)) \equiv t$. Let Ω be a centered, generically arranged mapping tower of height d, with multisize $\geq (\alpha_1, \cdots, \alpha_d)$ and monotonic multilength $\geq (\ell_1, \cdots, \ell_d)$. Let $\tau = (\tau_1, \cdots, \tau_d)$ be an arbitrary element of Ω and define the special towers $\Omega^\dagger(\rho_1, \rho_2)$ consisting of all points $(t_1, \cdots, t_d) \subset \mathbb{R}^d$ such that $|t_j - \tau_1| \leq \rho_1$ whenever j is odd, and $|t_j - \tau_2| \leq \rho_2$ whenever j is even.

Lemma 6.3. *Let $N \in \mathbb{N}$. Then there exist $c, c' > 0$ such that for any polynomial P of degree $\leq N$ and any towers $\Omega, \Omega^\dagger$ of height d as described above,*

$$\sup_{\Omega_d} |P| \geq c \sup_{\Omega_d^\dagger(\ell_1, \ell_2)} |P|. \tag{6.4}$$

Consequently

$$\int_{\Omega_d} |P(t)|\, dt \gtrsim \prod_{n>2\text{even}} \frac{\alpha_n}{\alpha_2} \cdot \prod_{n>1\text{odd}} \frac{\alpha_n}{\alpha_1} \cdot \int_{\Omega_d^\dagger(\alpha_1, \alpha_2)} |P(t)|\, dt. \tag{6.5}$$

The first conclusion follows from the same reasoning as in the proof of Lemma 6.2, taking into account the fact that a fiber F_n has measure $\gtrsim \alpha_n$ and that $\alpha_n \gtrsim \alpha_k$ where $k \in \{1, 2\}$ has the same parity as n. Then by (6.3),

$$\int_{\Omega_d} |P(t)|\, dt \sim |\Omega_d| \sup_{\Omega_d} |P| \gtrsim |\Omega_d| \cdot \sup_{\Omega_d^\dagger(\ell_1, \ell_2)} |P| \gtrsim |\Omega_d| \cdot \sup_{\Omega_d^\dagger(\alpha_1, \alpha_2)} |P| \geq \frac{|\Omega_d|}{|\Omega^\dagger(\alpha_1, \alpha_2)|} \int_{\Omega^\dagger(\alpha_1, \alpha_2)} |P|,$$

and (6.5) follows by substituting for the ratio of $|\Omega_d|$ to $|\Omega_d^\dagger(\alpha_1, \alpha_2)|$.

6.3. **Comparison with canonical towers.**

Lemma 6.4. *Suppose that $\mathcal{I}$ possesses rotational curvature. Then there exists $C < \infty$ such that for any mapping tower Ω and mapping Φ constructed via the algorithm specified in §5,*

$$\int_{\Omega_d} |\det(D\Phi)| \gtrsim (\alpha\alpha^\star)^{C\delta_0} \int_{\Omega^\dagger(x, x^\star, (\alpha\alpha^\star)^{C\delta_0}\alpha, (\alpha\alpha^\star)^{C\delta_0}\alpha^\star)} |\det(D\Phi)|. \tag{6.6}$$

Here c, C depend on the constants N, c in the definition of weak comparability.

Proof. By the construction, all α_n are weakly comparable to $\alpha, \alpha^\star$ as well. Moreover $\alpha_n \gtrsim (\alpha\alpha^\star)^{C\delta_0}\alpha_1$ whenever n is odd, and $\alpha_n \gtrsim (\alpha\alpha^\star)^{C\delta_0}\alpha_2$ whenever n is even.

Consider first the case where the Jacobian determinant $J = \det(D\Phi)$ is a polynomial. Then by (6.5),

$$\int_{\Omega_d} |J| \gtrsim \prod_{n>2 \text{ even}} \frac{\alpha_n}{\alpha_2} \prod_{n>1 \text{ odd}} \frac{\alpha_n}{\alpha_1} \int_{\Omega^\dagger(x, x^\star, \alpha_1, \alpha_2)} |J| \gtrsim (\alpha\alpha^\star)^{Cd\delta_0} \int_{\Omega^\dagger(x, x^\star, \alpha_1, \alpha_2)} |J|,$$

for certain points $x \in E, x^\star \in E^\star$. And clearly

$$\int_{\Omega^\dagger(x, x^\star, \alpha_1, \alpha_2)} |J| \gtrsim (\alpha\alpha^\star)^{C\delta_0} \int_{\Omega^\dagger(x, x^\star, \alpha(\alpha\alpha^\star)^{C\delta_0}, \alpha^\star(\alpha\alpha^\star)^{C\delta_0})} |J|,$$

since $\alpha_1, \alpha_2 \gtrsim \alpha, \alpha^\star$ up to factors of $(\alpha\alpha^\star)^{C\delta_0}$. The constants in these inequalities may be taken to be independent of the polynomial J, so long as its degree remains uniformly bounded.

In the general case where J is merely a C^∞ function, the result follows from the crude localization introduced in §4.1, by expanding J as a Taylor polynomial of fixed high degree K plus remainder. Indeed, the integral of the remainder is $O(\alpha\alpha^\star)^K$; if K is sufficiently large, this is negligible relative to $\int_{\Omega^\dagger(x,x^\star,\alpha,\alpha^\star)} |J|$, which, under the curvature hypothesis, is bounded below by a constant multiple of some positive power of $\alpha\alpha^\star$. □

The small argument in the preceding paragraph explains the fundamental role played by polynomials of bounded degree throughout this theory.

7. Lower bound for $|\Phi(\Omega_d)|$

We next establish a lower bound for $|\Phi(\Omega_d)|$, for general mappings Φ and towers Ω_d. The analysis is quite simple when Φ is a polynomial.

7.1. **The polynomial case.**

Suppose temporarily that there exist coordinates and parametrizations by which $\gamma, \gamma^\star$ are expressed as polyomials. Then Φ and its Jacobian determinant $J = \det(\partial\Phi/\partial\tau)$ are likewise polynomials. By the curvature hypothesis, J is not identically zero as a function of τ; see [11].

By Bezout's theorem, there exist $m < \infty$, depending only on the degree of J and the ambient dimension d, and an algebraic subvariety $Z \subset X = \mathbb{R}^d$ such that any point $y \notin Z$ has at most m pre-images in $\mathbb{R}^d$ under Φ. Thus

$$|E^\star| \ge |\Phi(\Omega_d)| \ge m^{-1} \int_{\Omega_d} |J(\tau)|\, d\tau. \tag{7.1}$$

7.2. **The general smooth case.**

Bezout's theorem does not apply directly when $\mathcal{I}$ is merely a C^∞ manifold satisfying possessing rotational curvature, so a more elaborate argument is required. The following lemma yields the same lower bound for $|\Phi(\Omega_d)|$ as in the polynomial case. Its many hypotheses will be satisfied in the eventual application.

Lemma 7.1. *Denote by $\mathcal{Q}$ the closed unit cube in $\mathbb{R}^d$. Consider a family $\mathcal{F}$ of C^2 mappings $\Phi : \mathcal{Q} \to \mathbb{R}^d$. Let $J_\Phi = \det D\Phi$ denote the Jacobian determinant of Φ and define $\mathcal{J}_\Phi = \max_{\tau\in\mathcal{Q}} |J_\Phi(\tau)|$.*

Suppose that there exist numbers $N < \infty$, $c_0 > 0$, and $C < \infty$ such that for all $\Phi \in \mathcal{F}$
(i) $\|\Phi\|_{C^2} \le C$.
(ii)$\mathcal{J}_\Phi \ne 0$ for all $\Phi \in \mathcal{F}$, but not necessarily with any uniform lower bound.
(iii) For each $\Phi \in \mathcal{F}$ there exists a polynomial mapping $\Psi : \mathcal{Q} \to \mathbb{R}^d$, whose components all have degrees $\le N$, such that

$$\|\Phi - \Psi\|_{C^2(\mathcal{Q})} \le c_0 \mathcal{J}_\Phi^2. \tag{7.2}$$

Let there also be given constants $c_1, c_2 \in \mathbb{R}^+$ and a collection $\mathcal{S}$ of subsets $\Omega \subset \mathcal{Q}$ such that for every polynomial $P : \mathbb{R}^d \to \mathbb{R}$ of degree $\le N^d$,

$$\left|\{\tau \in \Omega : |P(\tau)| \ge c_1 \sup_{t\in\mathcal{Q}} |P(t)|\}\right| \ge c_2|\Omega|. \tag{7.3}$$

If the constant c_0 is sufficiently small, depending only on N, d, then there exists $c = c(d, N, C, c_1, c_2) > 0$ such that for all $\Phi \in \mathcal{F}$ and $\Omega \in \mathcal{S}$

$$|\Phi(\Omega)| \ge c \int_\Omega |J_\Phi| \sim |\Omega|\mathcal{J}_\Phi. \tag{7.4}$$

Note the exponent 2 on the right-hand side of (7.2).

Proof. By a constant we will mean a positive quantity which depends only on the allowed quantities d, N, C, c_1, c_2. Let Φ, Ω be given and choose a polynomial mapping Ψ, of degree not exceeding N, satisfying the stated conditions. Let $\mathcal{J}_\Psi = \max_{\tau\in\mathcal{Q}} |J_\Psi(\tau)|$. If c_0 is chosen to be small relative to the constant C in hypothesis (i), then it follows that $\|\Psi - \Phi\|_{C^2} \le c_0\mathcal{J}_\Phi^2$ is much smaller than $\mathcal{J}_\Phi$. In particular, $\frac{1}{2}\mathcal{J}_\Phi \le \mathcal{J}_\Psi \le 2\mathcal{J}_\Phi$.

By (7.3) there exists a measurable subset $\omega \subset \Omega$ satisfying $|\omega| \ge c_2|\Omega|$, such that for all $\tau \in \omega$, $|J_\Psi(\tau)| \ge c_1\mathcal{J}_\Psi$. If c_0 is sufficiently small it follows that for all $\tau \in \omega$, $|J_\Psi(\tau)| \ge c_3\mathcal{J}_\Phi$ and $|J_\Phi(\tau)| \ge c_3\mathcal{J}_\Phi$, for a certain constant c_3.

There exists a constant c_4 such that for any $\bar{x} \in \omega$, both Φ, Ψ are injective on the ball $B(\bar{x}, 4c_4d^{1/2}\mathcal{J}_\Phi)$. This follows from an inspection of standard proofs of the inverse function theorem (see for instance [5]).

Partition $\mathbb{R}^d$, except for a subset of measure zero, into cubes Q_j of sidelengths $c_4\mathcal{J}_\Phi$. Denote by Q_j^* the ball of radius $(c_5 + c_4d^{1/2})\mathcal{J}_\Phi$ whose center equals the center of Q_j, where c_5 is to be specified below. Then

$$|\Phi(\Omega \cap Q_j)| \ge |\Phi(\omega \cap Q_j)| = \int_{\omega\cap Q_j} |J_\Phi| \gtrsim \mathcal{J}_\Phi|\omega \cap Q_j|, \tag{7.5}$$

because the restriction of Φ to Q_j is injective if Q_j intersects ω.

If Φ were globally injective, we could complete the proof by summing over j. It clearly suffices to show, as a substitute, that there exists a constant m such that for almost every $y \in \mathbb{R}^d$, y belongs to $\Phi(\omega \cap Q_j)$ for at most m distinct indices j.

Discard all Q_j which do not intersect ω. Since Ψ is of bounded degree and is injective on each Q_j, it follows from Bezout's theorem that no $y \in \mathbb{R}^d$ can belong to $\Psi(\omega \cap Q_j^*)$ for more than $m = m(N, d)$ indices j. Therefore it suffices to show that if $y \in \Phi(\omega \cap Q_j)$, then $y \in \Psi(\omega \cap Q_j^*)$.

Let $\bar{x} \in \omega \cap Q_j$ satisfy $\Phi(\bar{x}) = y$. Let $B \subset Q_j^*$ be the ball centered at x of radius $c_5\mathcal{J}_\Phi$. We will show that the constants c_j can be chosen so as to ensure that $y \in \Psi(B) \subset \Psi(Q_j^*)$, which suffices since $B \subset Q_j^*$.

Recall that if $D \subset \mathbb{R}^n$ is a closed ball with boundary ∂D, if $F : D \to \mathbb{R}^n$ is continuous and if $F|_{\partial D}$ vanishes nowhere, then associated to F is its degree as a mapping from the sphere ∂D to $\mathbb{R}^n \setminus \{0\}$; this equals the degree of $x \mapsto F(x)/|F(x)|$ as a mapping from ∂D to S^{n-1}. The degree is a homotopy invariant. If this degree is nonzero, then $0 \in F(D)$, for otherwise $F|_{\partial D}$ would be homotopic to a constant map via the homotopy $\partial D \times [0, 1] \ni (x, r) \mapsto F(rx)$. Any embedding F of ∂D into $\mathbb{R}^n \setminus \{0\}$ with $0 \in F(D)$ must have degree ± 1.

We will show that

$$|\Phi(x) - \Psi(x)| < |\Phi(x) - y| \text{ for all } x \in \partial B. \tag{7.6}$$

Thus Ψ and Φ are homotopic as mappings from ∂B to $\mathbb{R}^d \setminus \{y\}$, hence have the same topological degree. Since $\Phi|_B$ is a diffeomorphism, and $y \in \Phi(B)\setminus\Phi(\partial B)$, $\Phi : \partial B \to \mathbb{R}^d\setminus\{y\}$ has topological degree ± 1, and therefore $\Psi(B)$ contains y.

It remains only to prove (7.6). By hypothesis,

$$|\Phi(x) - \Psi(x)| \le c_0\mathcal{J}_\Phi^2. \tag{7.7}$$

On the other hand, if $\tilde{\Phi}$ denotes the Taylor polynomial of degree one for Φ at $\bar{x}$ then

$$|\tilde{\Phi}(x) - y| = |\tilde{\Phi}(x) - \Phi(\bar{x})| \ge cC^{-d+1}|J_\Phi(\bar{x})| \cdot |x - \bar{x}| \ge cC^{-d+1}c_1c_5\mathcal{J}_\Phi^2$$

where c depends only on d. The remainder satisfies

$$|\Phi(x) - \tilde{\Phi}(x)| \le cC|x - \bar{x}|^2 \le cCc_5^2\mathcal{J}_\Phi^2$$

for another constant c which depends only on d. Thus

$$|\Phi(x)-y| = |\Phi(x)-\Phi(\bar{x})| \ge cC^{-d+1}c_1c_5\mathcal{J}_\Phi^2 - cCc_5^2\mathcal{J}_\Phi^2.$$

Comparing this to (7.7), we see that (7.6) follows if $(cC^{-d+1}c_1c_5 - cCc_5^2)\mathcal{J}_\Phi^2 \ge c_0\mathcal{J}_\Phi^2$. This can be achieved by choosing first c_5, then c_0 to be sufficiently small. □

7.3. **Measures of images of towers.** Collections of mappings satisfying the hypotheses of Lemma 7.1 arise quite naturally in our context. Consider a single C^∞ mapping ϕ from any open subset $U_0 \subset \mathbb{R}^d$ to $\mathbb{R}^d$. Suppose that its Jacobian determinant $J_\phi = \det(D\phi)$ does not vanish to infinite order at any point of U_0, that is, there exists k such that for each $z \in U_0$, $\partial^\alpha J_\phi/\partial x^\alpha(z) \ne 0$ for some multi-index satisfying $0 \le |\alpha| \le k$.

Fix a compact subset $U \subset U_0$. Fix a small constant $\rho_0 > 0$. To each $z \in U$ and each $r = (r_1, \cdots, r_d)$ with all $r_j \in (0, \rho_0)$ associate the mapping $\Phi_{x,r} : \mathcal{Q} \to \mathbb{R}^d$ defined by $\Phi_{x,r}(\tau) = \phi\big(x + (r_1\tau_1, \cdots, r_d\tau_d)\big)$. These mappings are C^∞ uniformly in x, r, since all r_j are bounded above. Define $\mathcal{J}_{\Phi(x,r)} = \max_{\tau\in\mathcal{Q}} |\det(D\Phi_{x,r})|$. If all r_j are mutually weakly comparable then the nondegeneracy condition $\partial^\alpha J_\phi/\partial x^\alpha \ne 0$ implies a lower bound

$$\mathcal{J}_{\Phi_{x,r}} \gtrsim \prod_j r_j^C \tag{7.8}$$

for some finite exponent C. This follows from Taylor expansion about $x = 0$, using the condition that some partial derivative $\partial^\alpha J_{\Phi(x,r)}$ is bounded away from zero with α in a certain finite set, since the weak comparability assumption ensures that a sufficiently high order remainder is negligible relative to the contribution of any Taylor polynomial of sufficiently high order.

Let $c_0 > 0$ be given. Let N be a large positive integer, and define $\Psi_{x,r}$ to be the Taylor polynomial of degree N for $\Phi_{x,r}(\tau)$ about $\tau = 0$. If the scaling factors r_i are mutually weakly comparable, if N is chosen to be sufficiently large relative to both k and to the parameters in the definition of weak comparability, and if ρ_0 is chosen to be sufficiently small, then the lower bound (7.8) ensures that

$$\|\Phi_{x,r} - \Psi_{x,r}\|_{C^2} \le c_0\mathcal{J}_{\Phi_{x,r}}^2 \tag{7.9}$$

uniformly for all x, r in the regions specified.

By Lemma 6.3, any centered, generically arranged tower Ω satisfies (7.3). Thus we conclude

Lemma 7.2. *Let ϕ be a smooth mapping from a subset of $\mathbb{R}^d$ to $\mathbb{R}^d$, whose Jacobian determinant J does not vanish to infinite order at any point. Then there exists $C < \infty$ such that the following inequality holds. Let Ω be any centered, generically arranged tower of any multisize $(\alpha_1, \alpha_2, \cdots, \alpha_d) \gtrsim (\cdots, \beta^\star, \beta, \beta^\star, \beta)$ and any monotonic multilength ℓ. Let $s \in \Omega_1$, and $(s, s^\star) \in \Omega_2$. Define*

$$\Omega^\dagger = \cdots \times [s^\star - \beta^\star, s^\star + \beta^\star] \times [s - \beta, s + \beta] \times [s^\star - \beta^\star, s^\star + \beta^\star] \cdots \times [s - \beta, s + \beta]$$

with d factors in the Cartesian product. Then

$$|\phi(\Omega_d)| \ge C^{-1}|\Omega_d| \max_{\tau\in\Omega^\dagger} |J(\tau)|$$

provided that $\beta, \beta^\star$ and all α_n, ℓ_j are weakly comparable to one another and are all sufficiently small.

Here C does depend on the constants in the definition of weak comparability. The leftmost factor in the Cartesian product is either $[s^\star-\beta^\star, s^\star+\beta^\star]$ or $[s-\beta, s+\beta]$, depending on the parity of d.

7.4. **Provisional conclusion.** Thus far we have proved

Proposition 7.3. *Suppose that $A(p,p^\star)<\infty$. Then for any $\varepsilon>0$ there exists $c_\varepsilon>0$ such that for any Borel sets $E,E^\star$ satisfying $\mathcal{T}(E,E^\star)>\frac{1}{2}A(p,p^\star)|E|^{1/p}|E^\star|^{1/p^\star}$,*

$$|E^\star| \ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon \lambda(c(\alpha\alpha^\star)^\varepsilon\alpha, c(\alpha\alpha^\star)^\varepsilon\alpha^\star). \tag{7.10}$$

We emphasize that c_ε does not depend on the value of $A(p,p^\star)$. Here, as elsewhere, $\alpha,\alpha^\star$ denote the average numbers of incidences per point of $E,E^\star$, respectively.

Indeed, we have shown that there exists $z\in\mathcal{I}$ such that

$$|E^\star| \ge |\Phi(\Omega_d)| \ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon \int_{\Omega^\dagger(z,c(\alpha\alpha^\star)^\varepsilon\alpha, c(\alpha\alpha^\star)^\varepsilon\alpha^\star)} |\det(D\Phi)|,$$

which by definition is the right-hand side of (7.10).

8. Removal of the assumption that $A(p,p^\star)$ is finite

8.1. **A restriction on the exponents $p,p^\star$.** There is a universal upper bound on $\frac{1}{p}+\frac{1}{p^\star}-1$, depending on the dimension d but otherwise independent of the details of the geometry of $\mathcal{I}$, beyond which no bound of the desired type can hold.

Lemma 8.1. *A necessary condition for the inequality $\mathcal{T}(E,E^\star)\le C|E|^{1/p}|E^\star|^{1/p^\star}$ is that*

$$\frac{1}{p}+\frac{1}{p^\star}-1 \le \frac{1}{d}\min(\frac{1}{p},\frac{1}{p^\star}). \tag{8.1}$$

Proof. Consider any point $x_0^\star\in X^\star$, define $E^\star$ to be a Euclidean ball of small radius ε centered at $x_0^\star$, and define $E=\{x:\mathcal{M}_x\cap E^\star\ne\emptyset\}$. Then E contains a $c\varepsilon$-neighborhood of a curve of length ~ 1, so $|E|\sim\varepsilon^{d-1}$. Moreover if c is chosen to be sufficiently small, then $T(\chi_{E^\star})\gtrsim\varepsilon$ at each point of this tubular neighborhood. Thus $\mathcal{T}(E,E^\star)\gtrsim\varepsilon^d$, while $|E|^{1/p}|E^\star|^{1/p^\star}\lesssim\varepsilon^{(d-1)/p}\varepsilon^{d/p^\star}$. It is therefore necessary that $d\ge\frac{d-1}{p}+\frac{d}{p^\star}$, or equivalently, $\frac{1}{p}+\frac{1}{p^\star}-1\le\frac{1}{dp}$. The roles of $E,E^\star$ can be interchanged to obtain the other half of (8.1). □

We will later need the analogous fact for the balls $B,B^\star$.

Lemma 8.2. *Suppose that for each N,c there exists $C<\infty$ such that for all $z\in\mathcal{I}$ and all $0<r,r^\star\lesssim 1$ which are (N,c)–weakly comparable, the ball $\mathcal{B}=\mathcal{B}(z,r,r^\star)$ satisfies $|\mathcal{B}|\le C|\pi(\mathcal{B})|^{1/p}|\pi^\star(\mathcal{B})|^{1/p^\star}$. Then the exponents $\frac{1}{p},\frac{1}{p^\star}$ must satisfy* (8.1).

Proof of Lemma 8.2. Let $\eta\in(0,1)$ be small, fix any z_0, and for arbitrarily small $\delta>0$ consider $\mathcal{B}(z_0,\delta^\eta,\delta)$. Let $B,B^\star$ be the two associated projections. The two radii δ,δ^η are weakly comparable so long as η remains fixed. Lemma (9.2) below asserts that for weakly comparable bi-radii $r,r^\star$, $|B|\sim|\mathcal{B}|/\delta^\eta$, and $|B^\star|\sim|\mathcal{B}|/\delta$. Clearly $\mathcal{B}$ is contained within a tubular neighborhood of width $C\delta$ of a smooth curve of length $C\delta^\eta$, so since $\mathcal{I}$ is a manifold of dimension $d+1$, $|\mathcal{B}|\lesssim\delta^{d+\eta}\lesssim\delta^d$. Thus since $1-\frac{1}{p}+\frac{1}{p^\star}<0$,

$$\frac{|\mathcal{B}|}{|B|^{1/p}|B^\star|^{1/p^\star}} \gtrsim |\mathcal{B}|^{1-1/p-1/p^\star}\delta^{1/p^\star}\delta^{C\eta} \gtrsim \delta^{d(1-1/p-1/p^\star)}\delta^{1/p^\star}\delta^{C\eta}$$

for some $C\in\mathbb{R}^+$ independent of η; this inequality is uniform so long as η remains fixed. By letting $\delta\to 0$, then letting $\eta\to 0$, we deduce that if the ratio $\frac{|\mathcal{B}|}{|B|^{1/p}|B^\star|^{1/p^\star}}$ is bounded uniformly for all balls $\mathcal{B}$ with weakly comparable radii, then one half of (8.1) must hold. The other half follows by reversing the roles of $X, X^\star$. □

8.2. **An induction.** Let $(p, p^\star)$ belong to the interior of the region for which $\Lambda < \infty$. We now set up an induction on scales, for certain modified inequalities which are trivial at sufficiently small scales, so that the provisional assumption of the finiteness of $A(p, p^\star)$ can be eliminated. A simpler argument is sketched in Remark 8.1 below, but that argument is unavoidably restricted to pairs $(p, p^\star)$ in the interior of the set of all admissible exponents, whereas the induction on scales could be modified to work for endpoint estimates, if other steps of the argument did.

Fix coordinate systems for $X, X^\star$. Define

$$A_\varepsilon = \sup_{E, E^\star} \mathcal{T}(E, E^\star)\, |E|^{-1/p} |E^\star|^{-1/p^\star}, \tag{8.2}$$

where the supremum is taken over all pairs of sets $E, E^\star$ which are arbitrary disjoint unions of cubes of sidelengths ε in those fixed coordinate systems. A_ε is certainly finite, for all exponents $p, p^\star \in [1, \infty]$, and it suffices to prove that A_ε is bounded above by some finite constant independent of ε whenever there exist $q < p, q^\star < p^\star$ such that $\Lambda(q, q^\star) < \infty$.

Fix ε. We carry out the algorithm described in §5, with two changes. Firstly, whenever one of the cubes which E comprises intersects $E^{(n)}$, we include that entire cube in $E^{(n)}$, and likewise for $E^\star, E^{\star(n)}$. Secondly, at each stage of the construction we consider only intervals I_x or $I^\star_{x^\star}$ of length $> \varepsilon$. Thus when we attempt to apply the pigeonhole principle at some stage $d - n + 1$, there need not exist m_n satisfying the analogue of (5.12) with the additional constraint that $2^{m_n} > \varepsilon$. If such an m_n does exist at each stage, then the proof proceeds just as above.

If no such m_n exists at stage n, then we find subsets $E_\flat, E^\star_\flat$ of $E, E^\star$ respectively such that $\mathcal{T}(E_\flat, E^\star_\flat) \ge c(\alpha\alpha^\star)^{\delta_0} \mathcal{T}(E, E^\star)$ such that either $E_\flat$ is contained in $\Pi^{-1}(J)$ for some interval J of length comparable to ε, or $E^\star_\flat$ is contained in $\Pi^{-1}_\star(J^\star)$ for some interval $J^\star$ of length comparable to ε. Moreover $E_\flat, E^\star_\flat$ are finite unions of cubes of sidelength ε. It suffices to discuss the case where $E^\star_\flat$ is contained in $\Pi^{-1}_\star(J^\star)$, since the other case is equivalent to it under interchange of the roles of $X, X^\star$.

Choose exponents $q < p$ and $q^\star < p^\star$ satisfying $\min(\frac{1}{dq}, \frac{1}{dq^\star}) \ge \frac{1}{q} + \frac{1}{q^\star} - 1$. These exist, by (8.1), since $(p, p^\star)$ is not in the interior of the set of all admissible exponents.

We claim that there exists $A^\ddagger < \infty$ such that $\mathcal{T}(E_\flat, E^\star_\flat) \le A^\ddagger |E_\flat|^{1/q} |E^\star_\flat|^{1/q^\star}$. The constant $A^\ddagger$ depends only on geometric data and on $q, q^\star$, not on any assumption about $A(p, p^\star)$. $E^\star$ can be replaced by $E^\star \cap \Pi^{\star-1}(J^\star)$ for some interval $J^\star$ of length comparable to ε. δ_0 can be taken arbitrarily small, with $q, q^\star$ fixed, so the extra factor $|E_\flat|^{\frac{1}{q} - \frac{1}{p}} |E^\star_\flat|^{\frac{1}{q^\star} - \frac{1}{p^\star}}$ in this inequality compensates for the factor $(\alpha\alpha^\star)^{-\varepsilon}$ lost through the crude localization carried out before stage 1. Thus this bound suffices to complete the proof of an *a priori* upper bound for $A_\varepsilon(p, p^\star)$, independent of ε.

Decompose $E^\star_\flat = \cup_{i=1}^M Q_i$ where each Q_i is a cube of sidelength $\sim \varepsilon$, and these cubes have pairwise disjoint interiors. To each Q_i associate the tube $\tau_i = \{\gamma^\star(x^\star, t) : x^\star \in Q_i,\, t \in \mathbb{R}\} = \{x : \mathcal{M}_x \cap Q_i \ne \emptyset\}$. These tubes have bounded overlap; there exists $K < \infty$ such that no point of X belongs to more than K tubes τ_i, uniformly in ε. This holds because the mapping $(x^\star, t) \mapsto \gamma^\star(x^\star, t)$, restricted to any slice $\{x^\star : \Pi^\star(x^\star) = s\}$ for some $s \in \mathbb{R}^1$, is a diffeomorphism, uniformly in s. Thus for any $x \in X$, $|\{t : \gamma(x, t) \in E^\star_\flat\}| \lesssim \varepsilon$; so $T(\chi_{E^\star_\flat}) \lesssim \varepsilon \chi_{\cup_i \tau_i} \sim \varepsilon \sum_i \chi_{\tau_i}$. Therefore

$$\mathcal{T}(E_\flat, E^\star_\flat) = \langle \chi_{E_\flat}, T(\chi_{E^\star_\flat}) \rangle \lesssim \varepsilon |E_\flat \cap \cup_i \tau_i|.$$

Thus it suffices to show that if $E \subset \cup_i \tau_i$ then $\varepsilon|E| \lesssim |E|^{1/q}|E^\star|^{1/q^\star} \sim |E|^{1/q}M^{1/q^\star}\varepsilon^{d/q^\star}$. This is equivalent to $|E|^{1-1/q} \lesssim M^{1/q^\star}\varepsilon^{(d/q^\star)-1}$. Since $E \subset \cup_i\tau_i$, this is equivalent to $|\cup_i\tau_i|^{1-1/q} \lesssim M^{1/q^\star}\varepsilon^{(d/q^\star)-1}$. Since $|\cup_i\tau_i| \lesssim M\varepsilon^{d-1}$, it suffices to show that $(M\varepsilon^{d-1})^{1-1/q} \lesssim M^{1/q^\star}\varepsilon^{(d/q^\star)-1}$. Since $\frac{1}{q}+\frac{1}{q^\star} \ge 1$, the power of M on the left does not exceed the power on the right. Thus it suffices to compare the powers of ε. Since $\varepsilon \lesssim 1$, we need $(d-1)(1-\frac{1}{q}) \ge \frac{d}{q^\star}-1$, which is equivalent to $\frac{1}{dq} \ge \frac{1}{q}+\frac{1}{q^\star}-1$, which we chose $q, q^\star$ to satisfy. □

Remark 8.1. An alternative argument avoids the induction on scales. As is shown in §8.3 below, Proposition 7.3 can be reinterpreted as an *a priori* estimate of the form $A(p,p^\star) \le \frac{1}{2}A(p,p^\star) + C_\eta A^\dagger(p-\eta, p^\star-\eta)$ for any $\eta > 0$, where the constant $A^\dagger$ depends only on geometric data and the exponents indicated. But because a factor $(\alpha_\sharp\alpha_\sharp^\star)^{C_0\delta_0}$ is gained in (5.15), the same reasoning actually gives

$$A(p,p^\star) \le C'_\eta A(p+\eta', p^\star+\eta') + C_\eta A^\dagger(p-\eta, p^\star-\eta) \tag{8.3}$$

for any $\eta > 0$, where $\eta' > 0$ depends on η. Any pair $(p,p^\star)$ in the interior of the region of admissible exponents can be treated by finitely many iterated applications of (8.3).

However, we plan to use the induction on scales in future work to treat exponents on the border of the admissible region, where the alternative argument cannot be applied.

8.3. **Conclusion.** This completes the proof of a version of Proposition 7.3 without the hypothesis that $A(p,p^\star) < \infty$: For any two sets $E, E^\star$ which are finite unions of cubes of any common positive sidelength, for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that (7.10) holds; $|E^\star| \ge c_\varepsilon(\alpha\alpha^\star)^\varepsilon\lambda(c(\alpha\alpha^\star)^\varepsilon\alpha, c(\alpha\alpha^\star)^\varepsilon\alpha^\star)$.

Let $a, a_\star$ be the exponents associated to $(p,p^\star)$ as in (2.17). Suppose now that there exist $q < p$, $q^\star < p^\star$ such that $\Lambda(q,q^\star) < \infty$. By Lemma 2.3 applied to $(q,q^\star)$, there exist $\eta > 0$ such that for any (N,c) there exists $C < \infty$ such that whenever $\beta, \beta_\star \lesssim 1$ are (N,c)–weakly comparable,

$$\lambda(\beta,\beta_\star) \ge C^{-1}\beta^{a-\eta}\beta_\star^{a_\star-1-\eta}. \tag{8.4}$$

Choose ε to be sufficiently small relative to η, and set $\beta = c_\varepsilon(\alpha\alpha^\star)^\varepsilon\alpha$ and $\beta_\star = c_\varepsilon(\alpha\alpha^\star)^\varepsilon\alpha^\star$. Then there exist N, c depending on ε but not on $\alpha, \alpha^\star$ for which $\beta, \beta_\star$ are (N,c)–weakly comparable. Since ε is small relative to η, we conclude from (7.10) and transitivity that $|E^\star| \ge C^{-1}\alpha^a(\alpha^\star)^{a_\star-1}$. By Lemma 2.3, it follows that $\mathcal{T}(E,E^\star)$ is majorized by the desired quantity $|E|^{1/p}|E|^{1/p^\star}$. This *a priori* inequality for finite unions of cubes completes the proof of our main upper bound. □

9. Properties of balls in $\mathcal{I}$

It remains to establish Lemmas 2.2 and 2.3, which are essentially geometric, rather than analytic, results. Our discussion relies heavily on work of Nagel, Stein, and Wainger [23], who established foundational results concerning two-parameter Carnot-Caratheodory balls with weakly comparable radii, although they formulated their results only in the one-parameter case.

9.1. **Doubling estimates.** We assume throughout the discussion that the vector fields $V, V^\star$ on the incidence manifold $\mathcal{I}$ satisfy the bracket condition.

Lemma 9.1. *For any $N < \infty$ there exists $C < \infty$ such that for all $z = (x,x^\star) \in \mathcal{I}$ and all sufficiently small positive $r, r^\star$ which are weakly comparable with exponent N,*

$$|\mathcal{B}(z,2r,2r^\star)| \le C|\mathcal{B}(z,r,r^\star)|. \tag{9.1}$$

This lemma is essentially contained in the theory of Nagel, Stein, and Wainger [23] concerning balls and metrics associated to vector fields. Given a finite collection of vector fields $\{V_j\}$, those authors discuss certain properties of balls with arbitrary centers x in some open set and arbitrary radii $0 < r \le r_0$. This theory can be reformulated in terms of families of finite collections of vector fields $\{rV_{j,x}\}$ depending on parameters (r,x); here $V_{j,x}$ is defined in terms of V_j by an appropriate change of coordinates mapping x to 0. One then considers only the ball of radius 1 centered at 0, for each parameter (r,x). The analysis of [23] establishes a doubling inequality for volumes of balls with a *uniform* doubling constant C, for families of vector fields satisfying appropriate uniformity hypotheses, which are however not explicitly formulated since [23] is concerned explicitly only with a special case.

Properties of balls with two independent radii, associated to $\{rV, r^\star V^\star\}$, can be reformulated in this same way in terms of families of balls of radius 1 centered at 0. It can be verified by inspection that the proofs in [23] apply in this situation, under the crucial hypothesis that $r, r^\star$ remain weakly comparable. Among the key points which must be verified, and whose analogues do hold in the weakly comparable two-parameter setting under the bracket hypothesis, are the two inequalities (17) and (26) of [23] related to Taylor expansions. We will not give any further proof of Lemma 9.1 here, but refer the reader to [23] for details. Alternatively, Lemma 9.1 is proved by Tao and Wright [37].

Lemma 9.2. *Let $N, c < \infty$ be arbitrary. Then for all $(x,x^\star) \in \mathcal{I}$ and all sufficiently small $r, r^\star$ which are (N,c)-weakly comparable,*

$$r^\star|B^\star(x,x^\star,r,r^\star)| \sim |\mathcal{B}(x,x^\star,r,r^\star)| \tag{9.2}$$

$$r|B(x,x^\star,r,r^\star)| \sim |\mathcal{B}(x,x^\star,r,r^\star)| \tag{9.3}$$

$$|B(x,x^\star,2r,2r^\star)| \sim |B(x,x^\star,r,r^\star)| \tag{9.4}$$

$$|B^\star(x,x^\star,2r,2r^\star)| \sim |B^\star(x,x^\star,r,r^\star)|, \tag{9.5}$$

with uniform upper and lower bounds so long as N, c remain fixed.

Proof. The last two conclusions follow from the first two via (9.1). The second conclusion is the same as the first with the roles of $X, X^\star$ interchanged. To prove the first, note that the intersection of $\mathcal{B}(x,x^\star,Cr,Cr^\star)$ with the preimage under $\pi^\star$ of any point of $B^\star(x,x^\star,r,r^\star)$ contains an interval of length $r^\star$. Thus $r^\star|B^\star(x,x^\star,r,r^\star)| \lesssim |\mathcal{B}(x,x^\star,Cr,Cr^\star)| \sim |\mathcal{B}(x,x^\star,r,r^\star)|$. To prove the converse inequality fix a smooth function $h : \mathcal{I} \to \mathbb{R}$ such that $V(h) \equiv 0$ but $V^\star(h)$ vanishes nowhere. Any linear combination W of $rV, r^\star V^\star$ and their iterated Lie brackets satisfies $W(h) = O(r^\star)$, since $V(h) \equiv 0$, the coefficients of W are $O(r^\star)$ whenever W involves $r^\star V^\star$, and $r, r^\star \lesssim 1$. Therefore for any $z_1, z_2 \in \mathcal{B}(x,x^\star,r,r^\star)$, $|h(z_1) - h(z_2)| = O(r^\star)$. In particular, in the intersection of any fiber of $\pi^\star$ with $\mathcal{B}(x,x^\star,r,r^\star)$, h varies by at most $O(r^\star)$. Since $V^\star(h)$ never vanishes, this implies that any fiber has one-dimensional measure $O(r^\star)$, and hence $|\mathcal{B}(x,x^\star,r,r^\star)| \lesssim r^\star|B^\star(x,x^\star,r,r^\star)|$. □

9.2. **Balls versus images.** Let

$$Q_{r,r^\star} = \cdots \times [-r^\star,r^\star] \times [-r,r] \times [-r^\star,r^\star] \times [-r,r] \subset \mathbb{R}^{d+1} \tag{9.6}$$

with $d+1$ factors in the Cartesian product. Let $z \in \mathcal{I}$. For all sufficiently small $\tau \in \mathbb{R}^{d+1}$ define

$$\Phi_\natural(\tau) = e^{t_{d+1}V} e^{t_d V^\star} e^{t_{d-1}V} e^{t_{d-2}V^\star} \cdots e^{t_1 \tilde V}(z), \tag{9.7}$$

where $\tilde V = V$ if d is even, and $\tilde V = V^\star$ if d is odd.

Lemma 9.3. *Suppose that $r, r^\star$ are weakly comparable with respect to some finite exponent N. Then*

$$|\Phi_\natural(Q_{r,r^\star})| \sim |\mathcal{B}(z,r,r^\star)|, \tag{9.8}$$

uniformly in $z, r, r^\star$.

Proof. Consider the vector fields $V_1 = rV$, $V_2 = r^\star V^\star$. Define finite collections of vector fields $\mathfrak{g}_k$ by induction on k, by setting $\mathfrak{g}_1 = \{V_1, V_2\}$, and for each $k > 1$, defining $\mathfrak{g}_k$ to be the set of all commutators $[W, W']$ such that there exist m, n satisfying $m + n = k$ with $W \in \mathfrak{g}_m$ and $W' \in \mathfrak{g}_n$. The $\mathfrak{g}_k$ are finite sets, whose cardinalities are bounded above by quantities independent of $r, r^\star$.

Let K be sufficiently large, and choose vector fields $W_1, \cdots, W_{d+1} \in \cup_{k=1}^K \mathfrak{g}_k$ such that $|\det(W_1, \cdots, W_{d+1})(z)|$ is maximal among all possible choices of $W_1, \cdots, W_{d+1}$. These depend on $r, r^\star$, and while this maximal determinant is nonzero for any nonzero $r, r^\star$ by the bracket hypothesis, it satisfies no uniform lower bound.

Nonetheless there exists a neighborhood $U_0 \subset \mathbb{R}^{d+1}$ of the origin, independent of $r, r^\star$, such that the mapping $\Psi(s) = \exp(\sum_{j=1}^{d+1} s_j W_j)(z)$ is a diffeomorphism of U_0 with an open subset of $\mathcal{I}$. $\Psi(U_0)$ depends strongly on $r, r^\star$, of course. This is proved in [23] in the one-parameter setting $r = r^\star$; the same analysis applies provided that $r, r^\star$ remain weakly comparable with arbitrary but fixed parameters N, c. It is likewise shown in [23] that

$$|\mathcal{B}(z,r,r^\star)| \sim |Q_{r,r^\star}| \max_{s \in U_0} |\partial\Psi/\partial s|, \tag{9.9}$$

for weakly comparable $r, r^\star$.

Denote by Y_1, Y_2 the pullbacks of $V_1 = rV$, $V_2 = r^\star V^\star$ under Ψ. Then as shown in [23], if U_0 is chosen to be sufficiently small then Y_1, Y_2 are C^∞ in U_0, with upper bounds uniform in $r, r^\star, z$. Moreover, Y_1, Y_2 satisfy the bracket condition *with uniform lower bounds*; indeed the pullbacks of $W_1, \cdots, W_{d+1}$ are iterated Lie brackets of Y_1, Y_2 of uniformly bounded degrees, and these are the coordinate vector fields at the origin in the s coordinate system. Since they are uniformly C^∞, they also span the tangent space at every point of a uniform neighborhood U_0.

Consider now the mapping

$$\phi(\tau) = e^{t_{d+1}Y_2} e^{t_d Y_1} e^{t_{d-1}Y_2} \cdots (0),$$

for $\tau = (t_1, \cdots, t_{d+1}) \in U_0$; $\Phi_\natural = \Psi \circ \phi$. ϕ is C^∞, with upper bounds uniform in $z, r, r^\star$. For any particular $z, r, r^\star$, its Jacobian determinant does not vanish identically in any neighborhood of $\tau = 0$, by Lemma 9.6 below. We claim a uniform lower bound: For any $c_0 > 0$ there exists $c_1 > 0$ independent of $z, r, r^\star$ such that

$$\max_{|\tau| \le c_0} |\det(D\phi)(\tau)| \ge c_1. \tag{9.10}$$

Since $\det(D\Phi_\natural) = \det(D\Psi) \cdot \det(D\phi)$, (9.10) and (9.9) together yield the desired conclusion (9.8).

(9.10) is proved by contradiction. If it were false, then there would be a sequence of such structures, determined by a sequence of values of $r, r^\star$ tending to zero, for which the associated Jacobian determinants tended uniformly to zero. By the Arzela-Ascoli theorem and the uniform upper bounds on all C^M norms of the coefficients of Y_1, Y_2 in the s coordinate system, there would exist a subsequence $Y_1^{(\nu)}, Y_2^{(\nu)}$ converging in the C^∞ topology to some limiting vector fields $\overline{Y}_1, \overline{Y}_2$. Because $Y_1^{(\nu)}, Y_2^{(\nu)}$ satisfy the bracket hypothesis uniformly in ν, $\overline{Y}_1, \overline{Y}_2$ would still satisfy the bracket hypothesis, and the associated mapping $\overline{\phi}$ would have identically vanishing Jacobian determinant, in some neighborhood of the origin. This is a contradiction, so the claim is proved. □

Lemma 9.4. *Let $(x, x^\star) \in \mathcal{I}$. For any N, c there exists $C < \infty$ such that for all (N, c)–weakly comparable pairs $r, r^\star$,*

$$C^{-1}|B^\star(x, x^\star, r, r^\star)| \le |\Phi(\Omega^\dagger(x, x^\star, r, r^\star)| \le C|B^\star(x, x^\star, r, r^\star)|. \tag{9.11}$$

Proof. Consider the mapping from $\Omega^\dagger(x, x^\star, r, r^\star) \times [-r^\star, r^\star]$ to $\mathcal{I}$ defined for $\tau = (t_1, \cdots, t_d, t_{d+1})$ by $\Phi_\natural(\tau) = e^{t_{d+1}V^\star} e^{t_d V} e^{t_{d-1}V^\star} \cdots e^{t_1 \tilde{V}}(x, x^\star)$ where $\tilde{V}$ is either V or $V^\star$, as dictated by parity, and $e^W(\cdot)$ denotes the exponential mapping associated to a vector field W. This is a lift of the mapping $\tilde{\Phi}(\tau) = \gamma^\star(\Phi(t_1, \cdots, t_d), t_{d+1})$ in the sense that $\pi \circ \Phi_\natural \equiv \tilde{\Phi}$.

By definition of $V^\star$, there is the additional relation $\pi^\star(\Phi_\natural(\tau)) \equiv \Phi(t_1, \cdots, t_d)$, since $V^\star$ is tangent to the fibers of $\pi^\star$. The range of $\Phi_\natural$ is contained in $\mathcal{B}(x, x^\star, Cr, Cr^\star)$, so $\Phi(\Omega^\dagger(x, x^\star, r, r^\star)) \subset B^\star(x, x^\star, Cr, Cr^\star)$, and hence $|\Phi(\Omega^\dagger(x, x^\star, r, r^\star))| \lesssim |B^\star(x, x^\star, r, r^\star)|$.

To prove the converse inequality, observe that for any set $F \subset \mathcal{B}(x, x^\star, Cr, Cr^\star)$, $|\pi^\star(F)| \gtrsim |F|/r^\star$. This holds because the fibers of $\pi^\star$, intersected with $\mathcal{B}(x, x^\star, Cr, Cr^\star)$, are contained in intervals of diameter $Cr^\star$.

Now $\Phi(\Omega^\dagger(x, x^\star, r, r^\star)) \subset \pi^\star(F)$ where

$$F = \Phi_\natural(\Omega^\dagger(x, x^\star, r, r^\star) \times [-r^\star, r^\star]) \subset \mathcal{B}(x, x^\star, Cr, Cr^\star).$$

Therefore

$$|\Phi(\Omega^\dagger(x, x^\star, r, r^\star))| \gtrsim (r^\star)^{-1}|\Phi_\natural(\Omega^\dagger(x, x^\star, r, r^\star) \times [-r^\star, r^\star])|.$$

By the preceding lemma, this is $\gtrsim |\mathcal{B}(x, x^\star, r, r^\star)|/r^\star$, which in turn is comparable to $|B^\star(x, x^\star, r, r^\star)|$ by Lemma 9.2. □

The arguments in §7 establish

Lemma 9.5. *For any (N, c) there exists $C < \infty$ such that for all (N, c)–weakly comparable parameters $0 < r, r^\star \ll 1$ and all $z = (x, x^\star) \in \mathcal{I}$,*

$$C^{-1}\lambda(z, r, r^\star) \le |\Phi(\Omega^\dagger(z, r, r^\star)| \le C\lambda(z, r, r^\star). \tag{9.12}$$

This and Lemma 9.4 together imply the equivalence of the second condition of Lemma 2.3 with the third.

The next result was used in the proof of Lemma 9.3.

Lemma 9.6. *Let $\{Y_\alpha : \alpha \in A\}$ be a finite collection of C^∞ vector fields, satisfying the bracket condition at a point $z_0 \in \mathbb{R}^n$. For each $\beta = (\beta_1, \cdots, \beta_n) \in A^n$ define $\Phi_\beta(t_1, \cdots, t_n) = e^{t_n Y_{\beta_n}} e^{t_{n-1} Y_{\beta_{n-1}}} \cdots e^{t_1 Y_{\beta_1}}(z_0)$. Then for any $\delta > 0$ there exist $\beta \in A^n$ and $\overline{\tau} = (t_1, \cdots, t_n)$ satisfying $|\overline{\tau}| < \delta$ such that $\det(\partial\Phi_\beta/\partial\tau)(\overline{\tau}) \ne 0$. Moreover, for any $\alpha_0 \in A$ such that $Y_{\alpha_0}(z_0) \ne 0$, there exists such a multi-index β satisfying $\beta_1 = \alpha_0$.*

In the case of two vector fields Y_1, Y_2, the only possible ordered n-tuples β are $(1, 2, 1, 2, \cdots)$ and $(2, 1, 2, 1, \cdots)$; $D\Phi_\beta$ cannot have full rank if two successive indices β_j, β_{j+1} are equal. Since it is assumed in Lemma 9.3 that neither vector field vanishes anywhere, Lemma 9.6 implies the claim made in the proof of the former lemma.

Proof. Consider any $\alpha_0 \in A$ for which $Y_{\alpha_0}(z_0) \ne 0$, and set $\beta_1 = \alpha_0$. There must exist β_2 such that the derivative with respect to (t_1, t_2) of the mapping $\Psi_2(t_1, t_2) = e^{t_2 Y_{\beta_2}} e^{t_1 Y_{\beta_1}}(z_0)$ has rank 2 at a sequence of points tending to $(0, 0)$. For if not, then for sufficiently small $\delta > 0$, each vector field Y_α is tangent to the curve $M_1 = \{e^{t_1 Y_{\beta_1}}(z_0) : |t| < \delta\}$ in a relatively open subset of M_1 containing z_0. Therefore any iterated Lie bracket of the vector fields

Y_α is likewise tangent to M_1 in that same relatively open subset, contradicting the bracket hypothesis.

Fix such a β_2, and for any $\beta_3 \in A$ consider the mapping $\Psi_3(t_1,t_2,t_3) = e^{t_3 Y_{\beta_3}} e^{t_2 Y_{\beta_2}} e^{t_1 Y_{\beta_1}}(z_0)$. There must exist β_3 such that $D\Psi_3$ has rank 3 at a sequence of points (t_1,t_2,t_3) tending to $0 \in \mathbb{R}^3$. For if not, then in any neighborhood of z_0 there would exist a small open two-dimensional manifold M_2, parametrized by Φ_2 restricted to a certain open subset $U \subset \mathbb{R}^2$, such that $D\Psi_2$ has full rank at each point $(t_1,t_2) \in U$, and sucb that each vector field Y_α is tangent to M_2 at each point of M_2. Therefore the bracket hypothesis cannot hold at any point of M_2. Since the bracket hypothesis is satisfied at z_0, it must be satisfied at every point of some neighborhood of z_0, so again we have reached a contradiction.

Iterating this procedure up through dimension n yields the conclusion of the lemma. □

9.3. **The converse inequality in Theorem 2.4.** It remains to be shown that for any N,c, whenever $r,r^\star$ are (N,c)–weakly comparable, any ball $\mathcal{B} = \mathcal{B}(z,r,r^\star)$ satisfies

$$\frac{|\mathcal{B}|}{|\pi(\mathcal{B})|^{1/p}|\pi^\star(\mathcal{B})|^{1/p^\star}} \lesssim \sup_{E,E^\star} \frac{\mathcal{T}(E,E^\star)}{|E|^{1/p}|E^\star|^{1/p^\star}}, \tag{9.13}$$

where the supremum is taken over arbitrary Borel sets having strictly positive measures. To prove this set $E = B^\star(z,\frac12 r,\frac12 r^\star)$ and $E^\star = B(z,r,r^\star)$. Then $T(\chi_{E^\star}) \ge cr$ at every point of E. By Lemma 9.2 and the weak comparability assumption, $|\mathcal{B}| \sim r|E|$ and $|\pi(\mathcal{B})| \sim |E|$, while $E^\star$ is defined to be $\pi^\star(\mathcal{B})$. Consequently

$$\frac{|\mathcal{B}|}{|\pi(\mathcal{B})|^{1/p}|\pi^\star(\mathcal{B})|^{1/p^\star}} \sim \frac{r|E|}{|E|^{1/p}|E^\star|^{1/p^\star}} \lesssim \frac{\mathcal{T}(E,E^\star)}{|E|^{1/p}|E^\star|^{1/p^\star}}.$$

10. Superdoubling

The volume doubling property which is a pillar of the theory of one-parameter balls [23] does not hold, in general, for two-parameter balls in the presence of rotational curvature.

Proposition 10.1. *There exists a C^∞ incidence manifold $\mathcal{I} \subset \mathbb{R}^2\times\mathbb{R}^2$ possessing rotational curvature with a point $z_0 \in \mathcal{I}$ such that*

$$\limsup_{\max(r,r_\star)\to 0} \frac{|\mathcal{B}(z_0,2r,2r_\star)|}{|\mathcal{B}(z_0,r,r_\star)|} = \infty. \tag{10.1}$$

Proof. Let $\mathcal{I}$ be a small neighborhood of $z_0 = 0 \in \mathbb{R}^3$ equipped with coordinates (x,y,t). Let $a \in C^\infty(\mathbb{R})$ be an odd function which vanishes to infinite order at the origin, and is positive and strictly increasing on $\mathbb{R}^+$. On $\mathcal{I}$ define two vector fields $V = \partial_x$ and $V^\star = \partial_y + xy\partial_t + a(x)\partial_t$. There exist C^∞ submersions $\pi,\pi^\star$ from $\mathcal{I}$ to open sets $X, X^\star \subset \mathbb{R}^2$ such that the integral curves of $V, V^\star$ are the sets $\pi^{-1}(x)$ and $(\pi^\star)^{-1}(x^\star)$ for $x \in X$, $x^\star \in X^\star$.

Then $[V,V^\star] = (y + a'(x))\partial_t$, so $[V^\star,[V,V^\star]] = \partial_t$. Thus the bracket condition, equivalent to rotational curvature, is satisfied. Moreover $[V,[V,V^\star]] = a''(x)\partial_t$.

$\mathcal{B}(z_0,r,r_\star)$ is by definition the set of all points which can be reached by flowing from z_0 for time ≤ 1 along any real vector field $c(x,y,t)rV + c_\star(x,y,t)r_\star V^\star$ whose coefficients $c, c_\star$ are Lipschitz continuous and satisfy $c^2 + c_\star^2 \le 1$ at every point (x,y,t). In particular, the coordinates of any point $z = (x,y,t) \in \mathcal{B}(0,r,r_\star)$ satisfy $|x| \lesssim r$ and $|y| \lesssim r_\star$. It follows from the explicit form of $V, V^\star$ that

$$|t| \le rr_\star^2 + r_\star \max_{|x|\le r}|a(x)| = rr_\star^2 + r_\star \max_{|x|\le r}|a(x)| \sim rr_\star^2 + r_\star a(r).$$

In particular, if $rr_\star \le a(r)$ then every point $(x,y,t) \in \mathcal{B}(z_0,r,r_\star)$ satisfies $|t| \le 2r_\star a(r)$, so

$$\big|\mathcal{B}(z_0,r,r_\star)\big| \le 2rr_\star^2 a(r). \tag{10.2}$$

We claim that

$$\big|\mathcal{B}(z_0,5r,5r_\star)\big| \ge rr_\star^2 a(r). \tag{10.3}$$

If a is constructed to satisfy $\limsup_{x\to 0^+} a(2x)/a(x) = \infty$ then the conclusion (10.1) follows from from (10.2) and (10.3) together.

We show first that $\mathcal{B}(0,4r,4r_\star)$ contains each point $(0,0,t)$ with $|t| \le r_\star a(r)$. It suffices to prove this for $t > 0$, since a is odd. To reach any such point from $z_0 = 0$ by flowing along $V, V^\star$, first flow along rV for time 1 to $(r,0,0)$, then flow along $r_\star V^\star$. If this second flow is parametrized by s then one moves through points $(r, r_\star s, \tau(s))$ where $d\tau/ds = r_\star \cdot (rs + a(r)) \ge r_\star a(r)$. Therefore by flowing for some length of time ≤ 1 one reaches (r,y,t) for some $y \in [0, r_\star]$. Flowing along rV in reverse for time 1 brings us to $(0,y,t)$, and then flowing along $r_\star V^\star$ in reverse for time y brings us to $(0,0,t)$. The total time elapsed is ≤ 4.

The x coordinate remains constant along this motion, while the t coordinate varies at a rate $Ar_\star(a(cAr) + cAry)$, which is essentially $Ar_\star a(cAr)$. Thus we arrive at (cAr, y, t) with the desired value of t and some value of y. Then follow ArV in reverse to get to $(0,y,t)$, and then clearly following $Ar_\star V^\star$ in reverse gets us back to $(0,0,t)$ with the same value of t.

For any x,y satisfying $(x/r)^2 + (y/r_\star)^2 < 1$, there exists $s(x,y)$ such that $(x,y,s(x,y)) \in \mathcal{B}(0,r,r_\star)$. Since the coefficients of $V, V^\star$ are independent of the third coordinate, it follows that $(x,y,t+s(x,y)) \in \mathcal{B}((0,0,t),r,r_\star)$. If $|t| < r_\star a(r)$ then $(0,0,t) \in \mathcal{B}(0,4r,4r_\star)$, so $(x,y,t+s(x,y)) \in \mathcal{B}(0,5r,5r_\star)$. Thus $|\mathcal{B}(0,5r,5r_\star)| \ge 2\pi\, rr_\star^2 a(r)$. □

11. Remarks

Remark 11.1. There are three points in this argument at which an arbitrarily small positive power of $\alpha\alpha^\star$ is lost: (i) in the crude localization which replaces $E, E^\star$ by $E_\sharp, E_\sharp^\star$; (ii) in the application (5.12) of the pigeonhole principle, and (iii) in the replacement of $\tilde{F}_d(x)$ by a generic subset $F_d(x)$ (see (5.11)) whose measure need not quite be comparable to the measure of $\tilde{F}_d(x)$. We believe that we can show that the powers of $\alpha\alpha^\star$ lost in points (ii) and (iii) are more than adequately compensated for by a gain originating in the first inequality in (6.2) together with (11.1), below. We have not carried this out here, since we know of no way to avoid the loss (i) for general C^∞ incidence relations. In the real analytic case, though, the reduction to weakly comparable parameters should be unnecessary, and only the purportedly less serious losses (ii), (iii) should remain.

Remark 11.2. The assumption that the manifolds $\mathcal{M}_x, {\mathcal{M}^\star}_{x^\star}$ are one-dimensional is used in three distinct ways in the analysis. (i) It enters in the definition and use of genericity. It should be useful to extend this theory to higher dimensions, replacing intervals by subalgebraic sets of bounded, but arbitrarily large, degree and complexity. See Remark 11.6 below. (ii) It is essential in the main spatial localization which resulted in the orthogonality relation (4.5). Matters appear to be significantly more complicated in higher dimensions. (iii) When $\Phi : \omega \to \mathbb{R}^d$ and ω has dimension d, under favorable circumstances $|\Phi(\omega)| \gtrsim \int_\omega |\det D\Phi|$, but when ω has higher dimension than d, estimating the measure of $\Phi(\omega)$ with reasonable efficiency seems to be more difficult. The assumption that the submanifolds $\mathcal{M}, \mathcal{M}^\star$ are one-dimensional was essential in allowing us to restrict discussion to the situation when the domain and range of Φ have equal dimensions.

One can attempt to circumvent this difficulty by majorizing the measure of a union of images of d-dimensional slices by the *average* of their measures. In general this is devastatingly inefficient, but it nonetheless was used to obtain sharp estimates in [6], and again in [8], after appropriate preparations.

Remark 11.3. In this one-dimensional case, to every triple $(z, \alpha, \alpha^\star) \in \mathcal{I} \times \mathbb{R}^+ \times \mathbb{R}^+$ is associated an essentially unique canonical pair of sets, $\pi(\mathcal{B}(z, \alpha, \alpha^\star))$ and $\pi^\star(\mathcal{B}(z, \alpha, \alpha^\star))$. This uniqueness fails dramatically [8] for convolution with surface measure on a paraboloid in $\mathbb{R}^d$, for $d > 2$.

Conjecture 11.4. Let $X, X^\star$ be real analytic manifolds. If $\mathcal{I}$ is a real analytic submaniold of $X \times X^\star$ then we conjecture that endpoint strong type inequalities are valid. That is, if the ratios $|\mathcal{B}| \cdot |\pi(\mathcal{B})|^{-1/p} |\pi^\star(\mathcal{B})|^{-1/p^\star}$ are uniformly bounded, then for all functions $f, f^\star$, $\langle f, T(f^\star)\rangle \le C|E|^{1/p}|E^\star|^{1/p^\star}$, for the same exponents $p, p^\star$.

[8] shows how to pass from restricted weak type bounds to strong type $L^p \mapsto L^q$ bounds, even for endpoint exponents, for a related operator. Betsy Stovall [35] has extended this argument to the operators defined by convolution with arc length measure on finite portions of the curves $(t, t^2, t^3, \cdots, t^d)$ in $\mathbb{R}^d$ using the method of [6]. This paper is one step towards a proof of this conjecture, eliminating some but not all losses of $|E|^\varepsilon |E^\star|^\varepsilon$ from the analysis.

One remaining source of such a loss is the notion of a generic subset of an interval used here. An alternative notion introduced in [9] is also defective, though in a different way. We have found yet another alternative notion of genericity which avoids this loss. But other independent improvements are needed to complete the proof of the restricted weak type inequality, and not all details have yet been verified.

Question 11.5. Do there exist C^∞ incidence manifolds with rotational curvature, for which the ratios $|\mathcal{B}((x, x^\star), r, r^\star)|/|B(x, r, r^\star)|^{1/p}|B^\star(x^\star, r, r^\star)|^{1/p^\star}$ are uniformly bounded, yet the corresponding the restricted weak type inequality $\mathcal{T}(E, E^\star) \le C|E|^{1/p}|E^\star|^{1/p^\star}$ fails?

The counterexample of Proposition 10.1, concerning the volume doubling property, does not seem to lead directly to a counterexample to endpoint $L^p \to L^q$ inequalities.

Remark 11.6. There is a natural analogous notion of genericity for subsets of higher-dimensional Euclidean spaces. Let $\mathcal{A}$ be a subalgebraic[3] set of $\mathbb{R}^k$ of bounded degree and complexity, having positive Lebesgue measure. A subset $E \subset \mathcal{A}$ is said to be (ε, δ)–generic if $|E \cap \mathcal{A}^\star| \le \varepsilon |E|$ for every subalgebraic subset $\mathcal{A}^\star \subset \mathcal{A}$ of bounded degree and complexity satisfying $|\mathcal{A}^\star| \le \delta |\mathcal{A}|$.

Here "bounded degree" means that the inequalities used to define $\mathcal{A}$ involve only polynomials of degrees not exceeding a specified constant, and likewise "bounded complexity" means that at most a specified number of unions, complementations, and intersections are allowed in defining $\mathcal{A}$ from such inequalities. There is some latitude for discretion here; one might require the same maximal degree and complexity for $\mathcal{A}^\star$, or one might allow specified higher degree and complexity. One might work with sets parametrized by diffeomorphisms with rectangles, satisfying certain natural derivative bounds, rather than directly with sets defined by inequalities.

These notions underlie the analysis of a basic example in [8], for which general subalgebraic sets can be replaced by convex sets. A helpful property of convex sets is that in

[3]The theory ought to be C^∞ diffeomorphism invariant. But just as was done in §4.1, by sacrificing endpoints we can localize matters to a Euclidean ball of radius $\min(\alpha, \alpha^\star)^\varepsilon$. By Taylor approximation, everything can then be reduced to polynomials of bounded degree.

Euclidean space of any fixed dimension, any such set has a subset with uniformly comparable volume and uniformly bounded complexity.

Remark 11.7. The Jacobian determinant $J(\tau) = \det \partial\Gamma_d(x,\tau)/\partial\tau$ (or $J^\star(\tau)$, depending on the parity of d) satisfies

$$P(t) = 0 \text{ whenever there exists an index } n \in \{3, 4, \cdots, d\} \text{ such that } t_n = t_{n-2}\ . \tag{11.1}$$

Indeed if d is even then

$$\Gamma'_n(y, t_1, \cdots, t_{n-2}, t_{n-1}, t_{n-2}) = \Gamma'_{n-2}(y, t_1, \cdots, t_{n-2}) \text{ for all } t_{n-1}.$$

Therefore

$$\Phi_d(t_1, \cdots, t_{n-2}, t_{n-1}, t_{n-2}, t_{n+1}, \cdots, t_d) = \Gamma'(y, t_1, \cdots, t_{n-2}, t_{n-1}, t_{n-2}, t_{n+1}, \cdots, t_d)$$

is likewise independent of t_{n-1}. Thus the Jacobian matrix cannot have full rank when $t_{n-2} = t_n$.

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