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Workshop: Nonlinear Waves and Dispersive Equations

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Workshop: Nonlinear Waves and Dispersive Equations

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Abstracts

On Multilinear Oscillatory Integrals

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(joint work with Xiaochun Li, Terence Tao, Christoph Thiele)

Consider multilinear integral operators of the form

$$T_\lambda(f_1, \cdots, f_n) = \int_{\mathbb{R}^d} e^{i\lambda P(y)} \prod_{j=1}^{n} f_j \circ \ell_j(y)\, \eta(y)\, dy$$

where P is a real-valued polynomial, $\lambda \in \mathbb{R}$ is a large parameter, η is a smooth compactly supported cutoff function, and $\ell_j : \mathbb{R}^d \mapsto \mathbb{R}^{d_j}$ are surjective linear transformations. Is

$$|T_\lambda(\{f_j\})| \le C|\lambda|^{-\delta} \prod_j \|f_j\|_{L^\infty}$$

uniformly for all functions f_j as $|\lambda| \to \infty$?

The most fundamental example is the inequality

$$\Big| \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{-i\lambda x \cdot y} f(x) g(y) \eta(x,y)\, dx\, dy \Big| \le C|\lambda|^{-d/2} \|f\|_2 \|g\|_2,$$

which implies the L^2 boundedness of the Fourier transform. Here every point $x \in \mathbb{R}^d$ interacts with every point $y \in \mathbb{R}^d$. This talk, in contrast, is concerned with generalizations where the integral is taken over a d-dimensional linear subspace of $\prod_j \mathbb{R}^{d_j}$; most n-tuples of points $(x_1, \cdots, x_n) \in \prod_j \mathbb{R}^{d_j}$ do not interact.

In the linear/bilinear case $n = 2$ this problem has been studied intensively, in particular by Stein and by Phong-Stein but also by many others. For bilinear expressions $\iint_{\mathbb{R}^{d+d}} e^{i\lambda P(x,y)} f(x) g(y) \eta(x,y)\, dx\, dy$ with P polynomial, a power decay bound holds if and only if P is not of the form $p(x) + q(y)$. In the truly multilinear case quite little is known. The focus here is on the basic question of whether there is any decay at all.

From linear experience we expect the case of polynomial phases P to be fundamental. We're putting the strongest norm on the functions f_j not involving any smoothness, and aren't trying to quantify δ.

There is an obvious necessary condition: If $P = \sum_j q_j \circ \ell_j$ for some functions q_j then there's no decay (take $f_j = e^{-i\lambda q_j}$ to cancel out all the apparent oscillation).

Definition. P is nondegenerate relative to $\{\ell_j\}$ if P can not be represented as $\sum_j q_j \circ \ell_j$ for any functions q_j.

Question. Does power decay always hold for nondegenerate polynomial phase functions P? This remains open, even for quadratic polynomials in three variables.

Lemma. (Suppose P homogeneous, to simplify statements.) The following are equivalent:

(1) $P \ne \sum_j q_j \circ \ell_j$ for polynomials q_j of degrees $\le$ degree(P).
(2) $P \ne \sum_j h_j \circ \ell_j$ for any distributions h_j.

(3) There exists a constant-coefficient homogeneous linear partial differential operator $\mathcal{L}$ satisfying $\mathcal{L}(f_j \circ \ell_j) \equiv 0$ for all functions f_j, for all j and $\mathcal{L}(P) \neq 0$.

Warning: Nondegeneracy of P relative to $\{\ell_j : 1 \le j \le n\}$ imposes no bound whatsoever on n in terms of the degree of P and the ambient dimension d.

Definition. P is *simply nondegenerate* if there exists $\mathcal{L}$ of the form $\mathcal{L} = \prod_j (v_j \cdot \nabla)$ which kills all functions $f_j \circ \ell_j$, yet $\mathcal{L}(P)$ does not vanish identically.

Theorem. If P is simply nondegenerate then it satisfies a power decay bound.

Proposition. When each $d_j = d-1$, simple nondegeneracy is equivalent to nondegeneracy. Consequently nondegeneracy is equivalent to the power decay property in the codimension one case $d_j = d - 1$.

Theorem. If each $d_j = 1$ and if the number of functions n satisfies $n < 2d$ then any nondegenerate polynomial P satisfies a power decay bound (under an auxiliary general position hypothesis on $\{\ell_j\}$).

A more elementary question arises in several different ways in the discussion: For what exponents $p_j \in [1, \infty]$ does the multilinear expression make sense for all $f_j \in L^{p_j}$? Bennett, Carbery, and Tao analyzed the global version (for different reasons) and obtained a nice characterization:

Theorem. Let $\ell_j : \mathbb{R}^d \mapsto \mathbb{R}^{d_j}$ be surjective linear transformations. Then $\int_{\mathbb{R}^d} \prod_j |f_j \circ \ell_j|\, dy \le C \prod_j \|f_j\|_{L^{p_j}}$ if and only if $\sum_j p_j^{-1} d_j = d$ and $\sum_j p_j^{-1} \dim(\ell_j(V)) \ge \dim(V)$ for every subspace $V \subset \mathbb{R}^d$.

I've given an alternative proof which also establishes the following generalization:

Theorem.

$$\int_{\mathbb{R}^d \cap \{y : |\ell_0(y)| \le 1\}} \prod_{j=1}^{n} |f_j \circ \ell_j(y)|\, dy \le C \prod_{j=1}^{n} \|f_j\|_{L^{p_j}}$$

for all measurable f_j if and only if every subspace $V \subset \mathbb{R}^d$ satisfies $d - \dim(V) \ge \sum_j p_j^{-1}\big(d_j - \dim(\ell_j(V))\big)$ and furthermore $\sum_j p_j^{-1} \dim(\ell_j(V)) \ge \dim(V)$ if $V \subset \operatorname{kernel}(\ell_0)$.

The results stated above for multilinear oscillatory integrals fail to cover a well-known example, and the techniques don't yield optimal decay exponents δ. The twisted convolution inequality is $\big| \iint_{\mathbb{C}^n \times \mathbb{C}^n} e^{i\lambda \Im(z \cdot \bar{w})} f_1(z) f_2(w) f_3(z - w)\, dz\, dw \big| \le C|\lambda|^{-n/2} \prod_j \|f_j\|_2$. This inequality is self-dual in sense that when it is rewritten as a trilinear expression in the three Fourier transforms $\widehat{f_j}$, precisely the same expression is obtained, except for changes in various constants.

The last part of the talk is a preliminary report on joint work with Justin Holmer. We've analyzed the inequality

$$\big| \int_{\mathbb{R}^d} e^{i\lambda Q(y)} \prod_{j=1}^{n} f_j \circ \ell_j(y) \eta(y)\, dy \big| \le C|\lambda|^{-\delta_0} \prod_j \|f_j\|_{L^2}$$

where Q is a homogeneous quadratic polynomial, all $d_j = D$, all norms on the right-hand side are L^2 norms, and $\delta_0 = \frac{d}{2} - \frac{nD}{4}$ is the largest exponent for which such an estimate isn't ruled out by scaling considerations. Thus we're trying to characterize the maximally nondegenerate quadratic phase functions. We've established a sufficient condition which we believe is also necessary. Unfortunately, we don't yet have a palatable formulation of our sufficient condition, so I discuss only the method of proof without formulating the result.

Our analysis uses an FBI transform. Define $\mathcal{F}(f)(x,\xi) = \langle f, \varphi_{(x,\xi)}\rangle$ where $\varphi_{(x,\xi)}(y) = e^{iy\cdot\xi}e^{-|x-y|^2/2}$. There are a Plancherel identity and inversion formula analogous to those for the Fourier transform. Proving the desired multilinear L^2 bound is equivalent to proving a global inequality without any large parameter, of the form $\big|\int_{\mathbb{R}^d} e^{iQ}\prod_j f_j \circ \ell_j\big| \le C\prod_j \|f_j\|_{L^2}$. Here there is a preferred unit scale. With respect to the FBI transform there is no longer any self-duality.

Expressing each f_j in terms of $\mathcal{F}(f_j)$ yields $\int_{\oplus_j T^*(\mathbb{R}^D)} a(x,\xi)\prod_j \mathcal{F}(f_j)(x_j,\xi_j)\,dx\,d\xi$ where $(x,\xi) = (x_1,\xi_1,\cdots,x_n,\xi_n) \in (\mathbb{R}^{2D})^n$ and $|a(x,\xi)| \le Ce^{-c\,\mathrm{distance}((x,\xi),\Sigma)^2}$ where the linear subspace Σ equals the set of all (x,ξ) for which there exists $y \in \mathbb{R}^d$, necessarily unique, such that $\ell_j(y) = x_j$ for all j and $\nabla Q(y)+\sum_j \ell_j^*(\xi_j) = 0$. Moreover a exhibits no useful cancellation or decay on Σ. Thus a good model for this expression is $\int_\Sigma \prod_j \mathcal{F}(f_j)(x_j,\xi_j)\,d\sigma$ where σ is Lebesgue measure on Σ. This is a nonoscillatory multilinear integral operator of precisely the type discussed in the middle portion of this talk.

Under certain hypotheses of general position on $\{\ell_j\}$, the dimension of Σ is always half of the dimension of the ambient space $\oplus_j T^*(\mathbb{R}^{d_j})$. Thus scaling considerations are consistent with a bound $\big|\int_\Sigma \prod_j F_j(x_j,\xi_j)\,d\sigma\big| \le C\prod_j \|F_j\|_{L^2(T^*(\mathbb{R}^{d_j}))}$, and we have $F_j = \mathcal{F}(f_j) \in L^2$ if $f_j \in L^2$ by the Plancherel identity for the FBI transform.

Our preliminary theorem says that the original multilinear oscillatory integral operator satisfies the strongest possible L^2 decay estimate provided that Σ (that is, Σ together with the collection of mappings $\pi_j|_\Sigma$ where $\pi_j : \oplus_i T^*(\mathbb{R}^{d_i}) \mapsto T^*(\mathbb{R}^{d_j})$ is the canonical projection) satisfies the hypothesis of the theorem of Bennett, Carbery, and Tao with all exponents $p_j = 2$. Special cases include the inequality for twisted convolution, and Plancherel's inequality itself.

References

[1] M. Christ, X. Li, T. Tao, and C. Thiele, *On multilinear oscillatory integrals, nonsingular and singular*, preprint, math.CA/0311039.

Reporter: Martin Hadac