Convergence of Fourier Series

An Ancient Problem
Thrives in the Modern World

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Fourier coefficients and expansions

Periodic case $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$:

$$\widehat{f}(n) = (2\pi)^{-d} \int f(x) e^{-ix\cdot n}\, dx$$

$$f(x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n) e^{in\cdot x}.$$

Euclidean space case $\mathbb{R}^d$:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx$$

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\xi\cdot x\, d\xi}.$$

Fourier transform diagonalizes fundamentally important operators:

$$\frac{\partial^\alpha}{\partial x^\alpha} e^{ix\xi} = (i\xi)^\alpha e^{ix\xi}.$$

Fourier (1807) aimed to solve heat equation

$$u_t = u_{xx}$$
$$u(x, 0) = f(x)$$

by

$$u(x,t) = \sum_n e^{-tn^2} \widehat{f}(n) e^{inx}.$$

Method depends on expanding "any" periodic f as sum of its Fourier series.

Validity was seriously disputed at the time. (Unsurprisingly, since the appropriate definitions of "any" and "=" were lacking.)

History has vindicated both sides.

Convergence of Fourier series of functions on $\mathbb{T}^1$ and $\mathbb{R}^1$ was intensively studied for over a century. Define partial sums

$$S_N f(x) = \sum_{|n|\le N} \widehat{f}(n) e^{inx}.$$

More informative representation

$$S_N f(x) = (2\pi)^{-1} \int f(x-y) D_N(y)\, dy$$

where the Dirichlet kernels are

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)};$$

It's helpful to think of these as being essentially

$$\sim e^{iNx} x^{-1} \quad .$$

They **oscillate rapidly** and **decay** at a certain rate away from the singularity at 0.

Older results

• If f' is bounded then $S_N f(x) \to f(x)$, uniformly on $\mathbb{T}$. (Dirichlet)

• There exists $f \in C^0$ whose Fourier series diverges (at a point). (Du Bois-Reymond, 1876)

• Any f can be recovered from $\widehat{f}$; the Cesàro means

$$\frac{1}{N+1} \sum_{n=0}^{N} S_n f(x)$$

converge to f at any point of continuity. (Fejér)

• Likewise Abel means $\sum_n r^{|n|} \widehat{f}(n) e^{inx}$ converge well as $r \to 1^-$. (This is what Fourier needed for heat flow analysis.)

• For any $f \in L^2$, $S_N f \to f$ in L^2 norm. (Parseval, Plancherel 1910)

More sophisticated results

• There exists $f \in L^1$ whose Fourier series diverges at almost every point.
(Kolmogorov, 1923)

• For $1 < p < \infty$ and any $f \in L^p$, $S_N f \to f$ in L^p norm. (M. Riesz, 1927)

• If $f \in L^p$ for some $p > 1$ then $S_N f(x) \to f(x)$ almost everywhere.
($p = 2$ Carleson 1966; general p Hunt)

• Calderón-Zygmund (1952) introduced real-variable methods which generalized a large chunk of 1D Fourier analysis to $d > 1$ — but this generalization doesn't include the operators arising in convergence questions.

Partial sums and convergence in dimensions $d > 1$

What's a partial sum/integral?

$$(2\pi)^{-d} \int_{\max_j(|\xi_j|)<R} e^{ix\xi} \widehat{f}(\xi)\, d\xi$$

(integrate over a large cube)

$$(2\pi)^{-d} \int_{\max_j |\xi_j/R_j|<1} e^{ix\xi} \widehat{f}(\xi)\, d\xi$$

(integrate over a large rectangle)

$$(2\pi)^{-d} \int_{|\xi|<R} e^{ix\xi} \widehat{f}(\xi)\, d\xi$$

(integrate over a large ball)

- Cubical partial sums behave essentially as in dimension 1.
- Rectangular partial sums of a continuous function can <u>diverge everywhere</u>.

(Fefferman, 1971)

Spherical partial sums: Norm convergence

$$S_R f(x) = (2\pi)^{-d} \int_{|\xi|<R} e^{ix\xi} \widehat{f}(\xi)\, d\xi.$$

Elementary (Plancherel): $S_R f \to f$ in L^2 norm as $R \to \infty$, for any $f \in L^2$. Riesz (1927) proved same holds for L^p, for all $1 < p < \infty$.

A shock: **Theorem.** (Fefferman 1971)
For $d > 1$, spherical convergence in L^p norm holds **only** for $p = 2$.

Equivalently, the ratio $\|Bf\|_{L^p}/\|f\|_{L^p}$ can be arbitrarily large for **ball multiplier operator**

$$\widehat{Bf}(\xi) = \widehat{f}(\xi) \qquad \underline{\text{if } |\xi| < 1}$$

and $= 0$ otherwise.

(Thus integrating over all $|\xi| < 1$ is much worse than integrating over **all** ξ.)

The Besicovitch/Kakeya set

Proof based on link with an old problem.

Question: (Besicovitch, 1917)
Does there exist a closed set in $\mathbb{R}^d$ of Lebesgue measure zero, which **contains at least one unit line segment pointing in each direction**?

Answer: **YES** (Besicovitch 1924)

Construction

Build such a set with arbitrarily small measure by iterative process:

- Start with a triangle (containing unit segments in all directions over a fixed arc).
- Cut it into 2^N pieces.
- Start pushing those pieces together so they overlap an enormous amount.
- No directions "get lost", and the more overlap, the smaller the measure.

Connection?

The ball multiplier operator B may be expressed as an integral operator

$$Bf(x) = \int f(y)K(x-y)\,dy$$

where *essentially*

$$K(y) = e^{i|y|}|y|^{-(d+1)/2} \qquad \text{for } |y| \geq 1$$

(and $K \in C^\infty$, so only behavior for large $|y|$ is problematic)

The rate of decay $|x-y|^{-(d+1)/2}$ by itself would be catastrophic, but the **oscillatory factor** $e^{i|x-y|}$ plays an enormous role through considerations of **curvature and nonintegrability**.

• In dimension 1, the level sets of the functions $y \mapsto e^{i(x-y)}$ are *arithmetic progressions*, and a level set for one parameter x coincides with a level set for any other parameter x.

• In higher dimensions, the level sets are spheres centered at x; changing the center completely changes the level sets.

To fully quantity the implications of this curvature is the unsolved central problem.

When B acts on a function built out of *appropriately oscillating* building blocks living on many very thin *disjoint* rectangles, it simply *moves* each building block inside a (slightly thickened) Besicovitch/Kakeya set without otherwise changing it. Everything *piles up*, producing a huge norm.

End of story?

For 1D Fourier series/integrals, convergence is improved by a gentler touch:

$$\sum_{|n|\le N}(1-\frac{|n|}{N})\widehat{f}(n)e^{inx} \text{ (Cesáro means)}$$

converge better than ordinary partial sums.

Bochner-Riesz means:

$$S^\delta f(x) = (2\pi)^{-d}\int_{|\xi|<1}(1-|\xi|^2)^\delta \widehat{f}(\xi)e^{ix\xi}\, d\xi$$

where $\delta > 0$.

Conjecture:

- S^δ is bounded <u>for all $\delta > 0$</u>, if $\frac{2d}{d+1} < p < \frac{2d}{d-1}$.

- S^δ is bounded on L^p (whence L^p norm convergence holds for corresponding Fourier "partial sums") provided that $\delta > d|\frac{1}{2} - \frac{1}{p}| - \frac{1}{2}$.

Results

- Conjecture holds in dimension 2.

(Carleson-Sjölin 1972)

- Conjecture holds for $1 < p < \frac{2(d+1)}{d+3}$.

(Fefferman-Stein 1970's)

- Conjecture holds for slightly larger range of parameters (Bourgain 1990)

- Certain generalizations of the conjecture

(Hörmander 1971) are
True for $p < \frac{2(d+1)}{d+3}$ but
False for all larger p (Bourgain 1990's)

Counterparts for Besicovitch's problem

Question: Suppose that $E \subset \mathbb{R}^d$ contains at least one unit line segment L in each direction. How small can (Hausdorff) dimension of E be? (More generally, what if L merely intersects E in a set with positive (1D) measure?)

All known Besicovitch-Kakeya sets in $\mathbb{R}^d$ have (measure zero but) **full dimension** d.

Conjecture: E must have full dimension d.

Results

• Conjecture correct in dimension 2
(C. Fefferman+Cordoba, circa 1975)

• Dimension $\geq \frac{d+1}{2}$
(Bourgain 1990; Christ et. al. 1985)

• For certain generalizations, $\frac{d+1}{2}$ is optimal.
(Bourgain)

• Dimension $\geq \frac{d+2}{2}$ (Wolff 1995)

• Myriad subsequent small improvements in low dimensions
(various authors including Katz-Tao)

• Dimension $\geq \frac{6d+5}{11}$
(Bourgain and Katz-Tao 1999)

Proofs in dimensions ≥ 3 have been based on **combinatorial reasoning**.

Simple sample: Bourgain's "bush" argument:

Consider a family of rectangles in $\mathbb{R}^d$ of dimensions $(1, \delta, \ldots, \delta)$ where $\delta \ll 1$, pointing in a set of "δ–separated" directions.

Goal: Measure of their union is $\geq c\delta^{\gamma}$; the smaller the exponent $\gamma > 0$, the better the estimate.

Argument:

- If the union has very small measure then at least one point P lies in many rectangles.
- Consider the union of this subfamily.
- At distance $\geq \frac{1}{2}$ from P they are all (essentially) disjoint; so the measure of their union is reasonably large.

Wolff refined this by analyzing "hairbrushes", which result when many rectangles intersect a single "backbone" rectangle rather than a single point.

More intricate combinatorial arguments have since been devised, but overall the results remain fragmentary. Many other problems in this general category of "Radon-like transforms" remain unsolved.

Sums and Differences

A completely different line of reasoning was started by Bourgain, who observed the relevance of some ideas of Gowers.

Proposition. Let A, B be sets of cardinality $\leq N$ in some Abelian group, let $G \subset A \times B$, and suppose that the set of sums $\{a + b : (a, b) \in G\}$ also has cardinality $\leq N$. Then the set of differences $\{a - b : (a, b) \in G\}$ has cardinality $\leq N^{2-\varepsilon}$ where $\varepsilon = \frac{1}{6}$. $\square$

The best possible exponent ε in such an inequality is unknown, but it must be strictly less than 1.

Digression: Szemerédi's theorem

Theorem. Let $\delta > 0$ and let k be any positive integer. There exists $N = N(k, \delta)$ such that any subset E of $\{1, 2, \cdots, N\}$ of cardinality $\geq \delta N$ contains a finite **arithmetic progression** of length k.

Roth (1953) $k = 3$ using Fourier analysis;

Szemerédi (1969, 1975) all k by combinatorics;

Furstenberg (1977) using erdogic theory

Prior to Gowers, proofs gave terrible bounds for N in terms of δ; he gave a Fourier analysis proof and thereby vastly improved the bounds.

Connection with Besicovitch/Kakeya

Suppose that a **small** set $E \subset \mathbb{R}^d$ contains unit line segments in many directions.

- There must exist three parallel, equally spaced hyperplanes H_j such that $E \cap H_j$ is small for $j \in \{1, 2, 3\}$.
- Let $A = E \cap H_1$, $B = E \cap H_2$.
- A line passing through $a \in H_1$ and $b \in H_3$ crosses H_2 at $c = \frac{1}{2}(a + b)$.
- On the other hand, the **direction** of this line is determined by $a - b$.
- If there are many directions, and if $E \cap H_j$ is small for each j, this contradicts the sums/differences lemma.

Restriction theorem

Theorem.
(Fefferman-Stein 1970, Tomas-Stein 1979)
In $\mathbb{R}^d$, the Fourier transform **restricted to unit sphere** satisfies

$$\|\widehat{f}\|_{L^2(S^{d-1})} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

(for $f \in L^p \cap L^1$) for all $1 \leq p \leq \frac{2(d+1)}{d+3}$.

This has the shocking implication that $\widehat{f}$ is well-defined on a set of Lebesgue *measure zero*.
(Shocking because $\widehat{f}$ is defined by decomposing f as $L^1 + L^2$; $\widehat{L^2} = L^2$ functions are merely defined a.e.)

This is **not true** for all hypersurfaces; **curvature** of the sphere is essential.

This result is closely related to spherical summation because it captures an effect which occurs right at the boundary of a disk.

Extension operator

The adjoint of the operator $Rf = \widehat{f}\Big|_{S^{d-1}}$ is the **extension operator**

$$R^* f(x) = \int_{S^{d-1}} f(\xi) e^{ix\cdot\xi}\, d\sigma(\xi)$$

where σ is surface measure. It maps functions on S^{d-1} to functions on $\mathbb{R}^d$. The restriction inequality is equivalent to

$$\|R^* f\|_{L^q(\mathbb{R}^d)} \le C \|f\|_{L^2(S^{d-1})}$$

where $q = \frac{2(d+1)}{d-1}$.

The operator R^*R has a more familiar description:

$$(R^*R)f(x) = \int K(x-y) f(y)\, dy$$

where $K(x) \sim e^{i|x|}|x|^{-\rho}$ for large x; $\rho = \frac{d-1}{2}$. This has the same form as the disk multiplier operator, except for a different exponent ρ.

Thus R and R^* are in a sense **square roots** of an operator in the same family as the disk multiplier and Bochner-Riesz multipliers.

Application: Strichartz inequalities

In coordinates $(\xi_1, \ldots, \xi_d, \tau)$ in $\mathbb{R}^{d+1}$, Fourier restriction works equally well for the paraboloid $\tau = |\xi|^2$. The extension operator maps $F(\xi)$ to

$$R^*F(x,t) = \int_{\mathbb{R}^d} e^{it|\xi|^2} e^{ix\cdot\xi} F(\xi)\, d\xi.$$

Consider the (linear) Schrödinger equation

$$iu_t + \Delta_x u = 0$$

in $\mathbb{R}^d_x \times \mathbb{R}^1_t$ with initial condition $u(x,0) = f(x)$. The solution is $R^*(\widehat{f}\,)$! Consequently

Proposition. (Strichartz 1977)
The solution to the initial-value problem for the (free, time-dependent) Schrödinger equation satisfies

$$\|u\|_{L^q(dx\,dt)} \leq C\|f\|_{L^2}$$

where $q = \frac{2(d+2)}{d}$. In particular,

- $\int |u(x,t)|^q\, dx$ decays in a sense as $|t| \to \infty$, whereas
- $\int |u(x,t)|^2\, dx$ is independent of time.

Nonlinear application

The *nonlinear Schrödinger equation* is

$$iu_t + \Delta_x u = |u|^2 u$$

in one space variable, $d = 1$, with initial condition $u(x, 0) = f(x)$.

• Formally $\int |u(x,t)|^2 \, dx$ is still conserved, but this bound wouldn't suffice to ensure even that $|u|^2 u$ is locally integrable.
• The Strichartz inequality guarantees that $|u|^2 u \in L^2(dx\, dt)$.
• A typical application, based on a sophisticated perturbation theory utilizing a whole family of Strichartz-type inqualities:

Theorem. (Tsutsumi 1987)
For any initial datum $f \in L^2$ there exists a solution u which belongs to $L^2(dx)$ for each time t. $u(\cdot, t)$ depends continuously on t and on f, in the L^2 topology. This solution is unique in a certain sense ...

Application: Carleman inequalities

The spherical summation circle of ideas has led to sharper *Carleman inequalities*:

Theorem. (Kenig-Ruiz-Sogge 1987)
Let L be a constant-coefficient second-order partial differential operator on $\mathbb{R}^d$ ($d \geq 3$) of the form $Q + R$ where Q is the Laplacian or d'Alembertian and R is of first order. Then for any function u and any linear functional $\ell : \mathbb{R}^d \mapsto \mathbb{R}$,

$$\|e^{\ell(x)} u\|_{L^q} \leq C \|e^{\ell(x)} L u\|_{L^p}$$

where $p = \frac{2d}{d+2}$ and $q = \frac{2d}{d-2}$ is the exponent conjugate to p.

C depends only on the dimension, <u>no matter how large</u> the coefficients of R and ℓ may be.

Application of application: Unique Continuation

Carleman (1937) showed that if $u = 0$ in an open set and $|\bar{\partial} u(x)| \le C|u(x)|$, then $u \equiv 0$.

Corollary. Let Δ = Laplacian in $\mathbb{R}^d$, $d \ge 3$. Suppose that $|\Delta u(x)| \le |Vu(x)|$ for all x in some connected open set, where $V \in L^{d/2}$, and that $u \equiv 0$ in some nonempty open set. Then $u \equiv 0$. (The condition $V \in L^{d/2}$ is optimal within the scale of L^p spaces.)

How to get analytic continuation from Carleman inequalities: If $u(x) \equiv 0$ when $x_d < 0$, it suffices to prove

$$\int e^{\lambda x_d} |u(x)|^q \, dx \quad \text{remains uniformly bounded}$$

as $\lambda \to +\infty$, since the weight tends to ∞ where $x_d > 0$.

The uniform Carleman inequality, hypothesis $|\Delta u| \le |Vu|$, and gamesmanship with Hölder's inequality lead immediately to this uniform bound.

The weighted case of general ℓ reduces to the unweighted case $\ell = 0$ since

$$e^{\ell(x)} \circ L \circ e^{-\ell(x)} = L + R$$

where R is first-order with constant coefficients.

Consider the **resolvent** case $L^{-1} = (-\Delta - 1)^{-1}$. On Fourier side,

$$\Big((-\Delta - 1)^{-1} f\Big)^{\wedge} = (|\xi|^2 - 1)^{-1} \widehat{f}(\xi)$$

$(|\xi|^2 - 1)^{-1}$ is singular precisely on a sphere; $\Big((|\xi|^2 - 1)^{-1}\Big)^{\wedge}$ has the familiar behavior

$$e^{i|x|}|x|^{-(d-1)/2} \qquad \text{for large } |x|.$$

Thus $(-\Delta - 1)^{-1}$ is nearly the same as composition $R^* \circ R$ of the restriction and extension operators.

The general case does not reduce to this one, but can be analyzed by the same techniques.

Application: Eigenfunction bounds

Let Δ be a Laplace-Beltrami operator (that is, an analogue of the Laplacian) on a smooth compact manifold M (without boundary).

Theorem. (Sogge 1987)
Let M have dimension d, and let q be the now-familiar exponent $q = \frac{2(d+1)}{d-1}$. Let f be an eigenfunction with (large) eigenvalue λ. Then

$$\|f\|_{L^q} \leq C\lambda^{\delta}\|f\|_{L^2}$$

where $\delta = 1/q$.

Exponent δ is optimal.
($M =$ sphere S^d is an example.)

Interest: restricts extent to which eigenfunctions can concentrate near a point or submanifold (eg. geodesics).

Connection with our topic

• Proof based on bounds for **spectral projection** operators mapping

$$f = \sum_j f_j,$$

where f_j is an eigenfunction with eigenvalue λ_j, to

$$\sum_{R \le \lambda_j \le R+1} f_j.$$

• On flat torus, e^{inx} is an eigenvector with eigenvalue $|n|^2$. Thus this projection operator is multiplication on the Fourier coefficient side by characteristic function of an **annulus** whose thickness 1 is $\ll$ its inner radius R.

• Thus this theorem is essentially a generalization to manifolds of the (adjoint) **restriction theorem**.