COMPACTNESS IN THE $\overline{\partial}$-NEUMANN PROBLEM, MAGNETIC SCHRÖDINGER OPERATORS, AND THE AHARONOV-BOHM EFFECT

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Abstract. Compactness of the Neumann operator in the d-bar Neumann problem is studied for weakly pseudoconvex bounded Hartogs domains in two dimensions. A nonsmooth example is given in which condition (P) fails to hold, yet the Neumann operator is compact. The main result, in contrast, is that for smoothly bounded Hartogs domains, condition (P) is equivalent to compactness.

The analyses of both compactness and condition (P) boil down to properties of the lowest eigenvalues of certain sequences of Schrodinger operators, with and without magnetic fields, parametrized by a Fourier variable resulting from the Hartogs symmetry. The nonsmooth counterexample is based on the Aharonov-Bohm phenomenon of quantum mechanics. For smooth domains, we prove that there always exists an exceptional sequence of Fourier variables for which the Aharonov-Bohm effect is weak. This sequence can be very sparse, so that the failure of compactness is due to a rather subtle effect.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. The $\overline{\partial}$-Neumann Laplacian $\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is a formally self-adjoint operator acting on $(0,q)$-forms with L^2-coefficients satisfying certain boundary conditions. The Kohn Laplacian $\Box_b$ is a non-elliptic operator acting on forms on the boundary, defined under certain regularity assumptions on $b\Omega$ [26]. An extensive literature is devoted to the problem of relating complex-geometric properties of $\partial\Omega$ with analytical properties of the $\overline{\partial}$-Neumann problem and $\Box_b$. Kohn [25] analyzed the $\overline{\partial}$-Neumann problem for smoothly bounded strictly pseudoconvex domains, and subsequently the subelliptic theory of $\Box$ and $\Box_b$ has become in large part understood [6, 8, 13, 27]. However, various fundamental issues for domains of infinite type, for which no subelliptic estimates hold, remain unresolved. See for instance [5],[10],[14] for surveys of aspects of the $\bar{\partial}$-Neumann problem and Kohn Laplacian. Some recent work on global regularity and on C^∞ hypoellipticity, for domains of infinite type, is in [9, 11, 12].

In this paper, we study compactness of $\Box_b$ and the $\overline{\partial}$-Neumann problem. For smoothly bounded pseudoconvex domains, it is well-known that compactness is a property weaker than subellipticity, but stronger than C^∞ global regularity [28]. The well-known property (P) was first introduced by Catlin, who proved that it implies compactness for smoothly bounded pseudoconvex domains [7], and showed that it is implied by natural geometric conditions. It was later systematically studied by Sibony for all compact sets in $\mathbb{C}^n$ from the viewpoint of potential theory [32]. A compact set K in $\mathbb{C}^n$ is said to satisfy property (P) (or to be B-regular in the terminology of Sibony) if for any $M > 0$, there exist a neighborhood U of K and a function $\rho \in C^\infty(U)$, $0 \le \rho \le 1$, such that the complex

Date: April 21, 2003. Minor revision November 5, 2003.
M.C. was supported in part by NSF grant DMS 9970660.
S.F. was supported in part by NSF grant DMS 0070697 and by an AMS centennial fellowship.

Hessian $(\partial^2\rho/\partial z_j\partial\bar z_k)$ is $\geq M$ at every point of U. Straube proved that Catlin's result on the $\overline{\partial}$-Neumann Laplacian holds for all bounded pseudoconvex domains without any regularity assumption on the boundary [35].

It has long been known that compactness precludes the presence of complex discs in the boundary for domains in $\mathbb{C}^2$ (under minimal regularity assumptions on the boundary, say Lipschitz). The converse is not true; Matheos [30] constructed a smoothly bounded, pseudoconvex, complete Hartogs domain in $\mathbb{C}^2$ whose boundary contains no complex analytic disc but whose $\overline{\partial}$-Neumann Laplacian nevertheless does not have compact resolvent. (See [17] for a discussion of this and other results on compactness.) However, whether compactness is equivalent to property (P) for smoothly bounded pseudoconvex domains in $\mathbb{C}^2$ had remained an open question[1], which we explore and answer for Hartogs domains — both in the affirmative and in the negative — in this paper.

Matheos [30] exploited the equivalence between compactness in the $\overline{\partial}$-Neumann problem, for smoothly bounded Hartogs domains in $\mathbb{C}^2$, and a certain property of an associated one-parameter family of magnetic Schrödinger operators in $\mathbb{C}^1$. Fu and Straube [18] observed that for smoothly bounded, pseudoconvex, complete Hartogs domains in $\mathbb{C}^2$, this problem is closely related to topics discussed in the mathematical physics literature under the names *diamagnetism* and *paramagnetism.* More precisely, that property (P) implies compactness for Hartogs domains is a consequence of diamagnetism, and whether compactness implies property (P) is connected to paramagnetism. For more on diamagnetism and paramagnetism the reader may consult [33],[34],[3],[16].

Our results for compactness of the $\overline{\partial}$-Neumann problem and Kohn Laplacian are twofold. By a *Hartogs domain* we mean an open subset $\Omega \subset \mathbb{C}^{d+1}$ such that whenever $(z,w) \in \Omega$, likewise $(z, e^{i\theta}w) \in \Omega$ for every $\theta \in \mathbb{R}$. Such a domain is said to be complete if whenever $(z,w) \in \Omega$ and $|w'| \leq |w|$, $(z,w') \in \Omega$.

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex Hartogs domain. Then the following are equivalent:*

1. *The $\overline{\partial}$-Neumann Laplacian $\Box$ has compact resolvent in $L^2(\Omega)$.*
2. *The Kohn Laplacian $\Box_b$ has compact resolvent.*
3. *$b\Omega$ satisfies property (P).*

For the precise meaning of (2) see Definition 2.1 below.

The equivalence between compactness and property (P) is however a quantitative rather than a qualitative phenomenon, which breaks down for boundaries having very limited regularity.

Theorem 1.2. *There exists a pseudoconvex, complete Hartogs domain $\Omega = \{(z,w) \in \mathbb{C}^2 : |w| < e^{-\psi(z)},\ |z| < 2\}$, where ψ is continuous, $\nabla\psi \in L^2$ in any compact subset of $\{|z| < 2\}$, and $\Delta\psi \in L^1$ is lower semicontinuous, such that $b\Omega$ does not satisfy property* (P)*, yet a Kohn Laplacian is well-defined on $b\Omega$ and satisfies a compactness inequality.*

However, we must emphasize the distinction between "a Kohn Laplacian" and "the Kohn Laplacian". The latter is defined with respect to the Hilbert space structure $L^2(b\Omega)$ induced by surface measure on $b\Omega$, while the Kohn Laplacian of our theorem is defined in terms of a different measure, which is smooth when expressed in terms of certain natural coordinates for $b\Omega$, but is quite different from surface measure. Our example could alternatively be

[1] McNeal [31] has introduced a variant $(\tilde{P})$ of condition (P). (P) implies $(\tilde{P})$ for all smooth pseudoconvex domains, and the two are equivalent for Hartogs domains, which are the only domains discussed in this paper.

described as a nonsmooth three dimensional CR manifold with an S^1 action, or as the unit sphere bundle in a holomorphic line bundle over a one-dimensional base manifold, equipped with a nonsmooth metric.

It is well-known that Schrödinger operators with magnetic fields arise in connection with holomorphic line bundles over complex manifolds; see for instance [15]. They arise in connection with Hartogs domains for the same reasons. Our results both amount to a semi-classical analysis of certain magnetic Schrödinger operators. Let φ be a subharmonic function on the unit disc Ω_0 such that $\nabla\varphi \in L^2(\Omega_0)$ in the sense of distributions. Let S_φ be a Schrödinger operator formally given by[2]

$$S_\varphi = -[(\partial_x + i\varphi_y)^2 + (\partial_y - i\varphi_x)^2] + \Delta\varphi$$

and let $S^0_\varphi = -\partial_x^2 - \partial_y^2 + \Delta\varphi$ be the Schrödinger operator with the same electric potential but zero magnetic potential. Denote by $\lambda^{\mathrm{m}}_\varphi$ and $\lambda^{\mathrm{e}}_\varphi$ respectively the lowest eigenvalues of the Dirichlet realizations of S_φ and of S^0_φ. The diamagnetic inequality of Simon [33] (see also Kato [24] and Simon [34]) guarantees that $\lambda^{\mathrm{e}}_\varphi \le \lambda^{\mathrm{m}}_\varphi$ for any φ. For C^∞ pseudoconvex complete Hartogs domains which are strictly pseudoconvex in a neighborhood of $\{(z,w): w = 0\}$, Fu and Straube [18] proved that condition (P) is equivalent to $\lambda^{\mathrm{e}}_{n\varphi} \to \infty$ as $n \to +\infty$. It is implicit in the analysis of Matheos [30] that $\lambda^{\mathrm{m}}_{n\varphi} \to \infty$ is equivalent to compactness.

It follows from [32] that there exists a C^∞, pseudoconvex, complete Hartogs domain $\Omega \subset \mathbb{C}^2$ for which the set of weakly pseudoconvex boundary points has positive measure, yet $b\Omega$ satisfies property (P); and there exists another such domain for which the set of strictly pseudoconvex points is dense, yet $b\Omega$ does not satisfy property (P). In light of this and [18], there exists a C^∞ subharmonic function ψ for which $\Delta\psi = 0$ on a set of positive Lebesgue measure, yet $\lambda^{\mathrm{e}}_{n\psi} \to \infty$; and on the other hand there exists another such ψ for which $\{z : \Delta\psi > 0\}$ is dense, yet $\sup_n \lambda^{\mathrm{e}}_{n\psi} < \infty$.

By virtue of these equivalences, Theorem 1.2 amounts to:

Theorem 1.3. *There exists a continuous subharmonic function φ on the unit disk Ω_0 with $\nabla\varphi \in L^2(\Omega_0)$ in the sense of distributions and $\Delta\varphi \in L^1(\Omega_0)$ lower semicontinuous, such that* $\lim_{n\to\infty} \lambda^{\mathrm{m}}_{n\varphi} = \infty$ *but* $\lim_{n\to\infty} \lambda^{\mathrm{e}}_{n\varphi} < \infty$.

This degree of regularity is quite natural from the perspective of Schrödinger operators, as it guarantees that the magnetic Schrödinger form $Q(u,v) = \langle S_\varphi u,\, v\rangle$ is well-defined for all u, v in a standard dense subclass of $L^2(\Omega_0)$, namely $C^1_0(\Omega_0)$.

Theorem 1.3 is based on the Aharonov-Bohm effect [1], a quantum phenomenon in which a physical system not exposed to a magnetic *field* is nonetheless influenced by the associated magnetic *potential*. Avron and Simon [3] gave a counterexample, based in part on this effect, to a conjectured paramagnetic inequality, which when specialized to our situation would have implied that $\lambda^{\mathrm{m}}_{n\varphi}$ is always $\le \lambda^{\mathrm{e}}_{2n\varphi}$. Theorem 1.3 realizes a more extreme form of this phenomenon, providing an example where paramagnetism can fail more dramatically.

The following weaker variant for C^∞ structures is an easy consequence of a simpler form of the same construction.

Proposition 1.4. *There exists a C^∞ subharmonic function φ on the unit disk Ω_0 such that* $\sup_{n\to\infty} \lambda^{\mathrm{e}}_{n\varphi} < \infty$ *but* $\limsup_{n\to\infty} \lambda^{\mathrm{m}}_{n\varphi} = \infty$.

[2] This is essentially a Pauli operator. The two-dimensional Pauli operator $\sigma\cdot(\nabla - a)^2$, with $a = i(\varphi_y, -\varphi_x)$, splits into two direct summands, $S_\varphi - \Delta\varphi$ and $S_\varphi + \Delta\varphi$. Nonnegativity of $\Delta\varphi$ implies that $S_\varphi - \Delta\varphi \le S_\varphi \le 2(S_\varphi - \Delta\varphi)$.

One can even make $\lambda^{\mathrm{m}}_{n\varphi} \to \infty$ as $n \to \infty$ through a subset of $\mathbb{N}$ whose complement is quite sparse. But to control every value of n, without exception, is a different matter.

Theorem 1.5. *Let φ be subharmonic, and suppose that $\Delta\varphi$ is Hölder continuous of some positive order. If $\sup_n \lambda^{\mathrm{e}}_{n\varphi} < \infty$ then $\liminf_{n\to\infty} \lambda^{\mathrm{m}}_{n\varphi} < \infty$.*

Our analysis produces a concrete bound for the rate of growth of a subsequence (n_ν) for which $\lambda^{\mathrm{m}}_{n_\nu\varphi}$ remains bounded, but this bound allows for sequences of very large gaps and strongly suggests that there should exist domains for which such subsequence all have upper density zero. The pigeonhole principle plays a crucial role in the proof that such a subsequence must exist. Theorem 1.1 is a consequence of Theorem 1.5; it is only through the existence of possibly sparse sequences of exceptional values of n that the failure of property (P) implies the failure of compactness.

Although our work may be viewed as a semiclassical analysis of magnetic Schrödinger operators, the point of view is different than that ordinarily taken in mathematical physics. There one studies $(h\nabla - A)^2$ as h tends to zero. We instead analyze $(\nabla - h^{-1}A)^2$, as $h \to 0$, and are interested in whether the lowest eigenvalue tends to infinity. In semiclassical terms we have a situation where the lowest eigenvalue is positive and tends to zero with h, and we are interested in whether or not it is $O(h^2)$. The other distinction is that our magnetic field is an arbitrary nonnegative function (with certain regularity), rather than a function with special properties.

This paper is organized as follows. In Section 2, we recall necessary definitions and basic properties of Kohn Laplacians, the $\bar\partial$–Neumann problem, and Schrödinger operators. Section 3 contains some basic inequalities for $\mathbb{C}^1$, including Lemma 3.2, which quantifies the key magnetic effect on which Theorem 1.3 is based. The example of Theorem 1.3 is constructed in Sections 4 through 6. The verification that it possesses the desired properties is given in Section 7. It will be apparent that Proposition 1.4 follows from a simplification of the same construction, so we will not provide a formal proof. Theorem 1.5 is proved in Section 8. Theorem 1.2 is then proved in Section 9 by reducing questions concerning property (P) and compactness to semi-classical analysis of Schrödinger operators. Finally, the reduction of Theorem 1.1 to Theorem 1.5 is indicated in §10.

We are grateful to B. Simon for clarifying for us the history of Kato's inequality, and of diamagnetic and paramagnetic inequalities. The second author also thanks J. J. Kohn and E. Straube for encouragement and stimulating discussions.

2. Preliminaries

2.1. **Kohn Laplacians and notions of compactness.** Let $\Omega = \{(z,w) : \ |w| < e^{-\varphi(z)},\ z \in \Omega_0\}$ be a complete Hartogs domain in $\mathbb{C}^d \times C^1$ for some $d \geq 1$. Assume that $b\Omega$ is both smooth and strictly pseudoconvex in a neighborhood of $b\Omega \cap \{w = 0\}$. (This is equivalent to the conditions that Ω_0 has smooth boundary, $\lim_{z\to b\Omega_0} \varphi(z) = \infty$, and there exists a subdomain $\hat\Omega_0 \subset\subset \Omega_0$ such that $-e^{-2\varphi}$ has a smooth extension to a neighborhood of $\overline{\Omega_0}$, whose complex Hessian is strictly positive on $\overline{\Omega}_0 \setminus \hat\Omega_0$.) We will also assume that φ is subharmonic on Ω_0 (which is equivalent, under our other hypotheses, to pseudoconvexity of Ω_0) and $\partial\varphi/\partial z \in L^2$ in any compact subset of Ω_0, in the sense of distributions.) All this implies that there exists an open domain $\Omega_0' \Subset \Omega_0$ such that φ is C^∞ and strictly plurisubharmonic in $\Omega_0 \setminus \Omega_0'$ as well as in a neighborhood of the boundary of Ω_0'.

Let $M = b\Omega$. The portion of M on which $z \in \Omega_0'$ will be parametrized by the projection $z \in \Omega_0'$ and thus identified with Ω_0'. For $1 \leq j \leq d$ let $\bar L_j = \partial_{\bar z_j} - i\varphi_{\bar z_j}\partial_\theta$ and let L_j be its conjugate. L_j and $\overline{L}_j$ may be considered as operators defined in the sense of distributions on $L^2(\Omega_0 \times \mathbb{T})$. On M one has formally the usual complex of Cauchy-Riemann operators $\bar{\partial}_b$, mapping $(0,q)$ forms to $(0,q+1)$ forms. We equip $M = b\Omega$ with a measure which has a nonvanishing C^∞ density with respect to the induced surface measure wherever $z \notin \Omega_0'$, and which agrees with Lebesgue measure in the coordinate $z \in \mathbb{C}^d$ wherever z lies in a neighborhood of the closure of Ω_0'. Note that surface measure, in contrast, carries a factor related to $\nabla\varphi$ so that when $\nabla\varphi$ is merely square integrable, surface measure is not equivalent to the measure which we have chosen.

Denote by $B^{0,q}$ the bundle of $(0,q)$ forms. Any section can be expressed as $f = \sum_J f_J(z)\overline{dz}_J$, and $\bar{\partial}_b f = \sum_{j=1}^d \bar{L}_j f \overline{dz}_j \wedge \overline{dz}_J$. We choose a Hermitian metric for $B^{0,1}$ so that $\{\overline{dz}_J\}$ form an orthonormal basis for $B^{0,q}$ at each point of Ω_0'.

For $d \geq 2$ (that is, for domains in $\mathbb{C}^3$), let

$$Q_b(f,f) = \|\bar{\partial}_b f\|_{L^2}^2 + \|\bar{\partial}_b^* f\|_{L^2}^2$$

for all sections f of $B^{0,1}$ belonging to $C_0^1(\Omega_0')$. When $d \geq 2$, for smoothly bounded domains, this is equivalent to the usual notion of compactness for $\Box_b$, the operator related to the closed sesquilinear form Q_b by $\langle \Box_b f,\, f\rangle = Q_b(f,f)$; this equivalence is a consequence of well-known estimates since M is strictly pseudoconvex where $z \notin \Omega_0'$.

For $d = 1$ we define instead

$$Q_b(f,f) = \|\bar{L}f\|_{L^2}^2 + \|Lf\|_{L^2}^2,$$

for scalar-valued $f \in C_0^1(\Omega_0')$, where $\bar{L} = \bar{L}_1$.

Definition 2.1. Let $\Omega \subset \mathbb{C}^2$, that is, $d = 1$. We say that the Kohn Laplacian has compact resolvent in $L^2(b\Omega)$ if the set of all $f \in C_0^1(\Omega_0')$ for which $Q_b(f,f) \leq 1$ is precompact in $L^2(\Omega_0')$.

For $d = 1$, that is for smooth domains in $\mathbb{C}^2$, an alternative notion of compactness is that the set of all scalar-valued $f \in C^1$ which are orthogonal to the L^2 nullspace of $\bar{\partial}_b$ and satisfy $\|\bar{\partial}_b f\|_{L^2(b\Omega)} \leq 1$ should be precompact in $L^2(b\Omega)$. For smoothly bounded pseudoconvex domains in $\mathbb{C}^2$, the range of $\bar{\partial}_b$ in L^2 is known to be closed, and compactness in the sense of Definition 2.1 would thus imply compactness in this alternative sense.

By formulating compactness as in Definition 2.1, we have avoided discussing whether $\bar{\partial}_b$ has closed range for the class of nonsmooth Hartogs domains, Hilbert space, and Hermitian structures investigated here; we have likewise sidestepped the question of the relation between compactness for the boundary Kohn Laplacian, and compactness for the $\bar{\partial}$-Neumann problem for the interior domain. In particular, the question of whether surface measure, or our alternative measure, is the relevant measure to place on the boundary has not been analyzed.

Matheos [30] proved that for arbitrary bounded pseudoconvex domains $\Omega \subset \mathbb{C}^2$ with C^∞ boundaries, compactness holds in the $\overline{\partial}$-Neumann problem if and only if the following boundary estimate holds: For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|u\|^2 \leq \varepsilon(\|Lu\|^2 + \|\overline{L}u\|^2) + C_\varepsilon \|u\|_{-1}^2 \tag{2.1}$$

for all $u \in C^\infty(b\Omega)$. This is equivalent to compactness in the sense of Definition 2.1.

2.2. The $\overline{\partial}$-Neumann Laplacian.

Let Ω be a bounded domain in $\mathbb{C}^2$. Let $L_q^2(\Omega)$, $0 \leq q \leq 2$, be the space of $(0,q)$ forms with L^2-coefficients, equipped with the standard Euclidean metric.

Let $\overline{\partial}_q\colon L^2_q(\Omega) \to L^2_{q+1}(\Omega)$ be defined in the sense of distributions with $\mathrm{dom}(\overline{\partial}_q) = \{f \in L^2_q(\Omega):\ \ \overline{\partial}_q f \in L^2_{q+1}(\Omega)\}$. Let $\overline{\partial}^*_q$ be the adjoint of $\overline{\partial}_{q-1}$. Consider

$$Q(u,v) = (\overline{\partial}_1 u, \overline{\partial}_1 v) + (\overline{\partial}^*_1 u, \overline{\partial}^*_1 v)$$

with $\mathrm{dom}(Q) = \mathrm{dom}(\overline{\partial}_1) \cap \mathrm{dom}(\overline{\partial}^*_1)$. It is easily to see that Q is a densely defined, non-negative, closed sesquilinear form. Therefore it uniquely defines a densely defined, non-negative, self-adjoint operator $\Box\colon L^2_1(\Omega) \to L^2_1(\Omega)$ such that $\mathrm{dom}(\Box^{1/2}) = \mathrm{dom}(Q)$ and $Q(u,v) = (\Box u, v)$ for $u \in \mathrm{dom}(\Box)$ and $v \in \mathrm{dom}(Q)$. The operator $\Box$ is called the $\overline{\partial}$-Neumann Laplacian. It is said to have compact resolvent if $(I+\Box)^{-1}\colon L^2_1(\Omega) \to L^2_1(\Omega)$ is compact. This is in turn equivalent to the following compactness estimate: For any $\varepsilon > 0$ there exists $C_\varepsilon < \infty$ such that

$$\|u\|^2 \le \varepsilon Q(u,u) + C_\varepsilon \|u\|^2_{-1}, \qquad \text{for all } u \in \mathrm{dom}(Q).$$

It follows from the L^2-estimates of Hörmander [23] for $\overline{\partial}$ that when Ω is pseudoconvex, $\Box$ is 1-1 and onto, and therefore has a bounded inverse N, which is called the $\overline{\partial}$-Neumann operator. In this case, $\Box$ has compact resolvent if and only if N is compact.

2.3. **Schrödinger operators in $\mathbb{C}^1$.** Let ψ be a subharmonic function defined in a bounded domain $\Omega_0 \subset \mathbb{C}^1$. Assume that $\nabla\psi \in L^2(\Omega_0)$, in the sense of distributions. Let $D_\psi = (\partial_x + i\psi_y, \partial_y - i\psi_x)$. Let

$$Q_\psi(u,v) = \langle D_\psi(u), D_\psi(v)\rangle + \langle u, \Delta\phi v\rangle$$

be the closed, non-negative sesquilinear form on $L^2(\Omega_0)$ with core $C^\infty_0(\Omega_0)$. This sesquilinear form uniquely defines a non-negative, self-adjoint, densely defined operator S_ψ on $L^2(\Omega_0)$. S_ψ is the Schrödinger operator with magnetic potential $A = (-\psi_y, \psi_x) = -\psi_y dx + \psi_x dy$, magnetic field $dA = \Delta\psi dx \wedge dy$, and electric potential $V = \Delta\psi$. It is formally written as

$$\begin{aligned} S_\psi &= D^*_\psi \cdot D_\psi + \Delta\psi \\ &= -[(\partial_x + i\psi_y)^2 + (\partial_y - i\psi_x)^2] + \Delta\psi. \end{aligned}$$

Let $S^0_\psi = -\partial^2_x - \partial^2_y + \Delta\psi$ be the Schrödinger operator with the same electric potential but zero magnetic potential. The lowest eigenvalue of S^0_ψ is given by

Definition 2.2.

$$\lambda^{\mathrm{e}}_\psi = \inf\{\|\nabla u\|^2 + \|\sqrt{\Delta\psi} u\|^2;\ \ u \in C^\infty_0(\Omega_0), \|u\| = 1\}.$$

The lowest eigenvalue of S_ψ is

Definition 2.3.

$$\lambda^{\mathrm{m}}_\psi = \inf\{Q_\psi(u,u);\ \ u \in C^\infty_0(\Omega_0), \|u\| = 1\}.$$

$\lambda^{\mathrm{m}}_\psi$ may alternatively be expressed as

$$\begin{aligned} \lambda^{\mathrm{m}}_\psi &= \inf\{4\|L_\psi(u)\|^2;\ \ u \in C^\infty_0(\Omega_0), \|u\| = 1\} \\ &= \inf\{4\|u_z e^\psi\|^2;\ \ u \in C^\infty_0(\Omega_0), \|ue^\psi\| = 1\}, \end{aligned}$$

where $L_\psi = -\partial_z + \psi_z$. The last equality above follows from an easy substitution while the preceding equality follows from the integration by parts formula:

$$\langle S_\psi(u), u\rangle = 4\int_{\Omega_0} |L_\psi(u)|^2 = \int_{\Omega_0} |D_\psi(u)|^2 + \int_{\Omega_0} \Delta\psi |u|^2.$$

Another useful integration by parts formula is the following twistor formula.

$$\int_{\Omega_0} a|L_\psi u|^2 = \int_{\Omega_0} \left((2a\psi_{z\bar z} - a_{z\bar z})|u|^2 + a|\bar L_\psi u|^2\right) + 2\,\mathrm{Re}\int_{\Omega_0} u a_z \overline{L_\psi(u)} \tag{2.2}$$

for any $a \in C^2(\overline{\Omega}_0)$. Let $b \in C^2(\overline{\Omega}_0)$ and $b \le 0$. Using the above formula with $a = 1 - e^b$ and applying the Schwarz inequality to the last term, we then obtain

$$\int_{\Omega_0} |L_\psi u|^2 \ge \int_{\Omega_0} b_{z\bar z}|u|^2 e^b + \int_{\Omega_0} 2a\psi_{z\bar z}|u|^2 + \int_{\Omega_0} a|\bar L_\psi u|^2. \tag{2.3}$$

3. Basic inequalities

In this section, we collect several inequalities which will be used in the analysis. We start with the following well-known inequality of Kato (e.g. [24] and [34]), whose relevance to diamagnetism was observed by Simon [33]. Integrals are taken with respect to Lebesgue measure on $\mathbb{C}^1$, except where otherwise indicated.

Lemma 3.1. *Let ψ be a real-valued function on a domain $\Omega_0 \subset \mathbb{C}^1$ such that $\nabla\psi \in L^2$, in the sense of distributions. Let $u \in C^1(\Omega_0)$. Then $\big|\nabla|u|(z)\big| \le |D_\psi u(z)|$ for a.e. $z \in \Omega_0$. In particular,*

$$\int_{\Omega_0} |D_\psi u|^2 \ge \int_{\Omega_0} \big|\nabla|u|\big|^2. \tag{3.1}$$

A short proof is provided for the reader's convenience.

Proof. $|u|$ is Lipschitz continuous, hence is differentiable almost everywhere. The L^∞ function $\nabla|u|$ thus defined equals the gradient in the distribution sense, and $\big|\nabla|u|\big| \le |\nabla u|$ a.e.

At points where u vanishes, the magnetic gradient equals the ordinary gradient, so the conclusion holds. In the open set where $u \ne 0$, one can locally write $u(z) = r(z)e^{i\theta(z)}$ with $r, \theta \in C^1$. Then $\nabla|u| = \nabla r$, while $(\partial_x + i\psi_y)re^{i\theta} = (r_x + i\psi_y r)e^{i\theta}$ has magnitude $(|r_x|^2 + |\psi_y r|^2)^{1/2} \ge |r_x|$. Bounding $(\partial_y - i\psi_x)re^{i\theta}$ in the same way leads to the desired inequality. □

For any $x \in \mathbb{R}$ define

$$\|x\|_* = \text{distance}\,(x, \mathbb{Z}). \tag{3.2}$$

The next lemma indicates one situation in which the magnetic gradient is relatively powerful; in fact it will be the key ingredient in our proof that $\lambda^{\mathrm{m}}_{n\varphi} \to \infty$. The result is also not original; for much more general results of the same type see [29] and [4].

Lemma 3.2. *If $\Delta\psi \equiv 0$ in $\mathcal{A} = \{z : r < |z| < R\}$, then for any $u \in C^1$,*

$$\int_{\mathcal{A}} |D_\psi u|^2 \ge \|w(\psi)\|_*^2 \int_{\mathcal{A}} |z|^{-2}|u(z)|^2, \tag{3.3}$$

where the winding number $w(\psi)$ is given by

$$w(\psi) = \frac{1}{2\pi}\int_{|z|=\rho} -\psi_y dx + \psi_x dy \tag{3.4}$$

for any $\rho \in (r, R)$. More precisely,

$$\int_0^{2\pi} |D_\psi u(\rho e^{i\theta})|^2\, d\theta \ge \rho^{-2}\|w(\psi)\|_*^2 \int_0^{2\pi} |u(\rho e^{i\theta})|^2\, d\theta \tag{3.5}$$

for any $\rho \in (r, R)$.

This expresses one instance of the Aharonov-Bohm phenomenon. The magnetic field $\Delta\psi$ vanishes identically in $\mathcal{A}$, yet if $w(\psi) \neq 0$ then (roughly speaking) a quantum particle confined to $\mathcal{A}$ and governed by the Hamiltonian $D_\psi^* D_\psi$ experiences a measurable effect from the magnetic potential. For a semiclassical analysis of this effect in certain cases see [20].

By Stokes' theorem together with the assumption $\Delta\psi = 0$, the integral (3.4) defining the winding number is independent of $\rho \in (r, R)$. If ψ extends to a C^2 function in the disk $|z| < R$, harmonic where $|z| > r$, then there is the alternative expression

$$w(\psi) = (2\pi)^{-1} \int \Delta\psi \, dx \, dy. \tag{3.6}$$

Proof. Using the polar coordinates $z = re^{i\theta}$, a straightforward calculation gives

$$|D_\psi u|^2 = |u_r + ir^{-1}\psi_\theta u|^2 + |r^{-1}u_\theta - i\psi_r u|^2. \tag{3.7}$$

It suffices to prove (3.5), which directly implies (3.3). Let $\tilde{\psi}$ be the harmonic conjugate of $\psi - w(\psi)\log|z|$ on $\mathcal{A}$. Let $v = ue^{-i\tilde{\psi}}$. Then $|D_\psi u| = |D_{w(\psi)\log|z|}v|$, and $|v| \equiv |u|$. Write

$$v(\rho e^{i\theta}) = \sum_{k=-\infty}^{\infty} \hat{v}(k,\rho)e^{ik\theta}$$

where this expression defines the Fourier coefficients $\hat{v}$. It follows from (3.7) that

$$\begin{aligned}
\int_0^{2\pi} |D_{w(\psi)\log|z|} v(\rho e^{i\theta})|^2 d\theta &\geq \rho^{-2} \int_0^{2\pi} |v_\theta - iw(\psi)v|^2 d\theta \\
&= \rho^{-2} \int_0^{2\pi} |\sum_{k=-\infty}^{\infty} \hat{v}(k,\rho)(k - w(\psi))e^{ik\theta}|^2 d\theta \\
&\geq \rho^{-2} \|w(\psi)\|_*^2 \int_0^{2\pi} |v(\rho e^{i\theta})|^2 d\theta.
\end{aligned}$$

The lemma then follows. □

A more general result holds, although only the special case formulated in Lemma 3.2 will be needed in our analysis.

Lemma 3.3. *Let* Γ *be a rectifiable Jordan curve of length* ρ*, parametrized by arclength* $s \in [0, \rho]$*. Let* h *be a real-valued function on* Γ*, regarded as a function of* s*. Let* L *be the first-order differential operator* $\frac{d}{ds} + ih$*, acting on the space* L^2 *of* periodic *functions on* $[0, \rho]$*. Define the winding number* $w = (2\pi)^{-1}\int_0^\rho h(s)\,ds$*. Then for any periodic test function* $u \in C^1([0, \rho])$*,*

$$\|Lu\|_{L^2} \geq 4\|w\|_* \rho^{-1} \|u\|_{L^2}. \tag{3.8}$$

Proof. Writing $H(s) = \int_0^s h$ and $Lu = f$, we have $L = e^{-iH}\frac{d}{ds}e^{iH}$ so $\frac{d}{ds}(e^{iH}u) = e^{iH}f$, whence

$$e^{iH(s)}u(s) = e^{iH(0)}u(0) + \int_0^s e^{iH} f. \tag{3.9}$$

Therefore

$$|e^{i(H(s)-H(0))}u(s) - u(0)| \leq \int_0^s |f|. \tag{3.10}$$

Applying the same reasoning to the interval $[s, \rho]$ gives

$$|u(\rho) - e^{i(H(s)-H(\rho))}u(s)| \leq \int_s^\rho |f|. \tag{3.11}$$

By the triangle inequality and the periodicity assumption $u(\rho) = u(0)$, this implies

$$|u(s)| \cdot |e^{i(H(s)-H(0))} - e^{i(H(s)-H(\rho))}| \leq \int_0^\rho |f|, \tag{3.12}$$

which is equivalent to

$$|u|_{L^\infty} \cdot |e^{iH(\rho)} - e^{iH(0)}| \leq \|f\|_{L^1}. \tag{3.13}$$

Now $|e^{iH(\rho)} - e^{iH(0)}| \geq 4\|w\|_*$, and applying Cauchy-Schwarz twice gives two factors of $\rho^{1/2}$. □

This implies (3.5), except for a constant factor in the inequality, by taking L to be the component of the magnetic gradient tangent to Γ. For a general Jordan curve Γ, the winding number w which appears in Lemma 3.3 equals $\pi^{-1}\int_{\mathcal{R}} \Delta\psi$, where $\mathcal{R}$ is the region enclosed by Γ.

We will also need the following Poincaré-type inequalities. Denote by $B(z,r)$ the disk centered at z with radius r.

Lemma 3.4. *If $u \in C^0(\overline{B(0,R)}) \cap W^1(B(0,R))$, then*

$$\int_{B(0,R)} |u|^2 \leq R^2 \left(2\int_{B(0,R)} \big|\nabla|u|\big|^2 + \int_0^{2\pi} |u(Re^{i\theta})|^2 d\theta \right). \tag{3.14}$$

Let $\mathcal{A} = \{r < |z| < R\}$. If $u \in C^0(\overline{\mathcal{A}}) \cap W^1(\mathcal{A})$ then

$$\int_{\mathcal{A}} |u|^2 \leq (R^2 - r^2)\log(R/r) \int_{\mathcal{A}} \big|\nabla|u|\big|^2 + (R^2 - r^2) \int_0^{2\pi} |u(re^{i\theta})|^2\, d\theta. \tag{3.15}$$

Proof. $|u|$ likewise belongs to $C^0 \cap W^1$. Using Friederichs mollifiers permits us to assume that $|u| \in C^1$. Therefore, without loss of generality, we assume that $u \geq 0$ and $u \in C^1$.

To prove (3.15) we work in polar coordinates, and exploit only the radial component of the gradient. It thus suffices to show that

$$\int_r^R |f(\rho)|^2 \rho\, d\rho \leq \log(R/r)(R^2 - r^2) \int_r^R |f'(\rho)|^2\, \rho\, d\rho + (R^2 - r^2)|f(r)|^2$$

for any $f \in C^1(\mathbb{R})$. Express $f(\rho) = f(r) + \int_r^\rho f'(t)\, dt$ and note that $\int_r^R |f(r)|^2\, \rho\, d\rho = \frac{1}{2}(R^2 - r^2)|f(r)|^2$. The other term is

$$\begin{aligned} &\leq 2\int_r^R \Big(\int_r^\rho |f'(t)|\, dt\Big)^2 \rho\, d\rho \\ &\qquad \leq 2\int_r^R \Big(\int_r^\rho |f'(t)|^2 t\, dt\Big) \Big(\int_r^\rho t^{-1}\, dt\Big) \rho\, d\rho \leq \log(R/r)(R^2 - r^2) \int_r^R |f'|^2\, t\, dt, \end{aligned}$$

as claimed. The proof of (3.14) is similar and is left to the reader. □

4. Construction of thick sets in $\mathbb{C}^1$

In this section we explicitly construct sets in $\mathbb{C}^1$ that have empty Euclidean interior and non-empty fine interiors. This construction will later be used in the proof of Theorem 1.3.

Let B be an integer greater than 2. We always assume that B is chosen to be sufficiently large for various inequalities encountered below to be valid. For any positive integer k, let $\varepsilon_k = B^{-k}$ and let $\Lambda_k = B^{-k} \cdot (\mathbb{Z} + i\mathbb{Z})$ be the set of lattice points. Let ρ_k be a positive number of the form

$$\rho_k = e^{-\sigma_k k B^{2k}} \quad \text{where } \sum_k \frac{1}{k\sigma_k} < \infty \text{ and } \sigma_k \geq 1. \tag{4.1}$$

In particular, ρ_k is much smaller than any power of ε_k for large k.

Let Ω_0 be the unit disk. We choose $\{z_j^k\}_{j=1}^{m_k} \in \Lambda_k \cap \Omega_0$ by induction on k as follows. For $k = 1$, $\{z_j^1\}_{j=1}^{m_1}$ are chosen to be all points in $\Lambda_1 \cap \Omega_0$ such that $\overline{B(z_j^1, \varepsilon_1)} \subset \Omega_0$. Suppose the points z_j^l have been chosen for $l \leq k-1$. Then $\{z_j^k\}_{j=1}^{m_k}$ are chosen to be all those points in $\Lambda_k \cap \Omega_0$ such that $\overline{B(z_j^k, \varepsilon_k)} \subset \Omega_0$ and

$$\text{distance}\,(z_j^k, \cup_{l=1}^{k-1} \cup_{j=1}^{m_l} D_j^l) > \varepsilon_k$$

where $D_j^l = B(z_j^l, \rho_l)$. Note that $m_k \leq 4\varepsilon_k^{-2}$. Let

$$\Omega_k = \Omega_0 \backslash \cup_{l=1}^{k} \cup_{j=1}^{m_l} \overline{D_j^l} \quad \text{and} \quad W = \Omega_0 \backslash \cup_{l=1}^{\infty} \cup_{j=1}^{m_l} \overline{D_j^l} = \cap_{k=1}^{\infty} \Omega_k.$$

It is evident that the set Ω thus constructed has empty (Euclidean) interior. Moreover, Ω has positive Lebesgue measure because

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |D_j^k| \leq 4\pi \sum_{k=1}^{\infty} \rho_k^2 \varepsilon_k^{-2} \leq 4\pi \sum_{k=1}^{\infty} \varepsilon_k^2 \ll |\Omega_0|$$

provided that B is chosen sufficiently large.

Lemma 4.1. *When B is sufficiently large, $m_k \geq \varepsilon_k^{-2}/4$. Furthermore, for any $z \in \Omega_k$ satisfying* distance $(z, \partial\Omega_0) \geq \varepsilon_k^{1/2}$, *there exist $\geq \varepsilon_k^{-1}$ indices j such that $|z_j^k - z| < 4\varepsilon_k^{1/2}$.*

Proof. We use the simple fact that for any $r \geq 1$, the number of integer lattice points in a (closed or open) disk of radius r is bounded between r^2 and $5r^2$. It follows that the number of points of $\Lambda_k \cap \Omega_0$ that are not elements of $\{z_j^k\}$ is no more than

$$\sum_{1 \leq l < k} 4\varepsilon_l^{-2} \cdot 5\big(\frac{\varepsilon_k + \rho_l}{\varepsilon_k}\big)^2 \leq 40 \sum_{1 \leq l < k} B^{2l}\big(1 + B^{2k-4l}\big) \leq C\varepsilon_k^{-2} B^{-2}.$$

Since the cardinality of $\Lambda_k \cap \{|z| < 1 - \varepsilon_k\}$ is $\geq \varepsilon_k^{-2}/2$, we have $m_k \geq (1/2 - CB^{-2})\varepsilon_k^{-2} > \varepsilon_k^{-2}/4$.

We now prove the second statement. Indeed, the above reasoning still applies, unless $B(z, 4\varepsilon_k^{1/2})$ meets some D_i^l with $\varepsilon_l > 16\varepsilon_k^{1/2}$. In that case there can be only one such l and only one such i. Otherwise, consider the largest such l; by construction, the distance from D_i^l to any D_j^m with $m \leq l$ is $\geq \varepsilon_l - 2\rho_l \geq \varepsilon_l - 2\varepsilon_l^2$. Thus $\varepsilon_l - 2\varepsilon_l^2 \leq 8\varepsilon_k^{1/2} < \varepsilon_l/2$, which is impossible. Since $B(z, 4\varepsilon_k^{1/2})$ meets only one D_i^l with $\varepsilon_l > 16\varepsilon_k^{1/2}$, it must contain a disk $B(z', 2\varepsilon_k^{1/2})$ which does not intersect any D_i^l with $\varepsilon_l > 16\varepsilon_k^{1/2}$. As in the preceding

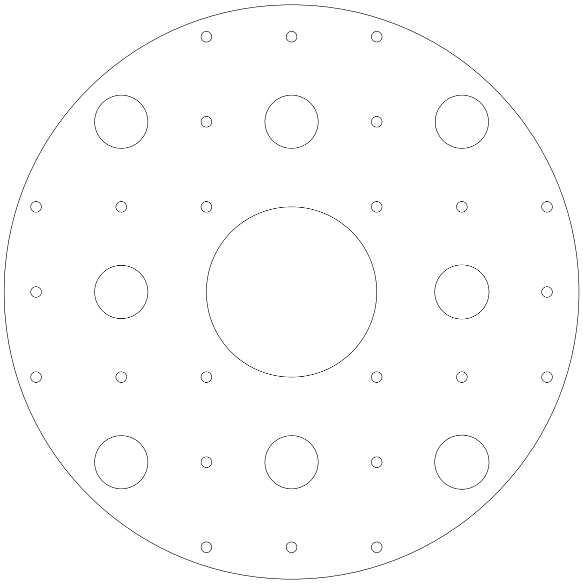

FIGURE 1. The domain Ω_0 and disks D_j^k of generations $k = 1, 2, 3$. Their centers z_j^k lie on a lattice of scale B^{-k}, whereas their radii ρ_k approach 0 at a doubly exponential rate.

paragraph, the number of points in $\Lambda_k \cap B(z', \frac{3}{2}\varepsilon_k^{1/2})$ that are not elements of z_j^k is no more than

$$\sum_{\substack{1 \le l < k \\ \varepsilon_l \le 16\varepsilon_k^{1/2}}} C\Big(\frac{2\varepsilon_k^{1/2}}{\varepsilon_l}\Big)^2 \cdot 5\Big(\frac{\varepsilon_k + \rho_l}{\varepsilon_k}\Big)^2 \le C\varepsilon_k \sum_{1 \le l < k} \varepsilon_l^{-2} \cdot \Big(\frac{\varepsilon_k + \rho_l}{\varepsilon_k}\Big)^2 \le C\varepsilon_k^{-1} B^{-2}.$$

Since the cardinality of $\Lambda_k \cap B(z', \frac{3}{2}\varepsilon_k^{1/2})$ is $\ge \frac{9}{4}\varepsilon_k^{-1}$, the number of points z_j^k in $B(z, 4\varepsilon_k^{1/2})$ is $\ge (\frac{9}{4} - CB^{-2})\varepsilon_k^{-1} \ge \varepsilon_k^{-1}$. □

Lemma 4.2. *If $\sum_k (k\sigma_k)^{-1} < \infty$ then there exist constants $c, C \in (0, \infty)$ and functions $F_k \in W_0^1(\Omega_k)$ such that for all k, $\|F_k\|_{L^2(\Omega_0)} \ge c$, $\|\nabla F_k\|_{L^2(\Omega_0)} \le C$, and $\|F_k\|_{L^\infty} \le C$.*

Proof. Define

$$f_j^l(z) = \frac{\log(|z - z_j^l|/\rho_l)}{\log(\varepsilon_l^2/4\rho_l)}$$

if $\rho_l \le |z - z_j^l| \le \varepsilon_l^2/4$, and $f_j^l(z) \equiv 1$ for $|z - z_j^l| \ge \varepsilon_l^2/4$. Note that $0 \le f_j^l(z) \le 1$ for all z. Let

$$F_k(z) = (1 - |z|^2) \min_{1 \le l \le k} \min_{1 \le j \le m_l} f_j^l(z).$$

If $z \in B(0, 1/2)$ and $|z - z_j^l| \ge \varepsilon_l^2/4$ for all $1 \le l \le k$ and all $1 \le j \le m_l$, then $F_k(z) = 1 - |z|^2 \ge 3/4$. Moreover $0 \le F_k(z) \le 1$ for all $z \in \Omega_0$. Since

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |B(z_j^k, \varepsilon_k^2/4)| \le C\pi \sum_k \varepsilon_k^{-2}\varepsilon_k^4,$$

we have $\|F_k\|^2 \ge \pi/2$ provided that B is chosen to be sufficiently large.

In estimating $\|\nabla F_k\|$ from above, we may disregard the harmless factor of $1-|z|^2$. Ω_k can be partitioned into finitely many pairwise disjoint subregions such that F_k is identically equal to some f_j^l on each subregion, and such that the supports of ∇f_j^l are mutually disjoint for a fixed l. Moreover, since F_k is continuous, $\|\nabla(1-|z|^2)^{-1}F_k\|^2$ equals the sum of the squares of the L^2 norms of its gradients over all these subregions. Therefore since F_k is bounded above in the supremum norm uniformly in k,

$$\|\nabla F_k\|^2 \le C + \sum_{l=1}^{k}\sum_{j=1}^{m_l}\int_{\Omega_l}|\nabla f_j^l|^2.$$

Since

$$\int_{\Omega_l}|\nabla f_j^l|^2 = 2\pi\int_{\rho_l}^{\varepsilon_l^2/4} r^{-2}[\log(\varepsilon_l^2/4\rho_l)]^{-2}\, r\,dr = 2\pi[\log(\varepsilon_l^2/4\rho_l)]^{-1} \le 4\pi(\log(1/\rho_l))^{-1},$$

we have

$$\|\nabla F_k\|^2 \le C\sum_{l=1}^{k}\varepsilon_l^{-2}\cdot(\log(1/\rho_l))^{-1} = C\sum_{l=1}^{k}B^{2l}\cdot B^{-2l}(l\sigma_l)^{-1} = C\sum_{l=1}^{k}(l\sigma_l)^{-1}.$$

□

Remark. The fine topology is the smallest topology on $\mathbb{C}$ with respect to which all subharmonic functions are continuous. We refer the reader to [22], Chapter 10, for an elementary treatise on the fine topology. It follows from [19] that the existence of functions F_k satisfying the conclusions of Lemma 4.2 is equivalent to W having nonempty fine interior.

5. Construction of the subharmonic function φ

We follow the construction in the preceding section, taking $\rho_k = \exp(-k\sigma_k B^{2k})$. Let $h \in C_0^\infty(\mathbb{C})$ be the radially symmetric function defined by $h(t) = c_0 e^{-1/(1-t)}$ for $0 \le t < 1$ and $h(t) = 0$ for $t \ge 1$, where the constant c_0 is chosen so that $\int_{\mathbb{C}} h = 1$. For $k \in \mathbb{N}$ and $1 \le j \le m_k$, define

$$\psi_j^k = 2\pi\mu_j^k\rho_k^{-2}h((z-z_j^k)/\rho_k),$$

where $\mu_j^k > 0$ satisfies

$$\mu_j^k \le \nu_k\varepsilon_k^2 \tag{5.1}$$

and the factors ν_k are chosen to satisfy

$$\sum_k (1+\sigma_k)k\nu_k < \infty. \tag{5.2}$$

Define also

$$\varphi_j^k = \frac{1}{2\pi}\log|z| * \psi_j^k.$$

Thus $\Delta\varphi_j^k = \psi_j^k$. Set

$$\varphi = \sum_k\sum_j \varphi_j^k;$$

the next lemma guarantees convergence of this sum.

Lemma 5.1. *If the sequences σ_k, ν_k satisfy (5.2), then φ is a subharmonic function on $\mathbb{C}$, $\Delta\varphi$ is lower semicontinuous, $\varphi \in C^0$, and $\nabla\varphi \in L^2(\Omega_0)$.*

Proof. Let $\psi = \sum_k \sum_j \psi_j^k$. Then

$$\|\psi\|_{L^1} = \sum_k \sum_j \|\psi_j^k\|_{L^1} = \sum_k \sum_j \mu_j^k \lesssim \sum_k B^{2k}\nu_k\varepsilon_k^2 = \sum_k \nu_k < \infty,$$

since $\sum_k k\nu_k$ converges by (5.2).

Write

$$\varphi_j^k(z) = \mu_j^k \int_{|t|<1} \log|\rho_k t + z_j^k - z| h(t).$$

If $|z - z_j^k| > 2\rho_k$, then

$$|\varphi_j^k(z)| \le C\mu_j^k \log(1/\rho_k).$$

If $|z - z_j^k| \le 2\rho_k$, then likewise

$$|\varphi_j^k(z)| \le C\mu_j^k\big(\log(1/\rho_k) + \int_{|t|<1} |\log(t + (z_j^k - z)/\rho_k)|\big) \le C\mu_j^k \log(1/\rho_k).$$

Moreover, whenever $z \notin B(z_j^k, \frac14\varepsilon_k)$, there is an improved bound $|\varphi_j^k(z)| \lesssim \mu_j^k \log(1/\varepsilon_k) \lesssim \nu_k\varepsilon_k^2 k$. For any z, k there exists at most one index j for which $|z - z_j^k| \le \varepsilon_k/4$. It follows that for any $z \in \mathbb{C}$, uniformly for any k,

$$\sum_j |\varphi_j^k(z)| \lesssim \nu_k\varepsilon_k^2\sigma_k k B^{2k} + \sum_{j:|z-z_j^k|\ge\varepsilon_k/4} \nu_k\varepsilon_k^2 k \lesssim \nu_k\varepsilon_k^2\sigma_k k B^{2k} + B^{2k}\nu_k\varepsilon_k^2 k,$$

and consequently

$$\sum_k\sum_j |\varphi_j^k(z)| \lesssim \sum_k (1+\sigma_k)k\nu_k < \infty$$

since $\varepsilon_k = B^{-k}$. Therefore the series defining φ is uniformly convergent to a continuous function. Since each φ_j^k is subharmonic, so is φ.

We now estimate the L^2 norm of φ. Write

$$\frac{\partial\varphi_j^k}{\partial z} = \tfrac12\mu_j^k \int_{|t|<1} \frac{h(t)}{z - z_j^k - \rho_k t}.$$

If $|z - z_j^k| > 2\rho_k$, then

$$\left|\frac{\partial\varphi_j^k}{\partial z}\right| \lesssim \mu_j^k |z - z_j^k|^{-1}.$$

If $|z - z_j^k| \le 2\rho_k$, then

$$\left|\frac{\partial\varphi_j^k}{\partial z}\right| \le \tfrac12\mu_j^k \int_{|t|<1} \frac{h(t)}{\rho_k|t - (z - z_j^k)/\rho_k|} \lesssim \mu_j^k\rho_k^{-1}\int_{|t|<3}\frac{1}{|t|} \lesssim \mu_j^k\rho_k^{-1}.$$

Therefore

$$\|\nabla\varphi_j^k\|_{L^2(B(0,1))} \le C\mu_j^k + C\mu_j^k\big(\int_{\rho_k}^1 r^{-2}\, r dr\big)^{1/2} \le C\mu_j^k(\log(1/\rho_k))^{1/2}.$$

As in the estimation of the supremum norm of φ, there is a stronger inequality

$$\|\nabla\varphi_j^k\|_{L^2\{|z-z_j^k|\ge\varepsilon_k/2\}} \lesssim \mu_j^k(\log(1/\varepsilon_k))^{1/2} \lesssim \mu_j^k k^{1/2}.$$

Since there are at most CB^{2k} indices j for each k,

$$\|\nabla\varphi\|_{L^2} \lesssim \sum_k \Big(B^{2k}(\nu_k\varepsilon_k^2)^2\log(1/\rho_k)\Big)^{1/2} + \sum_k\sum_j \nu_k\varepsilon_k^2 k^{1/2}$$
$$\lesssim \sum_k \big(B^{2k}\nu_k^2\varepsilon_k^4\sigma_k k B^{2k}\big)^{1/2} + \sum_k B^{2k}\nu_k\varepsilon_k^2 k^{1/2} \lesssim \sum_k \nu_k k^{1/2}\sigma_k^{1/2} + \sum_k \nu_k k^{1/2}.$$

The hypothesis (5.2) guarantees convergence of these sums. □

Remark. In order for $\Delta\varphi$ to be C^∞, or even Hölder continuous, it is necessary that the far more restrictive condition $\mu_j^k = O(\rho_k^\alpha)$ for some $\alpha > 2$ be satisfied.

6. The coefficients μ_j^k

Let B be any fixed positive integer, sufficiently large so that the hypotheses of Lemmas 4.1 and 4.2 are satisfied, and recall that $\varepsilon_k = B^{-k}$ and $\rho_k = \exp(-k\sigma_k B^{2k})$. The basic strategy in the proof of Theorem 1.3 is to combine (3.5) with (3.1), using the former to gain a strong bound over many circles, and the gradient estimate from (3.1) to then gain control over the remainder of Ω_0. For each large n, we want to find lots of (disjoint) circles Γ, for which (3.5), applied to $\psi = n\varphi$, gives a strong lower bound on $\int_\Gamma |u|^2$. (These circles can have different centers.) Then we use (3.1) on the complement of the union of all these circles, with (3.1) giving us good control on the boundary of the complement.

The factor of $\rho^{-2} = |z|^{-2}$ in (3.5) is important, since it tends to make $\int |u|^2$ much smaller than $\int |D_\psi u|^2$, provided the circle has small radius. It can also be used to gain satisfactory control of the boundary terms in (3.14) and (3.15), if for instance the annulus $r/2 \le |z| \le r$ is one on which (3.3) gives a good bound on u. On the other hand, we lose something in applying (3.15) to annuli for which $\log(R/r)$ is too large, relative to R^2. Thus for each large n, we want to have a large number of such good circles, and we want them to be fairly densely distributed in the sense that for each z and n there is such a circle within distance b_k of z, where $k = k(n)$ and $b_k \to 0$ at some rate to be specified.

We use the following setup. There will be a sequence of positive integers N_k converging rapidly to $+\infty$. To each $n \in [N_k, N_{k+1})$ we will associate a family $\mathcal{F}(n)$ of disks D_j^k with the following properties:

(1) For any $z \in \Omega_0$ there exists $D_j^k \in \mathcal{F}(n)$ satisfying distance$\,(z, D_j^k) \le C\varepsilon_k^{1/2}$.

(2) For any $D_j^k \in \mathcal{F}(n)$, $\|n\mu_j^k\|_* \ge \frac{1}{4}$, where $\|x\|_* = $ distance$\,(x, \mathbb{Z})$. A given disk D_j^k is permitted to belong to $\mathcal{F}(n)$ for many different values of n.

Define

$$N_k = 2^{B^{k-1}} \tag{6.1}$$

For any sufficiently large k, we construct $\{\mathcal{F}(n) : N_k \le n < N_{k+1}\}$, and $\{\mu_j^k\}$ as follows. We first cover Ω_{k-1} by $\sim \varepsilon_k^{-1}$ disks centered in Ω_k with radii $8\varepsilon_k^{1/2}$. We may arrange these covering disks so that each D_j^k belongs to at least one covering disk and the shrinking by half of each covering disk is disjoint from the other covering disks. By Lemma 4.1, we can then partition the disks D_j^k into $\sim \varepsilon_k^{-1}$ subfamilies, each of cardinality $\ge \varepsilon_k^{-1}$, so that for each subfamily, all of its member disks are contained in a common covering disk.

The number μ_j^k are chosen as follows, to ensure the existence of many disks with favorable winding numbers $n\mu_j^k$ for each integer $n \in [N_k, N_{k+1})$. Consider first $n = N_k$. Choose one disk from each subfamily, and let $\mathcal{F}(n)$ be the set of all disks thus chosen. For each disk $D_j^k \in \mathcal{F}(n)$, define the weight μ_j^k by

$$n\mu_j^k = \tfrac{1}{4}. \tag{6.2}$$

Then $\|n\mu_j^k\|_* = \frac{1}{4}$; moreover, $\|m\mu_j^k\|_* \geq 1/4$ for all $m \in [N_k, 2N_k)$. For each $m \in [N_k, 2N_k)$, set $\mathcal{F}(m) = \mathcal{F}(N_k)$.

Next consider $n = 2N_k$, and repeat the procedure: let $\mathcal{F}(n)$ be a collection consisting of one disk D_j^k from each subfamily, not previously chosen. Define μ_j^k by $n\mu_j^k = \frac{1}{4}$. Then $\|n\mu_j^k\|_* \geq \frac{1}{4}$ for all $n \in [2N_k, 4N_k)$. For $m \in [2N_k, 4N_k)$, let $\mathcal{F}(m) = \mathcal{F}(2N_k)$. The next iteration begins with $n = 4N_k$, and so on. Repeat the procedure until every integer $n \in [N_k, N_{k+1})$ has been considered, and $\mathcal{F}(n)$ and associated coefficients μ_j^k defined. There are sufficiently many disks D_j^k to allow this because $N_k \cdot 2^{\varepsilon_k^{-1}} > N_{k+1}$. Any disks D_j^k not in $\cup_{n=N_k}^{N_{k+1}} \mathcal{F}(n)$ play a lesser role in the analysis; we set $\mu_j^k = N_{k+1}^{-1}$ for those although any sufficiently small strictly positive quantity would suffice.

Thus

$$\mu_j^k \leq N_k^{-1} = 2^{-B^{k-1}}$$

for all sufficiently large k. On the other hand, we have already imposed the constraint (5.1) $\mu_j^k \leq \nu_k \varepsilon_k^2 = \nu_k B^{-2k}$, with which the above construction is consistent if

$$\nu_k \geq B^{2k} 2^{-B^{k-1}}.$$

In order to apply Lemma 5.1 to conclude that $\varphi \in C^0$ and $\nabla\varphi \in L^2$, we also need the constraints (4.1) $\sum k^{-1}\sigma_k^{-1} < \infty$ and (5.2) $\sum_k (1 + \sigma_k) k\nu_k < \infty$. All these are mutually compatible. Indeed if we fix any $\varepsilon > 0$ and set $\sigma_k = k^\varepsilon$ and $\nu_k = k^{-2-2\varepsilon}$ for large k, then $\nu_k \geq B^{2k} 2^{-B^{k-1}}$ with some room to spare.

The conclusions of this section are summarized in the following lemma.

Lemma 6.1. *Suppose that $\nu_m \geq B^{2m} 2^{-B^{m-1}}$ for all sufficiently large m. Then there exist coefficients $0 \leq \mu_j^m$ satisfying (5.1) such that for each sufficiently large positive integer n there exist an index $k = k_n$ and a collection J_n of indices j such that $k_n \to \infty$ as $n \to \infty$, and such that for each point $z \in \Omega_{k-1}$ there exists at least one $j \in J_n$ such that $|z - z_j^k| \leq C\varepsilon_k^{1/2}$ and* distance $(\pi^{-1} n \int_{D_j^k} \Delta\varphi, \mathbb{Z}) \geq \frac{1}{4}$.

Moreover it is possible to choose a sequence (ν_m) and an associated sequences (σ_m) such that (4.1) and (5.2) are also satisfied. Therefore φ is subharmonic, $\varphi \in C^0$ and $\nabla\varphi \in L^2$, and there exist functions F_k satisfying the conclusions of Lemma 4.2.

7. Proof of Theorem 1.3

We now proceed to prove that $\lambda_{n\varphi}^{\mathrm{m}} \to \infty$ as $n \to +\infty$. Given any large n, specify k by the relation $n \in [N_k, N_{k+1})$. Let D_j^k be any disk in $\mathcal{F}(n)$. Consider the annular region $\mathcal{A}_j^k = \{z : \rho_k < |z - z_j^k| < \varepsilon_k\}$. This region is disjoint from D_i^l for all $l \leq k$, except for $(l, i) = (k, j)$. Thus in $\mathcal{A}_j^k$, $\Delta\varphi \equiv \sum_{l>k} \sum_i \Delta\varphi_i^l$.

Define $\tilde{\varphi}_k = \sum_{l \leq k} \sum_i \varphi_i^l$. We have $\int_{D_j^k} n\Delta\tilde{\varphi}_k = n\mu_j^k \in [\frac{1}{4}, \frac{1}{2}]$, and hence for any test function $u \in C_0^\infty(\Omega_0)$, (3.3) gives

$$\int_{\mathcal{A}_j^k} |u(z)|^2 \leq 16\varepsilon_k^2 \int_{\mathcal{A}_j^k} |D_{n\tilde{\varphi}_k} u|^2.$$

Let E_j^k be the set of all radii $r \in [\rho_k, \frac{1}{2}\varepsilon_k]$ for which the circle $\Gamma_r = \{z : |z - z_j^k| = r\}$ intersects some closed disk $\overline{D_i^l}$ with $l > k$. The Lebesgue measure of E_j^k is $\le 4\sum_{l>k}\varepsilon_l^{-2}\rho_l = 4\sum_{l>k} B^{2l}e^{-\sigma_l l B^{2l}} \ll B^{-k} = \varepsilon_k$, provided k is sufficiently large. Moreover, for any $r \in [\rho_k, \varepsilon_k]$,

$$\int_{|z-z_j^k|<r} n\Delta(\varphi - \varphi_j^k) \le n\sum_{l>k}\sum_i \mu_i^l.$$

The sum over all $l \ge k+2$ contributes at most $4n\varepsilon_{k+2}^{-2}/N_{k+2} \le 4B^{2(k+2)}2^{-(B^{k+1}-B^k)} < 1/8$ for all sufficiently large k, since $n < N_{k+1}$. By construction, any point z_i^{k+1} satisfies $|z_i^{k+1} - z_j^k| \ge B^{-k-1} = B^{-1}\varepsilon_k$. Thus φ_i^{k+1} contributes nothing to the integral, provided that $r \le \frac{1}{2}B^{-1}\varepsilon_k$. We therefore conclude that for any $r \in [\rho_k, \frac{1}{2}B^{-1}\varepsilon_k]$ and any $m \ge k$,

$$\tfrac{1}{4} \le \int_{|z-z_j^k|<r} n\Delta\varphi_j^k \le \int_{|z-z_j^k|<r} n\Delta\tilde{\varphi}_m \le \int_{|z-z_j^k|<r} n\Delta\varphi_j^k + n\sum_{l\ge k+2}\sum_i \mu_i^l \le \tfrac{1}{2} + \tfrac{1}{8} \le \tfrac{5}{8},$$

provided as always that k is sufficiently large.

Therefore for any $r \in [\rho_k, \frac{1}{2}B^{-1}\varepsilon_k]\backslash E_j^k$,

$$\int_0^{2\pi} |u(z_j^k + re^{i\theta})|^2\,d\theta \le 16r^2\int_0^{2\pi} |D_{n\tilde{\varphi}_m}u(z_j^k + re^{i\theta})|^2\,d\theta$$

by (3.5). Hence

$$\int_{r\in[\rho_k,\frac{1}{2}B^{-1}\varepsilon_k]\backslash E_j^k}\int_0^{2\pi} |u(z_j^k + re^{i\theta})|^2\,d\theta\,rdr \le 4B^{-2}\varepsilon_k^2\int_{\mathcal{A}_j^k} |D_{n\tilde{\varphi}_m}u(z)|^2.$$

Since $\|\nabla\tilde{\varphi}_m - \nabla\varphi\|_{L^2} \to 0$, we may now conclude, by letting $m \to \infty$, that for each $n \in [N_k, N_{k+1})$, for each j such that $D_j^k \in \mathcal{F}(n)$,

$$\int_{r\in[\rho_k,\frac{1}{2}B^{-1}\varepsilon_k]\backslash E_j^k}\int_0^{2\pi} |u(z_j^k + re^{i\theta})|^2\,d\theta\,rdr \le 4B^{-2}\varepsilon_k^2\int_{\mathcal{A}_j^k} |D_{n\varphi}u(z)|^2. \tag{7.1}$$

We claim next that

$$\int_{B(z_j^k,\varepsilon_k)} |u|^2 \le C\varepsilon_k^2\int_{B(z_j^k,\varepsilon_k)} \big|\nabla|u|\big|^2 + C\int_{r\in[\rho_k,\frac{1}{2}B^{-1}\varepsilon_k]\backslash E_j^k}\int_0^{2\pi} |u(z_j^k + re^{i\theta})|^2\,d\theta\,rdr, \tag{7.2}$$

where C is a constant depend only on B. The proof of this claim follows from the Poincaré inequalities (3.14) and (3.15), as follows. By (3.14), for any $r \in [\frac{1}{4}B^{-1}\varepsilon_k, \frac{1}{2}B^{-1}\varepsilon_k]$,

$$\int_{B(z_j^k,\frac{1}{4}B^{-1}\varepsilon_k)} |u|^2 \le C\varepsilon_k^2\int_{B(z_j^k,\varepsilon_k)} \big|\nabla|u|\big|^2 + C\varepsilon_k^2\int_0^{2\pi} |u(z_j^k + re^{i\theta})|^2\,d\theta.$$

Integrating both sides with respect to r over $[\frac{1}{4}B^{-1}\varepsilon_k, \frac{1}{2}B^{-1}\varepsilon_k]\backslash E_j^k$, dividing both sides by ε_k, and using the fact that $|E_j^k| \ll \varepsilon_k$, we obtain

$$\int_{B(z_j^k,\frac{1}{4}B^{-1}\varepsilon_k)} |u|^2 \le C\varepsilon_k^2\int_{B(z_j^k,\varepsilon_k)} \big|\nabla|u|\big|^2 + C\int_{[\rho_k,\frac{1}{2}B^{-1}\varepsilon_k]\backslash E_j^k}\int_0^{2\pi} |u(z_j^k + re^{i\theta})|^2\,d\theta\,rdr. \tag{7.3}$$

Similarly, it follows from (3.15) that for any $r \in [\frac{1}{8}B^{-1}\varepsilon_k, \frac{1}{4}B^{-1}\varepsilon_k]$,

$$\int_{\frac{1}{4}B^{-1}\varepsilon_k<|z-z_j^k|<\varepsilon_k} |u|^2 \le C\varepsilon_k^2\int_{B(z_j^k,\varepsilon_k)} \big|\nabla|u|\big|^2 + C\varepsilon_k^2\int_0^{2\pi} |u(z_j^k + re^{i\theta})|^2\,d\theta.$$

Integrating both sides over $r \in [\frac{1}{8}B^{-1}\varepsilon_k, \frac{1}{4}B^{-1}\varepsilon_k]\setminus E_j^k$ and dividing by ε_k, we have

(7.4)
$$\int_{\frac{1}{4}B^{-1}\varepsilon_k<|z-z_j^k|<\varepsilon_k} |u|^2 \leq C\varepsilon_k^2 \int_{B(z_j^k,\varepsilon_k)} \left|\nabla|u|\right|^2 + C\int_{[\rho_k,\frac{1}{2}B^{-1}\varepsilon_k]\setminus E_j^k} \int_0^{2\pi} |u(z_j^k+re^{i\theta})|^2\, d\theta\, rdr.$$

Combining (7.3) and (7.4), we then obtain (7.2).

Applying (3.15) once more, we conclude that for any fixed finite constant $C'>0$,

(7.5)
$$\int_{\varepsilon_k<|z-z_j^k|<C'\varepsilon_k^{1/2}} |u|^2 \leq C\varepsilon_k \log(1/\varepsilon_k) \int_{\varepsilon_k<|z-z_j^k|<C'\varepsilon_k^{1/2}} \left|\nabla|u|\right|^2 + C\varepsilon_k\varepsilon_k^{-2}\int_{|z-z_j^k|\leq\varepsilon_k} |u|^2$$

By (7.2), (7.5), (7.1), and Lemma 3.1, we have

$$\begin{aligned}\int_{B(z_j^k,C'\varepsilon_k^{1/2})} |u|^2 &\leq C\varepsilon_k \log(1/\varepsilon_k)\int_{B(z_j^k,C'\varepsilon_k^{1/2})} \left|\nabla|u|\right|^2 + C\varepsilon_k^{-1}\int_{B(z_j^k,\varepsilon_k)} |u|^2 \\ &\leq C\varepsilon_k \log(1/\varepsilon_k)\int_{B(z_j^k,C'\varepsilon_k^{1/2})} |D_{n\varphi}u|^2 + C\varepsilon_k\int_{B(z_j^k,\varepsilon_k)} |D_{n\varphi}u|^2 \\ &\leq C\varepsilon_k \log(1/\varepsilon_k)\int_{B(z_j^k,C'\varepsilon_k^{1/2})} |D_{n\varphi}u|^2.\end{aligned}$$

This holds for any j such that $D_j^k \in \mathcal{F}(n)$.

It follows that
$$\int_{\cup_j B(z_j^k,C'\varepsilon_k^{1/2})} |u|^2 \leq CB^{-k/2}\int_{\Omega_0} |D_{n\varphi}u|^2,$$
where the union is taken over all j such that $D_j^k \in \mathcal{F}(n)$. Indeed, choose a maximal pairwise disjoint subfamily of all the balls $B(z_j^k, C'\varepsilon_k^{1/2})$, apply the preceding inequality to the tripled ball $B(z_j^k, 3C'\varepsilon_k^{1/2})$ for each element of the subfamily, and sum. No point of $\mathbb{C}$ belongs to more than a fixed number of the tripled balls, and each ball $B(z_j^k, C'\varepsilon_k^{1/2})$ is contained in at least one of the tripled balls.

The construction guarantees that any point in the complement in Ω_0 of $\cup_j B(z_j^k, 16\varepsilon_k^{1/2})$ either lies within distance $\varepsilon_k^{1/2}$ of the boundary of the unit ball Ω_0, or is contained in $\cup_{1\leq l<k}\cup_{i=1}^{m_l} D_i^l$. For the former region an application of the fundamental theorem of calculus, as in the proof of (3.15), gives the simple bound $\int_{1-\sqrt{\varepsilon_k}<|z|<1} |u|^2 \leq C\sqrt{\varepsilon_k}\int |\nabla|u||^2$, which is dominated by $CB^{-k/2}\langle S_{n\varphi}u,\, u\rangle$ by Kato's inequality. $k=k(n)\to\infty$ as $n\to\infty$, and $\varepsilon_k\to 0$ as $k\to\infty$, so this region is satisfactorily under control.

For any $n\in[N_k, N_{k+1})$ and any $l<k$, let $\tilde{\rho}_l = (1-2/\log n)\rho_l$ and $\hat{\rho}_l = (1-3/\log n)\rho_l$. Denote $\tilde{D}_i^l = B(z_i^l, \tilde{\rho}_l)$. The electric potential term gives us

$$\sum_{l<k}\sum_i \int_{\tilde{D}_i^l} n\rho_l^{-2}\mu_i^l\cdot c_0 n^{-1/2}|u|^2 \leq \langle S_{n\varphi}u,\, u\rangle. \tag{7.6}$$

Now
$$n\rho_l^{-2}\mu_i^l \geq ne^{2\sigma_l lB^{2l}}N_{l+1}^{-1} = ne^{2\sigma_l lB^{2l}}2^{-B^l} \geq n,$$
so we conclude that
$$\int_{\cup_{l<k}\cup_i\tilde{D}_i^l} |u|^2 \leq Cn^{-1/2}\langle S_{n\varphi}u,\, u\rangle.$$

Applying (3.15) to the annulus $\{r < |z - z_i^l| < \rho_l\}$ for all $r \in [\hat\rho_l, \tilde\rho_l]$, then integrating (after multiplying both sides by r), we obtain

$$\begin{aligned}\int_{\tilde\rho_l < |z-z_i^l| < \rho_l} |u|^2 &\le (\rho_l^2 - \hat\rho_l^2)\log(\rho_l/\hat\rho_l)\int_{D_i^l} \big|\nabla|u|\big|^2 + \frac{2(\rho_l^2-\hat\rho_l^2)}{\tilde\rho_l^2 - \hat\rho_l^2}\int_{\tilde D_i^l} |u|^2 \\ &\le C\big|\log(1-3/\log n)\big|\int_{D_i^l}\big|\nabla|u|\big|^2 + C\int_{\tilde D_i^l}|u|^2.\end{aligned}$$

Once again $\big|\nabla|u|\big|$ may be replaced by $|D_{n\varphi}u|$ in the last integral, by Lemma 3.1. Since every point of Ω_0 belongs either to $\{1-\sqrt{\varepsilon_k} < |z| < 1\}$, to some D_i^l with $l < k$, or to $B(z_j^k, 16\varepsilon_k^{1/2})$ for some j such that $D_j^k \in \mathcal{F}(n)$, we conclude finally that

$$\int_{\Omega_0} |u|^2 \le C\min(B^{-k/2}, n^{-1/2}, \big|\log(1-3/\log n)\big|)\langle S_{n\varphi}u,\, u\rangle,$$

where $k = k(n)$ is determined by the relation $n \in [N_k, N_{k+1})$.

It remains to prove that $\lim_{n\to\infty}\lambda_{n\varphi}^{\mathrm{e}} < \infty$. This is easy. Let F_k be the functions constructed in the proof of Lemma 4.2. Recall that F_k is piecewise smooth, vanishing outside Ω_k, $0 \le F_k(z) \le 1$, $\|F_k\|_{L^2} \ge c > 0$, and $\|\nabla F_k\|_{L^2} \le C < \infty$. Therefore

$$\lambda_{n\varphi}^{\mathrm{e}} \le C\Big(\int_{\Omega_k}|\nabla F_k|^2 + n\int_{\Omega_k}\Delta\varphi F_k^2\Big) \le C\big(1 + n\sum_{l>k}\sum_j \mu_j^l\big) \le C\big(1 + n\sum_{l>k}\nu_l\big).$$

This is $O(1)$ if k is chosen $\ge n$, since $\sum_k k\nu_k < \infty$. We thus deduce that $\lambda_{n\phi}^{\mathrm{e}} \le C < \infty$. This concludes the proof of Theorem 1.3. □

Remark. An interesting discussion of the lowest eigenvalue $\lambda_\psi^{\mathrm{m}}$ of S_ψ on multiply connected domains with finitely many holes, in the special case where the winding number corresponding to each hole is congruent to $\frac{1}{2}$ modulo $\mathbb{Z}$, appears in the work [21] of Helffer et. al, who lift the problem to a twofold covering surface on which each winding number belongs to $\mathbb{Z}$. It may very well be possible to show in this way that the lowest eigenvalue for S_ψ is large in certain situations, for instance when there are a large number of holes which are fairly densely distributed, if the winding numbers are half-integers. But our construction requires consideration of the case where the winding number varies over a $\frac{1}{N}$–dense subset of an interval $[\delta, 1-\delta]$ modulo $\mathbb{Z}$, where $N \to \infty$. We have followed a direct analytic path, based on Kato's inequality and the magnetic effect expressed through Lemma 3.2, whereby it is clear that the local effect produced by a single hole may be quantified in terms of the distance from a winding number to $\mathbb{Z}$, rather than its Diophantine character.

Remark. By refining the estimates of this section slightly one can carry out the construction so that $\Delta\varphi \in L(\log L)^\delta$ for any $\delta < 1$. It appears that one can get $\Delta\varphi \in L(\log L)(\log\log L)^{-C}$ for some finite C, but we have not verified this in detail.

8. Ground state energies in the smooth case

In this section we prove Theorem 1.5. We are given that $\Delta\varphi$ is Hölder continuous of order $\alpha > 0$, and that $\lambda_{n\varphi}^{\mathrm{e}}$ remains bounded as $n \to +\infty$.

Before embarking on the proof, we pause to explain the underlying issues. For large positive n, $\int|\nabla u|^2 + n\int|u|^2\Delta\varphi$ will be large relative to $\int|u|^2$, unless u is supported mainly where $\Delta\varphi$ is nearly zero. magnetic field into finitely many components for each n. Let W be any connected component of the open set $\{\Delta\varphi > 0\}$. For any n, $\Delta\varphi\big|_W$ gives rise to a magnetic field whose strength (if W is simply connected) on the complement of W is governed by the distance from $\int_W \Delta\varphi$ to $2\pi\mathbb{Z}$; if this distance is nearly zero modulo $2\pi\mathbb{Z}$ then this part of the magnetic field should not account for much of a discrepancy between $\lambda^{\mathrm{e}}_{n\varphi}$ and $\lambda^{\mathrm{m}}_{n\varphi}$. Since all that is relevant is $n\int \Delta\varphi$ modulo $2\pi\mathbb{Z}$, fields created by different components W can interfere destructively with one another. Moreover, even if the field due to W is strong, it is strong only near W; all that is required for $\lambda^{\mathrm{m}}_{n\varphi}$. to be not much larger than $\lambda^{\mathrm{e}}_{n\varphi}$ is for there to exist some suitably large subregion of Ω on which the net magnetic field is not very strong. Thus it can be advantageous in the analysis to group components W into clusters. Moreover, two or more components separated by narrow necks will tend to act like a single larger component, since a Brownian particle is unlikely to pass through a narrow neck without straying into one of the components bounding it. Thus breaking the support of $\Delta\varphi$ into its topological components is inefficient. Figure 2 illustrates some of these points.

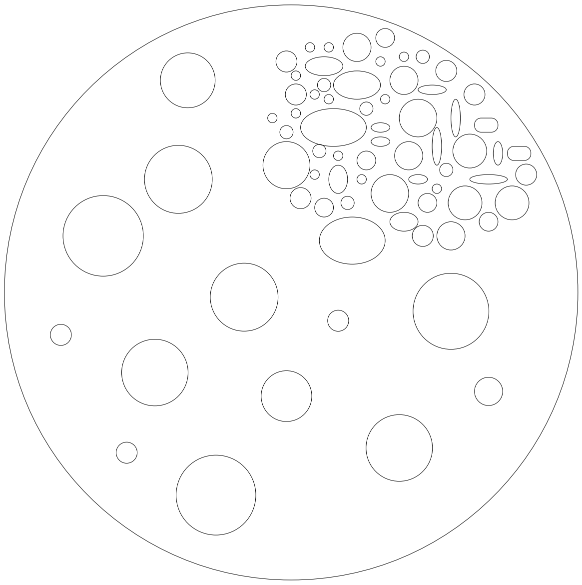

FIGURE 2. A situation in which the support of $\Delta\varphi$, represented by disks and ellipses, has many topological components, which can be effectively organized into a small number of clusters. There is a large subregion which is relatively empty of sources of magnetic field.

Lemma 8.1. *Let a subharmonic function φ be given, with $\Delta\varphi$ Hölder continuous of some order $\alpha > 0$. Suppose that $\lambda^{\mathrm{e}}_{n\varphi}$ remains bounded as $n \to +\infty$. Then there exists $C < \infty$ such that for any $\delta > 0$ there exists a real-valued function $u \in C^\infty$ such that u is supported in Ω_0, $u \geq 0$, $\|\nabla u\|_{L^2} \leq C$, $\|u\|_{L^\infty} = 1$, $\|u\|_{L^2} \geq C^{-1}$, and $u(z) = 0$ wherever $\Delta\varphi(z) \geq \delta$.*

Remark. It follows from results in potential theory [19], [18] that there exists u_0 with $\nabla u_0 \in L^2$, supported in $\{z \in \Omega_0 : \Delta\varphi(z) = 0\}$, so that $u_0 \neq 0$ on a set of positive Lebesgue measure. The desired function u may be obtained by suitably mollifying u_0. We have elected instead to give a self-contained proof.

Proof. We are given that there exists $B < \infty$ such that for any $M < \infty$ there exists a C^1 function v_M supported in Ω_0 such that $\|\nabla v_M\|^2 + M\int |v_M|^2\Delta\varphi \leq B^2$, and $\|v_M\|_{L^2} \sim 1$. By replacing v_M by its absolute value we may assume that $v_M \geq 0$. By replacing v_M by

$\max(v_M(z), c) - c$ for some sufficiently small $c > 0$ we may make v_M be supported in a compact subset of Ω_0, retaining uniform bounds on $v_M, \nabla v_M$.

Since $\|v\|_{L^4}$ is bounded by a constant times $\|\nabla v\|_{L^2} + \|v\|_{L^2}$, for large $\lambda > 0$ we have $\int_{|v(z)|>\lambda} |v|^2 \le \lambda^{-2} \int_{|v(z)|>\lambda} |v|^4 \le (B+1)C\lambda^{-2}$. Therefore if we fix a sufficiently large constant λ and replace v_M by $\min(v_M(z), \lambda)$, we still have $\|\nabla v_M\|^2 + M\int |v_M|^2 \Delta\varphi \le B^2$ and $\|v_M\|_{L^2} \ge \frac{1}{2}$, and have the additional property $\|v_M\|_{L^\infty} \le \lambda$ uniformly in M.

Choose a cutoff function $\eta = \eta_\delta \in C^\infty$, taking values in $[0,1]$, such that $\eta(z) \equiv 1$ wherever $\Delta\varphi(z) \le \delta/4$, and $\eta(z) \equiv 0$ wherever $\Delta\varphi(z) \ge \delta/2$. Consider $u = u_{M,\varepsilon} = \eta_\varepsilon v_M$ where $M = M(\delta)$ is to be chosen sufficiently large. $\|u\|_\infty$ is bounded above uniformly in M. $\|u\|_{L^2}$ is bounded below by a strictly positive constant, provided that $M(\delta)$ is sufficiently large, since

$$\int_{\Delta\varphi(z)\ge\delta/4} |v_M(z)|^2 \le M^{-1}4\delta^{-1} \int M\Delta\varphi |v_M|^2 \le M^{-1}2\delta^{-1}B^2$$

may be made as small as desired by choosing M sufficiently large. Finally $M\int |u|^2\Delta\varphi \le M\int |v_M|^2 \Delta\varphi$ is uniformly bounded, while

$$\|\nabla u\|_{L^2} \le \|\nabla v_M\|_{L^2} + \|v_M \nabla\eta\|_{L^2} \le B + \|\nabla\eta\|_{L^\infty} \|v_M\|_{L^2(\text{support}\,(\nabla\eta))}.$$

Since $\Delta\varphi \ge \delta/4$ on the support of $\nabla\eta$, the last term is $O(M\delta^{-1})^{-1/2}$, which tends to 0 as $M \to \infty$.

The final step is to convolve with an approximation to the identity to produce a C^∞ function; all the bounds continue to hold uniformly for a sufficiently fine approximation. □

Define $N_k = 2^k$ for each $k \in \mathbb{N}$; note that these differ from the quantities denoted N_k in previous sections. For each $k \in \mathbb{N}$ let $u_k \in C^\infty$ satisfy the conclusions of Lemma 8.1 with $\delta = N_k^{-2}$. By Sard's theorem, there exist regular values of u_k in $[\frac{1}{4}, \frac{1}{2}]$; choose any such regular value and denote it by c_k.

Consider all connected components V_j of $\{z \in \Omega_0 : u_k(z) < c_k\}$. Such a component is said to be harmless if $\sup_{z\in V_j} \Delta\varphi(z) \le N_k^{-2}$, and to be dangerous otherwise. Since c_k is a regular value, there are only finitely many such components, and each component of the boundary of any V_j is a smooth Jordan curve.

Let $\{W_i^k : 1 \le i \le M_k\}$ be the collection of all dangerous components V_j of $\{z \in \Omega_0 : u_k(z) < c_k\}$. M_k, the number of dangerous components, plays a central role in the analysis.

Lemma 8.2. *For each W_i^k,*

$$\int_{W_i^k} |\nabla u_k|^2 \gtrsim k^{-1},$$

uniformly in i, k. Consequently

$$M_k = O(k).$$

In contrast, if we were working with dangerous topological components of $\{z : \Delta\varphi(z) > 0\}$, the best bound would have roughly the form e^{ck}. The size of M_k will be the crucial element in the pigeonhole argument of Lemma 8.4.

Proof. To verify this recall that W_i^k contains a point w_i for which $\Delta\varphi(w) \ge N_k^{-2}$, and that $W_i^k = V_j$ for some j. The hypothesis of Hölder continuity implies that $\Delta\varphi(z) \ge \frac{1}{2}N_k^{-2}$ for all z in a disk D_i of radius $\gtrsim N_k^{-2/\alpha}$ centered at w_i. Therefore $u_k(z) = 0$ for all $z \in D_i$, and consequently $D_i \subset V_j \subset W_i^k$.

Now consider the function $\tilde{u}$ which equals u on W_i^k, and equals c_k on $\Omega^\dagger \setminus W_i^k$, where $\Omega^\dagger$ denotes some fixed open ball which contains the closure of Ω_0. This function $\tilde{u}$ vanishes on the boundary of D_i, and equals $c_k \in [\frac{1}{4}, \frac{1}{2}]$ on the boundary of $\Omega^\dagger$. Since $u_k \equiv c$ on the boundary of W_i^k, $\nabla\tilde{u} \in L^2$ in the sense of distributions. It follows from (3.15) that

$$\|\nabla\tilde{u}\|_{L^2}^2 \gtrsim 1/\log(1/\rho)$$

where ρ denotes the radius of D_i. Thus $\|\nabla\tilde{u}\|_{L^2}^2 \gtrsim 1/\log(N_k)$. But $\|\nabla u_k\|_{L^2(W_i^k)} = \|\nabla\tilde{u}\|_{L^2}$, so the first conclusion is established.

The second conclusion follows directly. Since the sets W_i^k are pairwise disjoint,

$$M_k = \sum_{i=1}^{M_k} 1 \lesssim \sum_{i=1}^{M_k} k \int_{W_i^k} |\nabla u_k|^2 \le k\|\nabla u_k\|_{L^2}^2 = O(k).$$

□

The bound $M_k = O(k)$ is the best possible bound of this type, but is insufficient for our purpose. The next lemma asserts an improvement for some subsequence.

Lemma 8.3. *There exists a strictly increasing sequence $k_\nu \to \infty$ such that*

$$M_{k_\nu} \le \frac{k_\nu}{\log(k_\nu)\log\log(k_\nu)}. \tag{8.1}$$

Our analysis requires a bound $M_{k_\nu} = o(k_\nu/\log k_\nu)$; the factor of $\log\log k_\nu$ in the denominator serves to guarantee this but is not otherwise needed.

Proof. Let $K \in \mathbb{N}$ be large. Choose $u_k = u$ and $c_k = c$ to be independent of k for all $k \le K$; these do still depend on K. For each $2 \le k \le K$ let $\{\tilde{W}_i^k : 1 \le i \le m_k\}$ be the collection of all connected components of $\{z : u(z) \le c\}$ for which $\max_{z\in\tilde{W}_i^k} \Delta\varphi(z) \in [N_k^{-2}, N_{k-1}^{-2}) = [2^{-2k}, 2^{2-2k})$. For $k = 1$ the latter condition is instead $\max_{z\in\tilde{W}_i^k} \Delta\varphi(z) \ge \frac{1}{4}$. Then for any $k \le K$, $\{W_i^k\} = \cup_{l\le k}\{\tilde{W}_i^l\}$, so $M_k = \sum_{l=1}^k m_l$.

Arguing as in the proof of Lemma 8.2 we find that

$$\sum_{k=1}^K k^{-1} m_k \lesssim \sum_{k=1}^K \sum_{i=1}^{m_k} \int_{\tilde{W}_i^k} |\nabla u|^2 \le \|\nabla u\|_{L^2}^2.$$

uniformly in K. Therefore $\sum_{k=1}^K k^{-1}m_k = O(1)$ uniformly in K.

By summation by parts, it follows that likewise $\sum_{k=1}^K k^{-2}M_k = O(1)$, since the boundary terms $k^{-1}M_k$ remain uniformly bounded by Lemma 8.2. Therefore for any given K', if K is sufficiently large there exists $k \in [K', K]$ such that $M_k \le k[\log k \cdot \log\log k]^{-1}$. Applying this for a sufficiently rapidly increasing sequence of values of K yields the lemma. □

Fix a sequence (k_ν) satisfying the conclusion of Lemma 8.3. Henceforth we consider only indices k belonging to this sequence, but omit the subscript ν in order to simplify notation. The possibly very sparse subsequence which yields the bounded limit infimum in Theorem 1.5 is obtained via the following application of the pigeonhole principle.

Lemma 8.4. *Let $A \in [1,\infty)$ be sufficiently large. Then for each sufficiently large ν, there exists $n_\nu \le N_{k_\nu}$ such that*

$$\text{distance}\,(n_\nu \int_{W_i^k} \Delta\varphi, 2\pi\mathbb{Z}) \le k_\nu^{-A} \qquad \textit{for all } 1 \le i \le M_k. \tag{8.2}$$

Proof. Write $k = k_\nu$ and let $\varepsilon_k = k^{-A}$. Consider the torus $T_k = (\mathbb{R}/2\pi\mathbb{Z})^{M_k}$, with one coordinate for each index $i \le M_k$. In T_k consider the sequence of points p_n, where the i-th component of p_n equals $n \int_{W_i^k} \Delta\varphi$ modulo $2\pi\mathbb{Z}$.

Partition T_k into $\lesssim \varepsilon_k^{-CM_k}$ cubes of sidelength $\le \frac{1}{2}\varepsilon_k$. $N_k/2$ is larger than the number of such cubes, since

$$\varepsilon_k^{-CM_k} \le k^{ACk/\log k \log\log k} \le e^{ACk/\log\log k} \ll \tfrac{1}{2}N_k = 2^{k-1}$$

for all sufficiently large k because of the extra factor of $\log\log k$. Therefore by the pigeonhole principle, there exist indices $1 \le n' < n'' \le N_k$ such that $p_{n'}, p_{n''}$ belong to the same cube. Setting $n_\nu = n'' - n'$, we conclude that n_ν has the desired property. □

Lemma 8.5. *There exists a sequence of natural numbers $n_\nu \le N_{k_\nu}$ satisfying* (8.2)*, such that $n_\nu \to \infty$ as $\nu \to \infty$.*

Proof. Modify the proof of Lemma 8.4 as follows: Let (b_k) be some nondecreasing sequence of natural numbers, such that $b_k \to \infty$ as $k \to \infty$, but $b_k/\log\log k \to 0$. In the proof of Lemma 8.4, consider only parameters n which are integral multiples of 2^{b_k}, $b_k = b_{k_\nu}$. The pigeonhole principle still applies, provided that

$$e^{ACk/\log\log k} < \tfrac{1}{2}N_k 2^{-b_k} = 2^{k-1-b_k}.$$

This holds for all sufficiently large k, since $b_k/\log\log k \to 0$. Therefore there exists an index n_ν satisfying (8.2), which is a positive integer multiple of $2^{b_{k_\nu}}$; in particular, $n_\nu \ge 2^{b_k}$. Thus $n_\nu \ge 2^{b_{k_\nu}} \to \infty$. □

Lemma 8.6. *Let $\Omega_0 \subset \mathbb{C}$ be open, and let $U_1, \dots U_N$ be pairwise disjoint open subsets of Ω_0, all with smooth boundaries. Set $\Omega = \Omega_0 \setminus \cup_j U_j$. Suppose that u is a real-valued harmonic function in Ω which has a multiple-valued real harmonic conjugate v in Ω such that e^{iv} is single-valued in Ω. Then for any function ψ with $\nabla\psi \in L^2(\Omega)$ and $\Delta\psi \in L^1(\Omega)$, the quadratic form S_ψ is unitarily equivalent in $L^2(\Omega)$ to $S_{\psi-u}$. That is, there exists a unitary mapping $\mathcal{U}$ on $L^2(\Omega)$ which preserves $C_0^1(\Omega)$ such that $Q_{\psi-u}(f,f) = Q_\psi(\mathcal{U}f, \mathcal{U}f)$ for all $f \in C_0^1(\Omega)$.*

Proof. $\mathcal{U}f(z) = e^{iv(z)}f(z)$ does the job, as one sees via the relations $v_y = u_x$, $v_x = -u_y$. □

Lemma 8.7. *Let Ω_0, U_j be as in Lemma 8.6, and suppose that they are all simply connected. Suppose that $u \in C^2(\Omega_0)$, that u is harmonic in $\Omega = \Omega_0 \setminus \cup_j U_j$, and that $\int_{U_j} \Delta u \in 2\pi\mathbb{Z}$ for each index j. Then u has a multiple-valued real harmonic conjugate v in Ω such that e^{iv} is single-valued in Ω.*

Proof. Define $\tilde{u}_j$ to be the Newtonian potential of $\Delta\varphi \cdot \chi_{U_j}$. Since $u - \sum_j \tilde{u}_j$ is harmonic in the simply connected domain Ω_0, it has a single-valued harmonic conjugate v_0 in Ω_0. Since U_j is simply connected, the fundamental group of $\mathbb{C} \setminus U_j$ is $\mathbb{Z}$, and hence the condition $\int_{U_j} \Delta\tilde{u}_j = \int_{U_j} \Delta u \in 2\pi\mathbb{Z}$ guarantees that $\tilde{u}_j$ has a multiple-valued harmonic conjugate v_j on $\mathbb{C} \setminus U_j$ such that e^{iv_j} is single-valued. Hence $v = v_0 + \sum_j v_j$ is a multiple-valued harmonic conjugate for u in $\Omega_0 \setminus \cup_j U_j$ such that e^{iv} is single-valued. □

Lemma 8.8. *If n_ν satisfies* (8.2) *for each ν, then the magnetic ground state eigenvalues $\sup_\nu \lambda^{\mathrm{m}}_{n_\nu\varphi}$ remain uniformly bounded as $\nu \to \infty$.*

Proof. Let $k = k_\nu$, and $n = n_\nu$. To each set W_i^k associate W_i^{k*}, the smallest open simply connected set containing W_i^k; this equals the union of W_i^k with all the bounded connected components of $\mathbb{C} \setminus W_i^k$. It may happen that one W_i^{k*} is properly contained in another; delete all such W_i^{k*} and retain only those which are maximal with respect to inclusion. Reorder so that those which remain are denoted $\{W_i^{k*} : i \le M_k^*\}$ where $M_k^* \le M_k$.

For each $1 \le i \le M_k^*$ there is some disk $D_i \subset W_i^{k*}$ of radius $\rho_i \gtrsim N_k^{-2/\alpha}$. Let $\tilde{D}_i$ be the disk concentric with D_i, with half as large a radius. Choose a function h_i supported on $\tilde{D}_i$ such that $\|h_i\|_{L^1} \lesssim k^{-A}$, and $\int_{W_i^{k*}} (n\Delta\varphi - h_i) \in 2\pi\mathbb{Z}$. Since distance $\left(\int_{W_i^{k*}} n\Delta\varphi,\ 2\pi\mathbb{Z}\right) = O(k^{-A})$, such functions clearly exist. Let H_i be the Newtonian potential $(2\pi)^{-1} h_i * \log|z|$ of h_i. Then because Ω_0 is bounded and every point of $\Omega \setminus D_i$ lies at a distance $\ge \frac{1}{2}\rho_i$ from the support of h_i,

$$\|\nabla H_i\|_{L^2(\Omega_0 \setminus D_i)} \le C(\log \rho_i)^{1/2} \|h_i\|_{L^1} = O(k^{1/2} k^{-A}).$$

By Lemmas 8.6 and 8.7, $S_{n\varphi}$ is unitarily equivalent, in $L^2(\Omega_0 \setminus \cup_{i=1}^{M_k^*} W_i^{k*})$, to S_{ψ_k} where ψ_k is the Newtonian potential of $n\Delta\varphi\chi_{\Omega_0 \setminus \cup_i W_i^{k*}} + \sum_i h_i$.

Consider the test function $f = u_k - c_k$ in $\Omega_0 \setminus \cup_i W_i^{k*}$, $f = 0$ in $\cup_i W_i^{k*}$. Since $u_k \equiv c_k$ on the boundary of each W_i^{k*}, $\|\nabla f\|_{L^2} \le \|\nabla u_k\|_{L^2}$, which is uniformly bounded. Moreover $\|f\|_{L^2}$ is bounded below by a strictly positive constant, uniformly in k.

Since $f = O(1)$ in L^∞, it suffices to show that $S_{\psi_k}(f, f)$ is bounded above, uniformly in k. Since $\|f\|_{L^\infty}$ and $\|\nabla f\|_{L^2}$ are uniformly bounded, it suffices to show that $\|\nabla \psi_k\|_{L^2}$ remains bounded as $k \to \infty$.

Now $n\Delta\varphi\chi_{\Omega_0 \setminus \cup_i W_i^{k*}} = O(2^{-k})$ in L^∞ norm, by the definition of harmless components V_j, so the gradient of its Newtonian potential is $O(2^{-k})$ in L^∞ and hence also in L^2. We have already noted that the gradient of the Newtonian potential H_i of h_i is $O(k^{-A+1})$ in L^2 norm on the complement of D_i, hence on the complement of W_i^{k*}. There are $M_k^* \lesssim k$ indices i, so in all, $\nabla\psi = O(k^{-A+2})$ in L^2 norm. By choosing $A > 2$ we can ensure that this is $O(k^{-1})$. Thus $\|\nabla \psi_k\|_{L^2} \to 0$ as $k \to \infty$. $\square$

Since $n_\nu \to \infty$ as $\nu \to \infty$, Lemma 8.8 implies Theorem 1.5.

Remark. The above arguments actually prove that for a subharmonic function φ such that $\Delta\varphi$ is Hölder continuous of some positive order, $\lim_{n\to\infty} \lambda^{\mathrm{e}}_{n\varphi} = \liminf_{n\to\infty} \lambda^{\mathrm{m}}_{n\varphi} = \lambda$, where λ is the first eigenvalue of the Dirichlet Laplacian of the fine interior of $\{\Delta\varphi = 0\}$. This is consistent with the well-known phenomenon that a strong magnetic field creates a Dirichlet boundary condition in the semi-classical limit.

Remark. Theorem1.5 also holds in the following slightly more general form. Let $A = (a_1, a_2) \in C^{1+\alpha}(\Omega_0, \mathbb{R}^2)$ and let $V = |\partial a_1/\partial y - \partial a_2/\partial x|$. Let $H_A = -(\nabla - iA)^2$ and $H_V^0 = -\Delta + V$. Let λ^m_{tA} and λ^e_{tV} denote respectively the first eigenvalues of the Dirichlet realizations of H_{tA} and H^0_{tV}. Then $\lim_{t\to+\infty} \lambda^m_{tA} = +\infty$ if and only if $\lim_{t\to+\infty} \lambda^e_{tV} = +\infty$. This can be proved by observing that after a gauge transformation, one may assume that $A = (-\varphi_y, \varphi_x)$ and $V = |\Delta\varphi|$ for some $\varphi \in C^{2+\alpha}(\Omega_0)$. Details are left to the interested reader.

9. Reduction to a problem in $\mathbb{C}^1$

The goal of this section is to reduce questions of compactness and property (P) on a complete Hartogs domain in $\mathbb{C}^2$ to questions concerning semi-classical limits of Schrödinger operators in $\mathbb{C}^1$. Theorem 1.2 is then an easy consequence of this reduction and Theorem 1.3.

Proposition 9.1. *Let $\Omega = \{(z, w) \in \mathbb{C}^2;\ |w| < e^{-\psi(z)},\ z \in \Omega_0\}$ be a complete Hartogs domain with smooth, strictly pseudoconvex boundary near $b\Omega \cap \{w = 0\}$. Suppose that ψ is a continuous subharmonic function on Ω_0 such that $\nabla\psi \in L^2_{loc}(\Omega_0)$ in the sense of distributions. Then*

(1) *If $b\Omega$ satisfies property (P) then $\lambda^{\mathrm{e}}_{n\psi}(\Omega_0) \to \infty$ as $n \to \infty$. If moreover $\Delta\psi$ is lower semicontinuous, then the converse also holds.*
(2) *The Kohn Laplacian has compact resolvent if and only $\lambda^{\mathrm{m}}_{n\psi}(\Omega_0) \to \infty$ as $n \to \infty$.*

Note that $\Delta\psi$ is indeed lower semicontinuous in the construction underlying Theorems 1.2 and 1.3.

When $b\Omega$ is smooth, the necessity in Theorem 9.1 (2) was first established by Matheos [30] and the other assertions of the lemma were proved in [18]. Although their results were stated only for the $\overline{\partial}$-Neumann Laplacian, their proofs contains the proof for the Kohn Laplacian as well. Since here we have only minimal assumption on regularity of the boundary, some modifications are needed. We provide details for the reader's convenience.

We first prove the forward implication in (1). Choose relatively compact subdomains Ω_2 and Ω_1, $\Omega_2 \subset\subset \Omega_1$, such that $b\Omega$ is strictly pseudoconvex over $\Omega_0 \setminus \Omega_2$.

For any $M > 0$, there exist a neighborhood U of $b\Omega$ and $g \in C^\infty(U)$ such that $-1 \le g \le 0$, and $\partial\overline{\partial}g \ge M$ on U. Replacing g by $\int_0^{2\pi} g(z, e^{i\theta}w)d\theta$ if necessary, we may assume that $g(z, e^{i\theta}w) \equiv g(z, w)$ for all z, w and all $\theta \in \mathbb{R}$. Let $\chi \in C_0^\infty(B(0, 1))$ be a Friedrich mollifier (*i.e.*, $\chi \ge 0$, $\chi(z) = \chi(|z|)$, and $\int_{\mathbb{C}} \chi = 1$). Let $\chi_\delta = (1/\delta^2)\chi(z/\delta)$ and $\psi_\delta = \chi_\delta * \psi$. Then $\{\psi_\delta\}$ is a decreasing sequence of smooth subharmonic functions converging locally uniformly to ψ as $\delta \to 0^+$. Furthermore, $\psi_\delta \to \psi$ in $L^1_{loc}(\Omega_0)$.

For $z \in \Omega_1$ and sufficiently small δ, let $h_\delta(z) = g(z, e^{-\psi_\delta(z)})$. A straightforward calculation using polar coordinates $w = re^{i\theta}$ yields that

$$(h_\delta(z))_{z\bar{z}} = \partial\overline{\partial}g(X, \overline{X}) - (\psi_\delta)_{z\bar{z}} g_r e^{-\psi_\delta}$$

where $X = (1, -2(\psi_\delta)_z e^{i\theta-\psi_\delta})$. Let $C_M = \max\{|g_r e^{-\psi}|;\ \ z \in \overline{\Omega}_1\}$; C_M depends on M since g does. Then $-1 \le h_\delta \le 0$ and $\Delta h_\delta \ge 4M - C_M \Delta\psi_\delta$ on $\overline{\Omega}_1$.

Let $\eta \in C_0^\infty(\Omega_1)$, $0 \le \eta \le 1$, and $\eta = 1$ on $\overline{\Omega}_2$. For any $f \in C_0^\infty(\Omega_0)$, substituting $u = f\eta$, $b = h_\delta$, and $\psi = 0$ into (2.3), we have

$$\int_{\Omega_0} |(f\eta)_z|^2 \ge \frac{1}{e}\int_{\Omega_0} (h_\delta)_{z\bar{z}} |f\eta|^2.$$

Therefore,

$$\begin{aligned}
\int_{\Omega_0} (|\nabla f|^2 + n\Delta\psi|f|^2) &\ge \int_{\Omega_0} (\frac{1}{2}|\nabla(f\eta)|^2 - |f\nabla\eta|^2 + n\Delta\psi|f|^2) \\
&\ge \frac{1}{8e}\int_{\Omega_0} \Delta h_\delta |f\eta|^2 + \frac{n}{2}\int_{\Omega_0} \Delta\psi|f|^2 \\
&\ge \frac{M}{2e}\int_{\Omega_0} |f|^2 + \int_{\Omega_0} (\frac{n}{4}\Delta\psi - \frac{C_M}{2e}\Delta\psi_\delta)|f|^2
\end{aligned}$$

when n is sufficiently large. Letting $\delta \to 0$, we obtain that

$$\int_{\Omega_0} (|\nabla f|^2 + n\Delta\psi|f|^2) \ge \frac{M}{2e}\int_{\Omega_0} |f|^2 + \int_{\Omega_0} (\frac{n}{4} - \frac{C_M}{2e})\Delta\psi|f|^2 \ge \frac{M}{2e}\int_{\Omega_0} |f|^2,$$

provided that n is sufficiently large relative to M. Therefore $\lambda^{\mathrm{e}}_{n\psi}(\Omega_0) \ge M/2e$ for all sufficiently large n, or equivalently, $\lim_{n\to\infty} \lambda^{\mathrm{e}}_{n\psi}(\Omega_0) = \infty$.

Now we prove the converse direction in (1). It suffices to prove that $K = b\Omega \cap \{(z,w) \in \mathbb{C}^2;\ z \in \overline{\Omega_2}\}$ satisfies property (P). Let $V = \{z \in \Omega_0;\ \Delta\psi > 0\}$; V is open since $\Delta\psi$ is assumed to be lower semicontinuous. Let $K_0 = \overline{\Omega_0} \setminus V$. Then K_0 is a compact subset of $\overline{\Omega_0}$ and $\Delta\psi \equiv 0$ on K_0. We claim that K_0 has empty fine interior. Otherwise, K_0 supports a function $\xi \in W^1$ which is nontrivial, that is, is nonvanishing on some set of positive Lebesgue measure (see [19]). It follows that $\lambda^{\mathrm{e}}_{n\psi}(\Omega_0) \le \|\nabla\xi\|^2/\|\xi\|^2 < \infty$, which contradicts our assumption. By Proposition 1.11 in [32], K_0 satisfies property (P). Therefore, for any $M > 0$, there exist a neighborhood U_0 of K_0 and $b \in C^\infty(U_0)$ such that $0 \le b \le 1$ and $\Delta b \ge M$. Since $\Delta\psi$ is lower semicontinuous and $\overline{\Omega_0} \setminus U_0$ is compact, there exists $\varepsilon_0 > 0$ such that $\Delta\psi \ge \varepsilon_0$ on $\Omega_0 \setminus U_0$. Let $g_\delta(z,w) = M(|w|^2 e^{2\psi_\delta} - 1) + b(z)$. Then when δ is sufficiently small, $|g_\delta| \lesssim 1$ and $\partial\overline{\partial} g_\delta \gtrsim M$ on K. Hence K satisfies property (P).

The proof of the necessity in (2) is easy: it suffices to plug $u(z)e^{in\theta}$ into the compactness estimate (2.1) and use the fact that $\|u(z)e^{in\theta}\|^2_{-1} \lesssim \|u\|^2/n^2$.

We now prove the sufficiency in (2) by establishing the compactness estimate (2.1). By assumption, there exists $N_\epsilon > 0$ such that when $n > N_\epsilon$,

$$\|v\|^2 \le \epsilon \|L_{n\psi} v\|^2, \quad \text{for all } v \in C_0^\infty(\Omega_0).$$

Taking the complex conjugate, we deduce that when $n < -N_\epsilon$,

$$\|v\|^2 \le \epsilon \|\bar{L}_{n\psi} v\|^2, \quad \text{for all } v \in C_0^\infty(\Omega_0).$$

Therefore, when $|n| > N_\epsilon$

$$\|v\|^2 \le \epsilon(\|L_{n\psi} v\|^2 + \|\bar{L}_{n\psi} v\|^2), \quad \text{for all } v \in C_0^\infty(\Omega_0).$$

For $u \in C_0^\infty(\Omega_0 \times T)$, write $u = \sum_{n=-\infty}^{\infty} u_n(z) e^{in\theta}$. Then

$$\begin{aligned} \|u\|^2 = \sum_{n=-\infty}^{\infty} \|u_n(z)\|^2 &\le \sum_{|n|\le N_\epsilon} \|u_n(z)\|^2 + \epsilon \sum_{|n|>N_\epsilon} (\|L_{n\psi} u_n\|^2 + \|\bar{L}_{n\psi} u_n\|^2) \\ &= \epsilon(\|Lu\|^2 + \|\bar{L}u\|^2) + \sum_{|n|\le N_\epsilon} \left(\|u_n\|^2 - \epsilon(\|L_{n\psi} u_n\|^2 + \|\bar{L}_{n\psi} u_n\|^2)\right). \end{aligned}$$

Since the inclusion $\{u \in L^2(\Omega_0);\ \|L_{n\varphi} u\|^2 + \|\overline{L}_{n\varphi} u\|^2 \le 1\} \subset L^2(\Omega_0)$ is compact(see [2]), the last sum in the above inequalities is less than or equal to $C_\epsilon \sum_{|n|\le N_\epsilon} \|u_n\|^2_{-1}$ for some sufficiently large C_ϵ, depending only on ϵ. The desired inequality (2.1) then follows from the fact that this last sum is controlled by $\|u\|^2_{-1}$.

We now indicate the standard procedure to construct the function ψ in Theorem 1.2 from the function φ in Theorem 1.3. Let χ_1 be a smooth function such that $0 \le \chi_1 \le 1$, $\chi_1 = 1$ for $t \le 4/3$, and $\chi_1 = 0$ for $t \ge 3/2$. Let χ_2 be a smooth function on $(0, 2)$ such that $\chi_2', \chi_2'' > 0$, $\chi_2 = 0$ for $t \le 1$, and $\chi_2(t) = -\frac{1}{2}\log(4 - t^2)$ when t is sufficiently closed to 2. We may choose such χ_2 so that its second derivative is large enough on $[4/3, 3/2]$ to guarantee that the Laplacian of $\psi(z) = \varphi(z)\chi_1(|z|) + \chi_2(|z|)$ is strictly positive when $1 < |z| < 2$.

That $b\Omega$ does not satisfy property (P) is then a consequence of Proposition 9.1 (1) and the fact that $\lambda^{\mathrm{e}}_{n\psi}(B(0,2)) \le \lambda^{\mathrm{e}}_{n\varphi}(B(0,1)) \lesssim 1$. It remains to prove that $\lambda^{\mathrm{m}}_{n\psi}(B(0,2)) \to \infty$ as $n \to \infty$. This is a consequence of the combination of the facts that $\lambda^{\mathrm{m}}_{n\varphi}(B(0,1)) \to \infty$, $b\Omega$ is strictly pseudoconvex over $1 < |z| < 2$, and $b\Omega \cap \{(z,\, w) \in \mathbb{C}^2;\ |z| = 1\}$ satisfies property (P); the details of this argument follow below.

Let χ_2 be a smooth function supported in $(-1, 1)$ such that $-1 \le \chi_3 \le 1$, $\chi_3 = t$ for $-1/2 \le t \le 1/2$. For any $M > 0$, let $b(z) = \chi_3(M(|z|^2 - 1)) - 1$. Applying (2.3) with this choice of $b(z)$, we obtain that for any $u \in C_0^\infty(B(0,2))$,

$$\int_{B(0,2)} |L_{n\psi}u|^2 \geq \frac{1}{e^2}\int_{B(0,2)} b_{z\bar{z}}|u|^2 \geq \frac{M}{e^2}\int_{||z|^2-1|<\frac{1}{2M}} |u|^2 - C_M\int_{||z|^2-1|\geq\frac{1}{2M}} |u|^2. \tag{9.1}$$

where C_M is a constant depend only on M. Let $\zeta(z) = (|z|^2-1)^2$. Then

$$\begin{aligned}\int_{B(0,1)} |L_{n\psi}u|^2 &\geq \int_{B(0,1)} |\zeta L_{n\psi}u|^2 \geq \frac{1}{2}\int_{B(0,1)} |L_{n\psi}(\zeta u)|^2 - \int_{B(0,1)} |u\frac{\partial\zeta}{\partial z}|^2 \\ &\geq \frac{1}{2}\lambda^{\mathrm{m}}_{n\varphi}(B(0,1))\int_{B(0,1)} |\zeta u|^2 - 4\int_{B(0,1)} |u|^2. \end{aligned}\tag{9.2}$$

Note also that

$$\int_{B(0,2)} |L_{n\psi}u|^2 \geq n\int_{1<|z|<2} \Delta\psi|u|^2. \tag{9.3}$$

Combining (9.1), (9.2), (9.3), and the facts that $\lambda^{\mathrm{m}}_{n\varphi}(B(0,1)) \to \infty$, $\Delta\psi > 0$ on $1 < |z| < 2$, and $\Delta\psi \to \infty$ as $|z| \to 2$, we obtain

$$\int_{B(0,2)} |L_{n\psi}u|^2 \geq \frac{M}{2e^2}\int_{B(0,2)} |u|^2$$

when n is sufficiently large. Therefore $\lambda^{\mathrm{m}}_{n\psi}(B(0,2)) \to \infty$ as $n \to \infty$.

10. Proof of Theorem 1.1

The equivalence of assertions (1) and (2) of Theorem 1.1 has been established by Matheos [30] and the implication (3) ⇒ (1) is a consequence of Catlin's theorem [7]. We need only show that (1) implies (3). This will be a consequence of Theorem 1.5; the proof of the reduction is divided into two lemmas. Let $\Omega = \{(z,w) \in \mathbb{C}^2;\ \rho(z,w) < 0\}$ where ρ is a smooth defining function that is invariant under rotations in w. Let

$$\begin{aligned} S_0 &= \{(z,w) \in b\Omega;\ \ \frac{\partial\rho}{\partial w}(z,w) = 0\}; \\ S_k &= \{(z,w) \in b\Omega;\ \ |\frac{\partial\rho}{\partial w}(z,w)| \geq 1/k\}. \end{aligned}$$

Then $b\Omega = S_0 \cup (\cup_{k=1}^\infty S_k)$. By Proposition 1.9 in [32], it suffices to prove that each S_k, $k = 0, 1, \ldots$, is B-regular.

Lemma 10.1. *If the $\bar\partial$-Neumann Laplacian $\Box$ has compact resolvent in $L^2(\Omega)$ then S_0 is B-regular.*

Proof. Let $\pi\colon S_0 \to \mathbb{C}$ be the projection to the z-plane. Let $\widehat{S}_0 = \pi(S_0)$. According to Proposition 1.10 in [32], it suffices to prove that all fibers $\pi^{-1}(z_0)$, $z_0 \in \widehat{S}_0$, as well as $\widehat{S}_0$ itself, are B-regular.

We identify $\pi^{-1}(z_0)$ with its projection to the w-plane. Note that $\pi^{-1}(z_0)$ is a union of circles centered at the origin. Since $b\Omega$ is variety-free, $\pi^{-1}(z_0)$ must have empty fine interior, and hence is B-regular. Otherwise, suppose w_0 is a fine interior point of $\pi^{-1}(z_0)$. Then $\pi^{-1}(z_0)$ contains a circle centered at w_0 (cf. [22], Theorem 10.14), which implies that $b\Omega$ contains an annulus (or a disc when $w_0 = 0$). But the presence of a complex variety in $b\Omega$ forces $\Box$ to have a noncompact resolvent (cf. [17]), which is a contradiction.

It remains to prove that $\widehat{S}_0$ is B-regular. In fact, we will prove that $\widehat{S}_0$ has zero Lebesgue measure. Let $(z_0, w_0) \in S_0$. Since $\rho_z(z_0, w_0) \neq 0$, we may assume without loss of generality that $\rho_y(z_0, w_0) \neq 0$. Then in a neighborhood U of (z_0, w_0), $b\Omega$ is given by $y = \tilde{\rho}(x, w)$ where $\tilde{\rho}$ is rotation-invariant with respect to w. It follows that locally

$$\pi(\overline{U} \cap S_0) = \{x + i\tilde{\rho}(x, |w|) \in \mathbb{C}^1 : \ \frac{\partial \tilde{\rho}}{\partial |w|}(x, |w|) = 0\},$$

which by Sard's theorem has zero Lebesgue measure.

□

Lemma 10.2. *Each S_k, $k = 1, 2, \ldots$, is B-regular.*

Proof. It suffices to prove that for any $(z_0, w_0) \in S_k$, there exists a neighborhood U of (z_0, w_0) such that $\overline{U} \cap S_k$ is B-regular.

Since $|\rho_w(z_0, w_0)| = |\rho_{|w|}(z_0, |w_0|)|/2 \geq 1/k$, $b\Omega$ is defined near (z_0, w_0) by a graph of the form $|w| = e^{-\varphi(z)}$. Assume that $\rho_{|w|}(z_0, |w_0|) < 0$. (The other case is treated similarly.) Then there exist $a, b > 0$ such that

$$\Omega \cap U_{a,b} = \{(z, w); \ \ z \in B(z_0, a), \ e^{-\varphi} < |w| < |w_0| + b\}.$$

where $U_{a,b} = B(z_0, a) \times \{|w_0| - b < |w| < |w_0| + b\}$. The pseudoconvexity of Ω implies that φ is superharmonic on $B(z_0, a)$. Shrinking a if necessary, we may also assume that $e^{-\varphi} \in (|w_0| - b/2, \ |w_0| + b/2)$ for all $z \in B(z_0, a)$.

For any $\beta \in C_0^\infty(B(z_0, a))$ and any positive integer n, consider the $(0,1)$-form

$$u_n = \begin{cases} \beta(z) w^{-n} d\bar{z}, & |w_0| - b < |w|; \\ 0, & \text{otherwise.} \end{cases}$$

Then $u_n \in C^\infty(\overline{\Omega})$ and $\overline{\partial} u_n = 0$. Since the $\overline{\partial}$-Neumann operator has compact resolvent, the canonical solution operator S is likewise compact (cf. [17]). It follows that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $f_n(z, w) = S(u_n)$ satisfies

$$\|f_n\|_\Omega^2 \leq \varepsilon \|u_n\|_\Omega^2 + C_\varepsilon \|u_n\|_{-1,\Omega}^2.$$

Since $\|u_n\|_{-1,\Omega}^2 \lesssim (1/n^2)\|u_n\|_\Omega^2$, there exists $N_\varepsilon > 0$ such that

$$\|f_n\|_\Omega^2 \leq 2\varepsilon \|u_n\|_\Omega^2, \quad n \geq N_\varepsilon. \tag{10.1}$$

Since $\partial f_n / \partial \bar{w} = 0$ for $(z, w) \in \Omega$ with $z \in B(z_0, a)$, $f_n(z, w)$ is holomorphic in w on $e^{-\varphi} < |w| < |w_0| + b$ for any fixed $z \in B(z_0, a)$. Furthermore, $f_n(z, w) = g_n(z) w^{-n}$ where $\partial g_n / \partial \bar{z} = \beta(z)$ on $\Omega \cap U_{a,b}$. Note that the left hand side of (10.1) is

$$\begin{aligned} \|f_n\|_\Omega^2 &\geq \int_{B(z_0,a)} |g_n(z)|^2 dxdy \int_{e^{-\varphi}}^{|w_0|+b} r^{-2n+1} dr \\ &= \frac{1}{-2(n-1)} \int_{B(z_0,a)} |g_n(z)|^2 \left(e^{2(n-1)\varphi} - (|w_0| + b)^{-2(n-1)} \right) dxdy, \end{aligned} \tag{10.2}$$

and the right hand side of (10.1) is

$$\begin{aligned} 2\varepsilon \|u_n\|^2 &\leq 2\varepsilon \int_{B(z_0,a)} |\beta(z)|^2 dxdy \int_{e^{-\varphi}}^{A} r^{-2n+1} dr \\ &\leq \frac{2\varepsilon}{-2(n-1)} \int_{B(z_0,a)} |\beta(z)|^2 \left(A^{-2(n-1)} - e^{2(n-1)\varphi} \right). \end{aligned} \tag{10.3}$$

(Here we assume that $\Omega \subset \{|w| < A/2\}$ for some constant $A > 0$.) Since

$$e^{2(n-1)\varphi} - (|w_0| + b)^{-2(n-1)} = e^{2(n-1)\varphi}\left(1 - \left((|w_0| + b)^{-1}e^{-\varphi}\right)^{2(n-1)}\right)$$

and

$$(|w_0| + b)^{-1}e^{-\varphi} \leq \frac{|w_0| + b/2}{|w_0| + b} < 1,$$

it follows from (10.1), (10.2), and (10.3) that for any $\beta \in C_0^\infty(B(z_0, a))$, there exists $g_n(z)$ such that $\partial g_n/\partial \bar{z} = \beta(z)$ and

$$\int_{B(z_0,a)} |g_n(z)|^2 e^{2(n-1)\varphi} \leq 3\varepsilon \int_{B(z_0,a)} |\beta(z)|^2 e^{2(n-1)\varphi}$$

when $n \geq N_\varepsilon$. A duality argument then yields that

$$\int_{B(z_0,a)} |\alpha(z)|^2 e^{-2(n-1)\varphi} \leq 3\varepsilon \int_{B(z_0,a)} |\alpha_z(z)|^2 e^{-2(n-1)\varphi}, \quad \forall \alpha \in C_0^\infty(B(z_0, a)), n \geq N_\varepsilon.$$

Substituting $\alpha = ue^{(n-1)\varphi}$ and then replacing $n - 1$ by n, this becomes

$$\int_{B(z_0,a)} |u(z)|^2 \leq 3\varepsilon \int_{B(z_0,a)} |(\partial_z + n\varphi_z)u|^2 \quad \forall u \in C_0^\infty(B(z_0, a)), n \geq N_\varepsilon.$$

Now φ is superharmonic; $-\varphi$ is subharmonic. $-\partial_z - n\varphi_z = L_{-n\varphi}$, so this last inequality is equivalent to $\lambda^{\mathrm{m}}_{-n\varphi}(B(z_0, a)) \to \infty$ as $n \to +\infty$, which by Theorem 1.5 implies that $\lambda^{\mathrm{e}}_{-n\varphi}(B(z_0, a)) \to \infty$. Therefore, by (the proof of) Proposition 9.1 (1), $b\Omega \cap \overline{U}_{a,b}$ satisfies property (P). $\square$

Remark. In the above proof we assume only that $b\Omega$ is of class $C^{2+\alpha}$, which is needed to invoke Theorem 1.5. In the C^∞ case, Lemma 10.2 could be proved more quickly by combining Theorem 1.5 with the equivalence, established by Matheos [30], between compactness in the $\bar\partial$-Neumann problem and boundary compactness in the sense of (2.1).

Remark. If $\{\Delta\varphi = 0\} = W$ is constructed as in Section 4, then the conclusion of Theorem 1.5 remains valid whenever $\Delta\varphi$ is lower semicontinuous and in L^p for some $p > 1$.

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