HYPOELLIPTICITY IN THE INFINITELY DEGENERATE REGIME

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Contents

Let $\{X_j\}$ be a collection of real vector fields with C^∞ coefficients, defined in a neighborhood of a point $x_0 \in \mathbb{R}^d$. Consider a second order differential operator $L = -\sum_j X_j^2 + \sum_j \alpha_j X_j + \beta$ where α_j, β are C^∞ real coefficients. A well known sufficient condition [17] for L to be C^∞ hypoelliptic is that the Lie algebra generated by $\{X_j\}$ should span the tangent space to $\mathbb{R}^d$ at x_0. This bracket condition is by no means necessary; no satisfactory characterization of hypoellipticity exists, and it appears unlikely that one could be found. The purposes of this note are:

(1) To establish sufficient conditions for hypoellipticity for operators such as $-\sum_j X_j^2$, for the Kohn Laplacian on pseudoconvex three-dimensional CR manifolds, and for the $\bar\partial$–Neumann problem in $\mathbb{C}^2$, in the case of infinite type.
(2) To point out an inequality weaker than the subelliptic estimates which implies hypoellipticity, and which is the weakest possible such inequality.
(3) To popularize within the $\bar\partial$–Neumann community certain ideas developed in another context.
(4) To emphasize the parallel between the theories of C^∞ and analytic/Gevrey class hypoellipticity.

This paper has undergone several revisions since its first version, written in the Spring of 1996. Although the results and methods employed here are new within the context of the $\bar\partial$–Neumann problem, I have subsequently learned that when viewed in the wider context of sums of squares of vector fields and related operators, they substantially overlap works of other authors, some earlier and some contemporaneous, including but not limited to

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[19],[26],[27]; other selected relevant works are listed in the bibliography. Thus this paper is in part expository.

It has subsequently been shown by Kohn [20] how some of these results may be extended to $\Box_b$ and the $\bar\partial$–Neumann problem in higher dimensions.

1. Examples

To orient the reader we review a few concrete examples of operators which need not satisfy the bracket hypothesis. Consider three classes of operators:

$$\begin{aligned} L_1 &= -\partial_x^2 - a^2(x)\partial_t^2 && \text{in } \mathbb{R}^2 \\ L_2 &= -\partial_x^2 - a^2(x)\partial_t^2 - \partial_y^2 && \text{in } \mathbb{R}^3 \\ L_3 &= -\partial_x^2 - a^2(x)\partial_t^2 - b^2(x)\partial_y^2 && \text{in } \mathbb{R}^3 \end{aligned}$$

Assume that $a, b \in C^\infty$, that a, b are even and nonnegative, and are nondecreasing on $[0, \infty)$, and that $a(x) = 0 \Leftrightarrow x = 0$. Concerning operators L_3, assume that $b(x) \ge a(x)$ for all x, and likewise that $b(x) = 0 \Leftrightarrow x = 0$.

Proposition 1.1. (Fediǐ [13]) *Under the above hypotheses, all operators L_1 are hypoelliptic.*

Proposition 1.2. (Kusuoka and Stroock [21]) *L_2 is hypoelliptic if and only if*

$$\lim_{x\to 0} x \log a(x) = 0. \tag{1.1}$$

This was originally proved by stochastic techniques; see Theorem 8.41 of [21]. To indicate how these examples fit within the general context discussed in this paper, we will indicate alternative proofs of Propositions 1.1 and 1.2.

Proposition 1.3. *Suppose that the coefficients $a, b \in C^\infty$ vanish where $x = 0$ but nowhere else, and that $b(x) \ge a(x) \ge 0$ for all x. If $\limsup_{x\to 0} |x \log a(x)| \ne 0$, and if the coefficient b satisfies*

$$\lim_{x\to 0} b(x)x|\log a(x)| = 0, \tag{1.2}$$

then L_3 is hypoelliptic. Moreover, if some partial derivative of b is nonzero at $x = 0$, then L_3 is hypoelliptic if and only if (1.2) is satisfied.

This result was discovered independently by other authors. What is interesting about this example is that increasing the size of b makes the operator L_3 stronger in the sense that the associated quadratic form $\langle L_3 f,\, f\rangle$ becomes larger for every f, but makes L_3 less likely to be hypoelliptic. In the context of analytic hypoellipticity such a phenomenon has already been observed: Métivier [23] has shown that $\partial_x^2 + (x\partial_t)^2 + (t\partial_t)^2$ is not analytic hypoelliptic, while the weaker operator $\partial_x^2 + (x\partial_t)^2$ is analytic hypoelliptic [15]. A parallel result for Gevrey hypoellipticity is that when $a(x) = x^q$ and $b(x) = x^p$ with $q \ge p \ge 1$, L_3 is hypoelliptic in the Gevrey class of order s if and only if $b \cdot a^{-1/s}$ is bounded as $x \to 0$ [7]; see also [5] and [24].

The next proposition is a special case of Theorem 2.5.3 of Oleĭnik and Radkevič[1] [28].

Proposition 1.4. *Suppose that $L = -\sum_j X_j^2$ where the X_j are smooth real vector fields in some open set V. Suppose that at each point of V, at least one X_j is nonzero. Suppose that $\{X_j\}$ satisfies the bracket hypothesis at all but finitely many points of V. Then L is hypoelliptic in V.*

The analogue for C^ω hypoellipticity is false; $\partial_x^2 + (x\partial_t)^2 + (t\partial_t)^2$ is elliptic except at a single point, yet is not analytic hypoelliptic [23]. For other such examples in the analytic and Gevrey class contexts see [8],[9].

2. Main results

Our results are most naturally formulated in a somewhat more general framework. Denote by $S^m_{\rho,\delta}$ the usual classes of symbols for pseudodifferential operators [31],[32]. Denote by $S^{m+}_{1,0}$ the intersection, over all $\varepsilon > 0$, of all classes $S^{m+\varepsilon}_{1-\varepsilon,\varepsilon}$. Given any class S of symbols, we denote by $\mathrm{Op}(S)$ the associated class of pseudodifferential operators, with respect to the quantization of the Kohn-Nirenberg calculus. In particular we consider classes $\mathrm{Op}(S^m_{\rho,\delta})$, $\mathrm{Op}(S^{m+}_{1,0})$ of operators.

All functions, distributions and symbols are permitted to be complex vector valued except where otherwise noted; $\langle u, v\rangle$ denotes either a Hermitian pairing of square integrable vector valued functions, or of test functions with distributions. The symbol $\|u\|$, with no subscript, denotes the L^2 norm. For any open subset $V \subset \mathbb{R}^d$, we denote by T^*V the complement of the zero section in the cotangent bundle of V. X^* denotes the formal adjoint of any operator X. $C_0^k(V)$ and $C_0^\infty(V)$ denote the classes of functions compactly supported in V and belonging to C^k, C^∞ respectively.

Fix an open neighborhood V of x_0, and suppose L to be a linear operator mapping $\mathcal{D}(V)$ to $\mathcal{D}'(V)$ taking the form

$$L = \sum_j X_j^* X_j + \sum_j A_j X_j + \sum_j X_j^* \tilde{A}_j + A_0 \tag{2.1}$$

where $X_j \in \mathrm{Op}(S^1_{1,0})$ and $A_j, \tilde{A}_j, A_0 \in \mathrm{Op}(S^0_{1,0})$. This sum and similar ones will always be taken over all $1 \le j \le J$ for some unspecified integer J. More generally, we consider any $L \in S^2_{1,0}$ whose full symbol equals the full symbol of such an operator, in some conic subset $\Gamma \subset T^*V$.

A fundamental example is the Kohn Laplacian $\Box_b = \bar\partial_b^*\bar\partial_b + \bar\partial_b\bar\partial_b^*$ defined on (p,q) forms on any CR manifold; this is a second order system of partial differential operators with complex coefficients.[2] A closely related second example (up to composition with a harmless elliptic factor) is the pseudodifferential Calderón operator for any smoothly bounded domain in $\mathbb{C}^2$, arising from application of the boundary reduction method to the $\bar\partial$–Neumann problem.

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[1]The theorem cited concerns *global* hypoellipticity, but for operators hypoelliptic in $U\backslash\{x_0\}$, hypoellipticity and global hypoellipticity in U are equivalent.

[2]It is not identical to the operator referred to by the same name in [22].

Write $\langle\xi\rangle = (e^2 + |\xi|^2)^{1/2}$ for any $\xi \in \mathbb{R}^d$. By $\log^2(y)$ we mean $(\log y)^2$. By a conic open subset of $T^*\mathbb{R}^d$ we will always mean such a subset disjoint from the zero section. Denote by H^s the usual Sobolev space of order s, and by $WF_{H^s}(u)$ the H^s wave front set of a distribution u. Denote by $\hat{u}$ the Fourier transform of u.

All operators studied in this paper will satisfy global inequalities of the form

$$\int_{\mathbb{R}^d} w^2(\xi)|\hat{u}(\xi)|^2 \, d\xi \le C \sum_j \|X_j u\|^2 + C\|u\|^2 \quad \text{for all } u \in C_0^1(V), \tag{2.2}$$

where V is some fixed open set in which hypoellipticity is to be studied, $C < \infty$ is a fixed constant, and w is a strictly positive, continuous function satisfying

$$w(\xi) \to \infty \quad \text{as } |\xi| \to \infty. \tag{2.3}$$

This will either be assumed explicitly, or will be a consequence of other hypotheses.

Theorem 2.1. *Let L take the form (2.1). Suppose that there exists a function w satisfying*

$$\frac{w(\xi)}{\log\langle\xi\rangle} \to \infty \quad \textit{as } |\xi| \to \infty \tag{2.4}$$

for which (2.2) holds. Then L is hypoelliptic in V. More precisely, for any $s \in \mathbb{R}$ and $u \in \mathcal{D}'(V)$,

$$WF_{H^s}(u) \subset WF_{H^s}(Lu). \tag{2.5}$$

The hypothesis (2.4) is the optimal condition of its type. For instance:

Proposition 2.2. *Operators of the type L_2 discussed in Proposition 1.2 are hypoelliptic if and only if they satisfy (2.4).*

Moreover, the operator $-\partial_{x_1}^2 - \partial_{x_2}^2 - e^{-2/|x_1|}\partial_{x_3}^2$ in $\mathbb{R}^3$ satisfies the inequality (2.2) with $w(\xi) = \log\langle\xi\rangle$, and fails to be hypoelliptic. See §5.

Theorem 2.1, and the observation that (2.4) is necessary for certain operators, are due originally to Y. Morimoto. See for instance [25], [37], [26].

Nonetheless, it is important to understand that (2.4) is very far from being a necessary condition for hypoellipticity. Consider for instance operators of the type L_1, with $a(x) = 0$ if and only if $x = 0$. These all satisfy the compactness inequality (2.2) for some weight w tending to ∞, but w may tend to infinity arbitrarily slowly; like operators L_2, they satisfy (2.4) if and only if $x \log a(x) \to 0$ as $x \to 0$. Yet they are all hypoelliptic. The same remarks apply to the operators with isolated degeneracies described in Proposition 1.4.

An equivalent formulation of (2.4) is that for each $\delta > 0$ there should exist $C_\delta < \infty$ such that for each real valued function $u \in C_0^2(V)$,

$$\int_{\mathbb{R}^d} \log^2\langle\xi\rangle \, |\hat{u}(\xi)|^2 \, d\xi \le \delta \sum_j \|X_j u\|^2 + C_\delta \|u\|^2.$$

Further definitions are required in order to formulate our main result. A point x_0 is said to belong to the complement of the H^s singular support of a distribution u if there exists a distribution $v \in H^s$ such that $v \equiv u$ in some neighborhood of x_0. For any conic subset $R \subset T^*\mathbb{R}^d$, write $u \in H^s(R)$ to mean that $WF_{H^s}(u) \cap R = \emptyset$. Similarly $u \in C^\infty(R)$ means that $WF(u) \cap R = \emptyset$.

A set of vector fields $\{X_j\}$ is said to satisfy the microlocal bracket hypothesis at a point (x_0, ξ_0) of the unit cosphere bundle S^*V if some iterated Poisson bracket

$$\{\sigma_1(X_{j_1}),\, \{\sigma_1(X_{j_2}),\, \dots\} \dots\}(x_0, \xi_0)$$

of their principal symbols $\sigma_1(X_j)$ is nonzero [4]. Write $X_j = \mathrm{Op}(\sigma(X_j))$. Recall that the Poisson bracket of two functions $f, g \in C^1(T^*\mathbb{R}^d)$ is defined to be

$$\{f,\, g\}(x, \xi) = \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} - \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial \xi_j}. \tag{2.6}$$

The principle underlying our analysis (when L is a sum of squares of vector fields X_j) is that hypoellipticity is governed by a semiglobal comparison of (i) the Hamiltonian vector fields associated to the principal symbols of $\{X_j\}$, with (ii) the size of those principal symbols. The larger the latter, the more likely is an operator to be hypoelliptic; the larger the former, the less likely. The following result and its variant Theorem 2.4, formulated below, express this principle. Together they subsume all other sufficient conditions for hypoellipticity formulated in this paper.

Main Theorem 2.3. *Let $R \subset T^*V$ be any ray. Assume that L takes the form* (2.1) *in some conic neighborhood of R, and satisfies* (2.2) *for some $w \in C^\infty$ such that $w(\xi) \to \infty$ as $|\xi| \to \infty$.*

Suppose that for each small conic neighborhood Γ of R there exist scalar valued symbols $\psi, p \in S^0_{1,0}$ such that ψ is everywhere nonnegative, $\psi \equiv 0$ in some smaller conic neighborhood of R, $\psi \geq 1$ on $T^*V \setminus \Gamma$, $p \equiv 0$ in a conic neighborhood of the closure of Γ, and such that for each $\delta > 0$ there exists $C_\delta < \infty$ such that for any relatively compact open subset $U \Subset V$ and for all $u \in C^2_0(U)$ and each index i,

$$\| \mathrm{Op}\left[\log\langle\xi\rangle\{\psi, \sigma(X_i)\}\right]u\|^2 \leq \delta \sum_j \|X_j u\|^2 + C_\delta\|u\|^2 + C_\delta\| \mathrm{Op}(p)u\|^2_{H^1}. \tag{2.7}$$

Then for any $u \in \mathcal{D}'(V)$, $Lu \in C^\infty(R) \Rightarrow u \in C^\infty(R)$. Likewise for each $s \in \mathbb{R}$ and any u, $Lu \in H^s(R) \Rightarrow u \in H^s(R)$.

Because $\{\psi, \sigma(X_i)\} \in S^0_{1,0}$, the symbol of $\mathrm{Op}\left[\log\langle\xi\rangle\{\psi, \sigma(X_i)\}\right]$ is bounded by $\log\langle\xi\rangle$. Hence Theorem 2.3 directly implies Theorem 2.1, by Gårding's inequality and pseudodifferential calculus.

Two points require clarification, however. First, all norms are taken over V, and the constants C_δ are permitted to depend on U. Note that because all pseudodifferential operators occurring here are pseudolocal, an equivalent statement is obtained by taking the L^2 and H^1 norms over an arbitrarily small neighborhood of the closure of U, rather than over V. Second, L is assumed to take the form (2.1) only in a small conic neighborhood of R; the symbols of the operators $X_j \in \mathrm{Op}(S^1_{1,0})$ are extended arbitrarily so that those operators become globally defined. The conclusion (2.7) is independent of the choices of extensions, because of the presence of the term involving $\mathrm{Op}(p)u$ on the right hand side.

The main hypothesis is the existence of ψ having a favorable commutation relation with each X_i. The symbol p plays a subsidiary role; its presence on the right hand side of the inequality means that commutation with ψ need be controlled only near R.

We do not believe that there can be any simple characterization on the level of symbols of those operators satisfying the hypotheses of Theorem 2.3, analogous to the bracket hypothesis for subellipticity; this is a substantial weakness in the theory. It is a more difficult problem to characterize pairs of pseudodifferential operators $A, B \in S^1_{1,0}$ for which B dominates A, in the sense that $\|Au\| \le C\|Bu\| + C\|u\|$ for all functions u, for general B than for subelliptic B, because the connection between the strength of an operator and the size of its symbol becomes more tenuous when the symbol is permitted to vanish somewhere to infinite order [14].

The next variant has weaker hypotheses and conclusion. An application will be given below in Theorem 3.5.

Theorem 2.4. *Let L take the form (2.1) in some open set $V \subset \mathbb{R}^d$, and let $x_0 \in V$. Suppose that for each neighborhood $U \Subset V$ of x_0 there exist scalar valued functions $\Psi \in C^\infty(V)$ and $\eta \in C^\infty(V)$ such that $\Psi \ge 0$, $\Psi \equiv 0$ in a neighborhood of x_0, $\Psi \ge c > 0$ on $V \backslash U$, and $\Psi > 0$ on a neighborhood of the support of η. Suppose further that for every $\delta > 0$ there exists $C_\delta < \infty$ such that $\psi(x,\xi) = \Psi(x)$ satisfies*

$$\|\operatorname{Op}\big[\log\langle\xi\rangle\{\psi, \sigma(X_i)\}\big]u\|^2 \le \delta \sum_j \|X_j u\|^2 + C_\delta\|u\|^2 + C_\delta\|\eta u\|^2_{H^1}, \tag{2.8}$$

uniformly for all $u \in C_0^\infty(U)$. Then for any $u \in \mathcal{D}'(V)$,

$$x_0 \notin \textit{singular support}\,(Lu) \Rightarrow x_0 \notin \textit{singular support}\,(u).$$

Moreover for any $s \in \mathbb{R}$, if Lu belongs to H^s in some neighborhood of x_0, then so does u.

Our analysis is based on conjugation with pseudodifferential operators of variable order. It is related to certain variants[3] of the FBI transform used to study Gevrey class hypoellipticity in [7]. The use of operators having favorable commutation relations with a given family of vector fields is a common technique, used for example in the work of Sjöstrand [29],[30] and of Tartakoff and other authors on analytic hypoellipticity, and of Boas and Straube [3] on global C^∞ regularity in the $\bar\partial$–Neumann problem. The same, or closely related, methods are employed in [25], [19], [26], [27], and other works.

3. Some applications

The motivation for this work was fourfold. First, a theorem of Derridj and Zuily [11] (together with its proof) asserts that for vector fields with real analytic coefficients[4], a subelliptic estimate of order ε implies that $\sum X_j^2$ is hypoelliptic in the Gevrey class G^s for all $s \ge \varepsilon^{-1}$; this exponent s is optimal, in general. Since $\varepsilon \to 0$ as $s \to \infty$, and since

[3] These variants take the form $\mathcal{F}u(x,\xi) = \int \exp(i(x-y)\cdot\xi - N\psi(x,\xi)\log\langle\xi\rangle)\, u(y)\eta(y)\,dy$, where $\eta \in C_0^\infty$ is a cutoff function, N is a large parameter, and $\psi \in C_0^1$ is an arbitrary nonnegative symbol.

[4] It suffices that the coefficients belong to G^s.

$G^s \subset C^\infty$ for all s, this suggested that some limiting bound weaker than subellipticity should suffice for C^∞ hypoellipticity.[5] Theorem 2.1 confirms this idea.

A second motivation was work of Bell and Mohammed [2], Kusuoka and Stroock [21] and Malliavin [22] establishing hypoellipticity in certain nonsubelliptic cases by stochastic methods. Consider any finite collection $\{X_j\}$ of C^∞ vector fields, defined in a neighborhood V of $x_0 \in \mathbb{R}^d$. For $k \geq 1$ denote by $\mathfrak{g}_k$ the $C^\infty(\mathbb{R}^d)$ module spanned by all Lie brackets of the X_j having less than or equal to k factors. Assume:

(1) $M \subset V$ is a C^∞ hypersurface.
(2) There exist k and a collection of vector fields $Z_{i,\alpha} \in \mathfrak{g}_k$, where $1 \leq i \leq d$ and α ranges over some finite index set, which spans the tangent space to $\mathbb{R}^d$ at each point of $V \backslash M$.
(3) At each point of M, at least one vector field X_j is transverse to M.
(4) The coefficient

$$\beta(x) = \sum_\alpha \left(\text{determinant}(Z_{i,\alpha})_{1 \leq i \leq d} \right)^2 (x) \tag{3.1}$$

degenerates weakly as $x \to M$ in the sense that

$$\lim_{V \ni x \to M} \text{distance}\,(x, M) \cdot |\log \beta(x)| = 0. \tag{3.2}$$

The theorem of Bell and Mohammed asserts that a slightly stronger version of these hypotheses implies hypoellipticity.[6]

Lemma 3.1. *Let $\{X_j\}$ satisfy the hypotheses enumerated above. Then for every relatively compact open subset $U \Subset V$ and each small $\delta > 0$ there exists $C_\delta < \infty$ such that for all $u \in C_0^\infty(U)$,*

$$\int \log^2 \langle \xi \rangle |\hat{u}(\xi)|^2 \, d\xi \leq \delta \sum_j \|X_j u\|^2 + C_\delta \|u\|^2. \tag{3.3}$$

Corollary 3.2. *If a collection $\{X_j\}$ of vector fields satisfies the hypotheses enumerated above, then $L = -\sum_j X_j^2$ is hypoelliptic.*

I emphasize that these first two motivations have retroactively proved spurious, in the sense that Theorem 2.1 was already known [25] to other authors.

A third motivation was the search for sufficient conditions for hypoellipticity of the $\bar{\partial}$–Neumann problem for smoothly bounded pseudoconvex domains of infinite type, a topic closely linked to the theory of sums of squares of real vector fields, but calling for generalizations involving complex vector fields, pseudodifferential operators, and systems, and

[5] The analogy between the hypotheses of Derridj and Zuily and those of Theorem (2.1) is imperfect; inequality (2.2) with $w(\xi) = \langle \xi \rangle^\varepsilon$ suffices for $G^{1/\varepsilon}$ hypoellipticity, while $w(\xi) = \log\langle \xi \rangle$ is not quite sufficient for C^∞ hypoellipticity. This has to do with the different natures of the toplogies for G^s and for C^∞, the former being a countable union of Banach spaces, the latter a countable intersection.

[6] In [2] it was assumed that $\beta(x) \geq \exp(-\,\text{distance}\,(x, M)^{-p})$ as $x \to M$, for some $p < 1$. Morimoto [25] had earlier established hypoellipticity under the sharp hypothesis distance $(x, M) \cdot |\log \beta(x)| \to 0$, in the special case $k = 1$ where all $Z_{i,\alpha}$ in hypothesis (3.1) belong to the span of the X_j.

thus perhaps not amenable to stochastic methods. The next result is a direct consequence of Theorem 2.3, Lemma 3.1, and the method of reduction to the boundary as in [6].[7]

Corollary 3.3. *Let Ω be a smoothly bounded pseudoconvex relatively compact open subset of $\mathbb{C}^2$. Suppose that near $x_0 \in \partial\Omega$, the set of weakly pseudoconvex points of $\partial\Omega$ is contained in a smooth hypersurface*[8] *M, which is everywhere transverse to $T^{1,0} \oplus T^{0,1}(\partial\Omega)$. Suppose also that the Levi form satisfies*

$$\lim_{\partial\Omega \ni x \to M} \text{distance}\,(x, M) \cdot \log \lambda(x) = 0.$$

Then the $\bar\partial$–Neumann problem for $(0,1)$ forms on Ω is hypoelliptic in a neighborhood of x_0.

Here the Levi form is identified with a real-valued function.

This result is derived from the following consequence of Theorem 2.1.

Corollary 3.4. *Let $\Omega \subset \mathbb{C}^2$ be a bounded open set with C^∞ boundary. Suppose that there exists a symbol $\mu \in S^1_{1,0}(\partial\Omega)$ which in any local coordinate system is real and nonnegative modulo addition of a symbol in $S^0_{1,0}$ and satisfies*

$$\mu(x,\xi)/\log\langle\xi\rangle \to \infty \quad \text{as } \langle\xi\rangle \to \infty,$$

such that for every $u \in C^1_0(\partial\Omega)$,

$$\|\operatorname{Op}(\mu)u\| \le C\|\bar\partial_b u\|_{L^2} + C\|\bar\partial_b^* u\| + C\|u\|_{L^2}$$

for some $C < \infty$. Then the $\bar\partial$–Neumann problem for Ω is C^∞ hypoelliptic.

J. J. Kohn has asked whether hypoellipticity always holds in this context, provided that the set of weakly pseudoconvex points consists of a smooth real curve that is everywhere transverse to the complex tangent space, but without any restriction on the rate at which the Levi form degenerates. In a forthcoming paper we will show that this is not the case, even though quite weak supplemental hypotheses can suffice to imply hypoellipticity.

A fourth motivation was the hope that a more penetrating study of C^∞ hypoellipticity might shed light on the more intrinsically interesting C^ω case. The present paper achieves for the C^∞ theory a rough parity with what is presently known, in the positive direction, about analytic hypoellipticity.[9]

We now formulate a few variants of the main results. The next theorem and corollary are rudimentary results, in which the symbols ψ required for the application of Theorems 2.3 and 2.4 may be constructed so that their gradients are supported in regions where L is elliptic, or at least is subelliptic. The commutator bounds (2.7) and (2.8) then hold with a great deal of room to spare.

[7] Our technique appears applicable to some extent in $\mathbb{C}^d$ for $d > 2$, but we have not carried out the details of the boundary reduction. The formulation of Corollary 3.3 would require certain modifications; for instance, for hypoellipticity on the level of $(0,1)$ forms, M should be assumed to have real dimension d, and to be totally real.

[8] M is a submanifold of $\partial\Omega$ of real codimension one.

[9] A C^ω analogue of Theorem 2.3 appears to be implicit in work of Grigis and Sjöstrand [16], although we have not yet verified this in detail.

Theorem 3.5. *Suppose that $L = -\sum_j X_j^2$ where the X_j are smooth real vector fields in some open set V, satisfying (2.2) for some $w \in C^\infty$ such that $w(\xi) \to \infty$ as $|\xi| \to \infty$. Suppose that the set of all points of V at which $\{X_j\}$ fails to satisfy the bracket hypothesis is totally disconnected. Then L is hypoelliptic in V. Moreover for any $s \in \mathbb{R}$, for any $u \in \mathcal{D}'(V)$, the H^s singular support of u is contained in the H^s singular support of Lu.*

If in addition the set of points of S^*V at which the microlocal bracket hypothesis fails to be satisfied is totally disconnected, then the H^s wave front set of u is contained in the H^s wave front set of Lu.

This implies a more precise formulation of Proposition 1.4.

Corollary 3.6. *Suppose that $L = -\sum_j X_j^2$ where the X_j are smooth real vector fields in some open set V. Suppose that at each point of V, at least one X_j is nonzero. Suppose that $\{X_j\}$ satisfies the bracket hypothesis at all but finitely many points of V. Then for any $s \in \mathbb{R}$, for any $u \in \mathcal{D}'(V)$, the H^s singular support of u is contained in the H^s singular support of Lu.*

If in addition the set of points of S^*V at which the microlocal bracket hypothesis fails to be satisfied is finite, then the H^s wave front set of u is contained in the H^s wave front set of Lu.

Our final theorem is one of various possible parabolic analogues. Let $(x,t) \in \mathbb{R}^d \times \mathbb{R}^1$ be coordinates in $\mathbb{R}^{d+1}$.

Theorem 3.7. *Let $\{X_j\}$ be a collection of vector fields in $\mathbb{R}^d$ satisfying the hypotheses of Theorem 2.1. Then $\partial_t - \sum_j X_j^2$ is hypoelliptic in $\mathbb{R}^{d+1}$.*

Throughout the discussion we assume L to satisfy a compactness inequality (2.2) with $w \to \infty$. It is likely that a variant of Theorem 2.3 may be formulated and proved, in which only an inequality $\|u\|_{H^{-m}} \le C\|Lu\|_{H^0}$ is assumed; a correspondingly stronger hypothesis depending on m must be imposed on ψ and the commutator operator to which it gives rise. Although examples are certainly known of hypoelliptic operators that satisfy only such weaker inequalities, we have not investigated the usefulness of such a generalization in the analysis of concrete examples.

It is not clear to this author to what extent the hypothesis $\psi \in S^0_{1,0}$ in Theorem 2.3 is natural. Perhaps a variant in which ψ is permitted to belong to a less restricted symbol class might also be useful in this context. Such variants appear for instance in [19],[25],[26].

For remarks and speculation concerning conditions on symbols characterizing hypoellipticity, see [34] and [10].

4. Proofs of Theorems 2.1 and 2.3

By a pseudodifferential operator we will always mean an operator of the form

$$\mathrm{Op}(a)f = \int_{\mathbb{R}^d} e^{ix\xi}\, a(x,\xi)\hat{f}(\xi)\, d\xi,$$

associated to the symbol a via the Kohn-Nirenberg calculus. These operators act on compactly supported distributions. We do not always assume them to be properly supported in the sense that $a(x,\xi)\equiv 0$ for x outside a compact subset of $\mathbb{R}^d$. However, pseudolocality ensures that if $U_1 \Subset U_2 \subset \mathbb{R}^d$ are open, and if A,B are pseudodifferential operators whose symbols each belong to one of the standard symbol classes $S^m_{\rho,\delta}$ with $\rho>0$, and if $\eta\in C_0^\infty(\mathbb{R}^d)$ is $\equiv 1$ in a neighborhood of the closure of U_2, then as an operator from $\mathcal{E}'(U_1)$ to $\mathcal{D}'(U_2)$, $A\circ\eta\circ B$ is independent of η, modulo an operator mapping $\mathcal{E}'(U_1)$ to $C^\infty(U_2)$. Consequently when analyzing hypoellipticity in an open set V we will sometimes write $A\circ B$ to mean $A\circ\eta\circ B$, where η is a cutoff function that is identically equal to one in a neighborhood of $\overline{V}$.

Denote by $\sigma(P)$ a symbol of a pseudodifferential operator P. We say that an operator is smoothing of order M in the Sobolev scale if it maps H^s to H^{s-M} for all $s\in\mathbb{R}$. A natural setting[10] for much of our reasoning will be the classes $S^{m,n}\subset S^{m+}_{1,0}$, which by definition consist of all $a\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ satisfying

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}[\log\langle\xi\rangle]^{n+|\alpha|+|\beta|}$$

for all $\alpha,\beta,(x,\xi)$. Certain manipulations performed below without comment are justified by well known basic properties of the operators associated to symbols in the latter class, such as the symbolic calculus and pseudolocality [32].

We begin with the proof of Theorem 2.1, then will later indicate how it should be modified to derive the more general Theorem 2.3. Let $u\in\mathcal{D}'(V)$ and $s\in\mathbb{R}$ be given. Suppose the H^s wave front set of Lu to be disjoint from some conic open neighborhood Γ_0 of $(x_0,\xi_0)\in T^*V$. Without loss of generality we may assume $u\in\mathcal{E}'(V)$. Fix K such that $u\in H^{-K}$. Given any $s\in\mathbb{R}$, our aim is to show that $(x_0,\xi_0)\notin WF_{H^s}(u)$ by constructing a pseudodifferential operator Λ which is elliptic of order s in a conic neighborhood of (x_0,ξ_0), and for which it can be shown that $\Lambda u\in H^0(\mathbb{R}^d)$.

To do this fix a small conic open neighborhood Γ_1 of (x_0,ξ_0) whose intersection with the unit cosphere bundle is a compact subset of Γ_0. Fix an auxiliary function $\phi(x,\xi)\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d\backslash\{0\})$ that is homogeneous of degree zero with respect to ξ, is everywhere nonnegative, vanishes identically in a small conic neighborhood of (x_0,ξ_0), and is strictly positive on the complement of Γ_1.

Define a symbol of nonconstant order, depending on parameters s,N_0, by

$$\lambda(x,\xi)=|\xi|^s e^{-N_0\log|\xi|\phi(x,\xi)} \tag{4.1}$$

where $|\xi|\ge e$, and define λ for $|\xi|<e$ so as to be C^∞ and nowhere vanishing. The nonnegativity of ϕ implies that $\lambda\in S^{s+}_{1,0}$. Moreover, $\lambda\in S^{s,0}$.

With s fixed, there exists $\delta>0$ such that for each N_0, $\lambda\in S^{-\delta N_0+}_{1,0}$ on the closure of the complement of Γ_1. Choose N_0 so that $-\delta N_0<-K$. Then $\Lambda u\in H^{-K+\delta N_0}\subset H^0$ microlocally on the complement of Γ_1.

Now L acts on sections of some bundle, and each X_j maps sections of that bundle to sections of some bundle W_j, whose rank may well depend on j. Restricting the analysis to a small open subset of $\mathbb{R}^d$, we may assume all these bundles to be trivial, and we fix

[10] Instead of the symbol class $S^{m,n}$ and Kohn-Nirenberg quantization one could employ the Weyl calculus of Hörmander [18], associated to the metric $g=\log^2\langle\xi\rangle\,dx^2+\log^2\langle\xi\rangle\langle\xi\rangle^{-2}\,d\xi^2$.

trivializations of them. Define $\Lambda = \mathrm{Op}(\lambda \cdot I)$ where I denotes the identity matrix. Likewise define $\Lambda_j = \mathrm{Op}(\lambda \cdot I_j)$, acting on sections of W_j, where I_j is an identity matrix whose dimension equals the rank of W_j.

Fix cutoff functions $\eta_k \in C_0^\infty(\mathbb{R}^d)$ such that $\eta_2 \equiv 1$ in a neighborhood of the support of u, $\eta_1 \equiv 1$ in a neighborhood of the support of η_2, and η_1 is supported in V.

Recall that if a, b are symbols in some classes $S^m_{\rho,\delta}$ and $S^n_{\rho,\delta}$, if b is properly supported, and if $\rho > \delta$, then $\mathrm{Op}(a) \circ \mathrm{Op}(b)$ has a symbol $a \odot b$ with an asymptotic expansion

$$a \odot b(x,\xi) \sim \sum_\alpha c_\alpha \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi) \tag{4.2}$$

where $c_\alpha = (\alpha!)^{-1}(-i)^\alpha$. The notation $\sim$ indicates convergence in the usual asymptotic sense: for any positive integer N, the difference between $\mathrm{Op}(a) \circ \mathrm{Op}(b)$ and an operator associated to the symbol $\sum_{|\alpha|<N} c_\alpha \partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi)$ is smoothing of order $m+n-N(\rho-\delta)$ in the scale of Sobolev spaces.

Lemma 4.1. *There exists an operator, denoted Λ^{-1}, belonging to $S^{m+}_{1,0}$ for some $m = m(s)$, such that $\Lambda \circ \Lambda^{-1} - I$ is smoothing of infinite order in the Sobolev scale. Moreover such an operator may be constructed with a symbol of the form $(1+f)\lambda^{-1} \cdot I$, where $f \in S^{-1,2}$. Likewise there exist operators Λ_j^{-1} of the same form $(1+f_j)\lambda^{-1} \cdot I_j$, such that each $\Lambda_j \circ \Lambda_j^{-1} - I_j$ is smoothing of infinite order.*

Proof. Write $f \sim \sum_{k \ge 1} f_k$. Solve the equation $\lambda \odot ((1+f)\lambda^{-1}) \sim 1$, using (4.2) and the usual iterative procedure. One obtains $f_1 \in S^{-1,2}$, and by induction, each $f_k \in S^{-k+}_{1,0}$. Choose Λ to be an operator whose full symbol has asymptotic expansion $\sum_{k\ge1} f_k$; such operators exist [31], [32]. The same method of construction yields Λ_j. □

Lemma 4.2. *Suppose that L takes the form (2.1). Then there exists a pseudodifferential operator G of the form*

$$G = \sum_j B_j \circ X_j + \sum_j X_j^* \circ \tilde{B}_j + B_0 \tag{4.3}$$

where $B_0 \in \mathrm{Op}(S^{0,2})$ and $B_j, \tilde{B}_j \in \mathrm{Op}(S^{0,1})$ for each $j \ge 1$, such that

$$(L+G)\eta_1 \Lambda \eta_2 = \eta_1 \Lambda L \eta_2 + R \tag{4.4}$$

for some R belonging to $\mathrm{Op}(S^{-M+}_{1,0})$ for every $M < \infty$.

Proof of Lemma 4.2. Throughout the argument, R denotes an operator belonging to $S^{-M+}_{1,0}$ for all M, which may change from one line to the next. In constructing the symbol of G we work formally, ignoring the cutoff functions η_j; this is permissible by pseudolocality, because $\eta_2\eta_1 \equiv \eta_2$. The desired equation $(L+G)\Lambda = \Lambda L + R$ is then equivalent to

$$\begin{aligned} G = \Lambda L \Lambda^{-1} - L + R\Lambda^{-1} &= \sum_j \left[\Lambda X_j^* X_j \Lambda^{-1} - X_j^* X_j\right] + R\Lambda^{-1} \\ &= \sum_j \left[(\Lambda X_j^* \Lambda_j^{-1})(\Lambda_j X_j \Lambda^{-1}) - X_j^* X_j\right] + R\Lambda^{-1}. \end{aligned}$$

Consider $\Lambda_j X_j \Lambda^{-1} = X_j + (\Lambda_j X_j - X_j\Lambda)\Lambda^{-1} + R$. The symbol of $\Lambda_j X_j - X_j \Lambda$ divided by λ equals the Poisson bracket $\{\log\lambda,\, \sigma(X_j)\}$ plus an element of $S^{-1,2}$, as follows from (4.2). Applying the composition formula (4.2) once more we conclude

$$\Lambda_j X_j \Lambda^{-1} = X_j + \mathrm{Op}(\{\log\lambda,\, \sigma(X_j)\}) \qquad \text{modulo } \mathrm{Op}(S^{-1+}_{1,0}). \tag{4.5}$$

A corresponding assertion holds for $\Lambda X_j^* \Lambda_j^{-1}$, with $\sigma(X_j^*)$ substituted for $\sigma(X_j)$ in the Poisson bracket. Since $\{\log\lambda, \sigma(X_j)\}$ and $\{\log\lambda,\, \sigma(X_j^*)\}$ belong to $S^{0,1}$, inserting these equations into the identity derived for G in the preceding paragraph completes the proof. □

Lemma 4.3. *Let G be a pseudodifferential operator of the form* (4.3)*. Then for any fixed relatively compact subset $U \subset V$, any $\delta > 0$ and any $f \in C_0^\infty$ supported in U,*

$$|\langle Gf,\, f\rangle| \le C_\delta \int \log^2\langle\xi\rangle\, |\hat f(\xi)|^2\, d\xi + \delta \sum_j \|X_j f\|^2_{L^2}. \tag{4.6}$$

This lemma follows from Gårding's inequality (and its proof, adapted to $\mathrm{Op}(S^{1+}_{1,0})$). One uses the nonnegativity of $C\log^2\langle\xi\rangle$ minus the symbol of B_0, provided that C is chosen to be sufficiently large, along with the inclusions $B_j, B_j^* \in S^{0,1}$. The detailed verification is left to the reader.

Lemma 4.4. *Let L take the form (2.1) and satisfy (2.4). Let $s, M \in \mathbb{R}$ be fixed. If N_0 is chosen to be sufficiently large in the definition of Λ, then for any fixed relatively compact subset $U \Subset V$ and any $u \in C^{s+3}(U)$,*

$$\|\eta_1 \Lambda u\|_{L^2} \le C\|\eta_1\Lambda L u\|_{L^2} + C\|u\|_{H^{-M}}. \tag{4.7}$$

All norms without subscripts in the following argument are $L^2 = H^0$ norms.

Proof. We have $(L+G)\eta_1\Lambda\eta_2 u = \eta_1\Lambda L\eta_2 u$, and $\eta_2 u \equiv u$. Thus setting $v = \eta_1\Lambda u \in C^2$,

$$\big\langle (L+G)v,\, v\big\rangle = \sum_j \|X_j v\|^2 + \|v\|^2\big) + O\big(\|v\|\cdot\|Gv\|\big).$$

Invoking Lemma 4.3 we find that

$$\begin{aligned}\sum_j \|X_j v\|^2 &\le \|\eta_1\Lambda L u\|^2 + \|Ru\|^2 + C\|v\|^2 + C\|Gv\|^2 \\ &\le \|\eta_1\Lambda L u\|^2 + \|Ru\|^2 + C\|v\|^2 + C_\delta \int \log^2\langle\xi\rangle\, |\hat v(\xi)|^2\, d\xi + \delta\sum_j \|X_j v\|^2.\end{aligned}$$

The error term R maps H^{-M} to L^2, and $\|v\|^2$ is majorized by $\int \log^2\langle\xi\rangle |\hat v|^2 d\xi$. Choosing $\delta < 1$, the last term on the right hand side may be absorbed into the left, yielding

$$\sum_j \|X_j v\|^2 \le C\int \log^2\langle\xi\rangle\, |\hat v(\xi)|^2\, d\xi + \|\eta_1\Lambda L u\|^2 + C\|u\|^2_{H^{-M}}.$$

We finally invoke the hypothesis (2.4) in the form

$$\sum_j \|X_j v\|^2 \geq A \int \log^2\langle\xi\rangle\, |\hat{v}(\xi)|^2\, d\xi - C_A\|v\|^2$$

for arbitrarily large A to deduce that

$$\int \log^2\langle\xi\rangle\, |\hat{v}(\xi)|^2\, d\xi \leq C\|\eta_1\Lambda L u\|^2 + C\|u\|^2_{H^{-M}} + C\|v\|^2$$

for some constant C. $\|v\|$ may be majorized by an arbitrarily small constant times the left hand side plus a large constant times the H^{-M-s} norm of v. $\eta_1\Lambda\eta_2$ maps H^{-M} boundedly to H^{-M-s}, so $\|v\|_{H^{-M-s}} \leq C\|u\|_{H^{-M}}$. Thus

$$\|v\|^2 \leq \int \log^2\langle\xi\rangle|\hat{u}(\xi)|^2\, d\xi \leq C\|\eta_1\Lambda L u\|^2 + C\|u\|^2_{H^{-M}}.$$

□

Proof of Theorem 2.1. It remains to remove the smoothness assumption on u and to convert the *a priori* estimate to the desired conclusion $\Lambda u \in H^0$. To accomplish this fix an auxiliary function $r \in C^\infty(\mathbb{R}^d)$ that is strictly positive, and satisfies $r(\xi) \equiv |\xi|^{-1}$ for all $|\xi| \geq 2$, and $r(\xi) \equiv 1$ for all $|\xi| \leq 1$. Fix a large exponent q. For all small $\varepsilon > 0$ define a mollified symbol

$$\lambda_\varepsilon(x,\xi) = r^q(\varepsilon\xi)\cdot\lambda(x,\xi) \tag{4.8}$$

where λ is as defined in (4.1). and let $\Lambda_\varepsilon = \mathrm{Op}(\lambda_\varepsilon)$. The symbols $r_\varepsilon = r^q(\varepsilon\xi)$ satisfy

$$|\partial_\xi^\alpha r_\varepsilon / r_\varepsilon| \leq C_{\alpha,q}|\xi|^{-|\alpha|}, \quad \text{uniformly in } \varepsilon \text{ and } \xi \in \mathbb{R}^d. \tag{4.9}$$

If q is chosen to be sufficiently large relative to the order of the distribution u, then $\Lambda_\varepsilon u \in C^2$ for all $\varepsilon > 0$, and because λ is elliptic of order s in a conic neighborhood of (x_0, ξ_0), it suffices to show that the L^2 norm of $\eta_1\Lambda_\varepsilon u$ remains uniformly bounded as $\varepsilon \to 0$. But Lemma 4.4 fails to apply directly, because u is merely known to be a distribution, not a function in C^{s+3} as hypothesized.

The parameter N_0 in (4.8) may be chosen sufficiently large that $\eta_1\Lambda L u \in L^2$, because ϕ is strictly positive in a conic neighborhood of the H^s wave front set of u, and hence Λ is regularizing there of order at least $s - \delta N_0$ for some constant $\delta > 0$. The L^2 norm of $\eta_1\Lambda_\varepsilon L u$ is bounded uniformly in ε and tends to the L^2 norm of $\eta_1\Lambda L u$.

As in the proof of Lemma 4.4 we have for each $\varepsilon > 0$ an operator G_ε and an identity $(L + G_\varepsilon)\eta_1\Lambda_\varepsilon u = \eta_1\Lambda_\varepsilon L u + Ru$ with both sides of the equation in C^2 for each $\varepsilon > 0$. Moreover (4.9) ensures that the proof of Lemma 4.2 carries through for each $\varepsilon > 0$ with Λ replaced by Λ_ε, so that G_ε takes the form (4.3), with each pseudodifferential coefficient B_j in the class indicated in Lemma 4.2, and with all bounds uniform in ε. All functions have sufficient differentiability for the proof of Lemma 4.4 and this identity to yield

$$\|\eta_1\Lambda_\varepsilon u\|_{L^2} \leq C\|\eta_1\Lambda_\varepsilon L u\|_{L^2} + C\|u\|_{H^{-M}},$$

uniformly as $\varepsilon \to 0$. Thus the L^2 norm of $\eta_1\Lambda_\varepsilon u$ remains bounded as $\varepsilon \to 0$. □

The proof of Lemma 4.2 yields the following more precise conclusion, which will be used to prove Theorem 2.3.

Lemma 4.5. *Let Λ be any pseudodifferential operator whose symbol λ takes the general form $\lambda(x,\xi) = |\xi|^s \exp\big(-N\log|\xi|\psi(x,\xi)\big)$ for large $|\xi|$ where $\psi \in S^0_{1,0}$. Suppose that L takes the form* (2.1)*. Define $b_j = \operatorname{Op}(\{\log\lambda, \sigma(X_j^*)\})$ and $\tilde b_j = \operatorname{Op}(\{\log\lambda, \sigma(X_j)\})$. Then there exists a pseudodifferential operator G of the form* (4.3) *satisfying* (4.4)*, with*

$$\begin{aligned} B_j &= b_j + c_j \quad \textit{and} \quad \tilde B_j = \tilde b_j + \tilde c_j \quad \textit{for every } j \geq 1 \\ B_0 &= \sum_j \big(b_j \circ \tilde b_j + A_j \tilde b_j + \tilde A_j b_j\big) \ \textit{modulo} \ \operatorname{Op}(S^{-1,2}) \end{aligned} \tag{4.10}$$

where each $c_j, \tilde c_j \in \operatorname{Op}(S^{-1,1})$, and where $A_j, \tilde A_j \in S^0_{1,0}$ are the coefficients in (2.1)*.*

Lemma 4.6. *Suppose that L, ψ, p satisfy the hypotheses of Theorem 2.3, and that*

$$\lambda(x,\xi) = |\xi|^s \exp(-N\log\langle\xi\rangle\psi(x,\xi)) \qquad \textit{for large } |\xi|. \tag{4.11}$$

Then for any $N \geq 0$ and for any fixed relatively compact subset $U \subset V$, any $\delta > 0$ and any $f \in C^{s+3}$ supported in U, the operator G constructed in Lemma 4.2 satisfies

$$|\langle Gf,\, f\rangle| \leq \delta \sum_j \|X_j f\|^2 + C_\delta \|f\|^2 + C_\delta \|\operatorname{Op}(p) f\|^2_{H^1}. \tag{4.12}$$

This follows directly from (4.10) and the hypothesis (2.7).

Lemma 4.7. *Let L, ψ, p satisfy the hypotheses of Theorem 2.3, and let λ be defined by* (4.11)*. Let s, M be fixed. If N is chosen to be sufficiently large in the definition of λ then for any fixed relatively compact subset $U \subset V$ and any $u \in C^\infty_0(U)$,*

$$\|\eta_1 \Lambda u\|_{L^2} \leq C\|\eta_1 \Lambda L u\|_{L^2} + C\|u\|_{H^{-M}}.$$

This follows from the same reasoning as in the proof of Lemma 4.4, using the hypotheses on ψ and Lemma 4.6. The smoothness assumption can be removed, and the a priori estimate converted to the conclusion $\eta_1 \Lambda u \in L^2$, as above. The term involving the H^1 norm of $\operatorname{Op}(p)\Lambda u$ may be absorbed into $\|u\|_{-M}$, because Λ may be made to be regularizing of arbitrarily high order in a conic neighborhood of the support of the symbol p, by choosing N to be sufficiently large. This completes the proof of Theorem 2.3. Theorem 2.4 is proved in the same way.

5. Analysis of examples

The purpose of this section is to analyze the three classes of examples L_j discussed in the introduction. It will be convenient to assume that $a(x) \to \infty$ as $|x| \to \infty$; this has no effect on the question of hypoellipticity. Fix a and define $\mathcal{L} = \mathcal{L}_\tau = -\partial_x^2 + \tau^2 a^2(x)$. Define $\lambda_0 = \lambda_0(\tau)$ to the infimum of $\langle \mathcal{L} f,\, f\rangle^{1/2}/\|f\|_{L^2}$ over all $0 \neq f \in C^2_0(\mathbb{R})$. The notation $A \sim B$ will mean that A/B is bounded above and below by positive constants, independent of τ as $\tau \to +\infty$.

Lemma 5.1.

$$\lambda_0 \sim \min_{0<y<\infty} \left(y^{-1} + \tau a(y)\right). \tag{5.1}$$

Proof. To obtain the upper bound for λ_0 it suffices to fix a test function $\varphi \in C_0^1(\mathbb{R})$ supported in $\{|x| < 1\}$, and to consider $\varphi(x/y)$ for arbitrary y. The conclusion is that $\lambda_0 \le C(y^{-1} + \max_{|x|\le y} \tau a(x))$, which is the required upper bound because of the monotonicity of a.

For a lower bound, consider any $y > 0$. For any $\varphi \in C_0^2$, $\langle \mathcal{L}\varphi,\, \varphi\rangle = \|\partial_x \varphi\|^2 + \int \tau^2 a^2(x)\, \varphi^2(x)\, dx$. Now

$$\int_{|x|\ge y} \tau^2 a^2 \varphi^2\, dx \ge \tau^2 [\min_{|x|\ge y} a^2(x)] \int_{|x|\ge y} \varphi^2\, dx.$$

Also

$$\int_{|x|<y} \varphi^2\, dx \le C \int_{y<|x|<2y} \varphi^2\, dx + Cy^2 \|\partial_x \varphi\|^2.$$

Thus

$$\|\varphi\|^2 \le C\big(\tau^{-2}[\min_{|x|\ge y} a(x)]^{-2} + y^2\big)\langle \mathcal{L}\varphi,\, \varphi\rangle \qquad \text{for any } y > 0. \tag{5.2}$$

We now invoke the monotonicity of a to choose the unique y satisfying $y^{-1} = \tau a(y)$. Then y^{-1} is comparable to the minimum in (5.1), since the functions x^{-1} and $\tau a(x)$ are respectively nonincreasing and nondecreasing on $\mathbb{R}^+$. Thus $(y^2 + \tau^{-2} a(y)^{-2}) \le C(y^{-1} + \tau a(y))^{-2}$, as desired. $\square$

Lemma 5.2. *Assume that a is even and nonnegative, vanishes only where $x = 0$, is nondecreasing on $[0, \infty)$ and that $a(x) \to \infty$ as $x \to +\infty$. Then*

$$\lim_{\tau\to\infty} \frac{\lambda_0(\tau)}{\log \tau} = \infty \Longleftrightarrow \lim_{x\to 0} x \log a(x) = 0.$$

Proof. Given any large $\tau \in \mathbb{R}^+$, define $y \in \mathbb{R}^+$ by $y^{-1} = \tau a(y)$. Then $\lambda_0(\tau) \sim y^{-1}$, as observed in the proof of Lemma 5.1. Thus

$$\log \tau / \lambda_0(\tau) \sim y \log \tau = y |\log[y a(y)]| \sim y \log a(y).$$

Thus if $x \log a(x) \to 0$ as $x \to 0$, then $\lambda_0(\tau)/\log \tau \to \infty$ as $\tau \to \infty$.

Conversely if a small $x > 0$ is given, define $\tau = x^{-1} a(x)^{-1}$. Then $\tau \to \infty$ as $x \to 0$, and again $\lambda_0(\tau) \sim x^{-1}$, so

$$\lambda_0(\tau)/\log \tau \sim x^{-1}/\log(x^{-1} a(x)^{-1}) \sim |x \log a(x)|^{-1}.$$

The converse implication follows directly. $\square$

In the next lemma we omit the hypothesis that a is nondecreasing.

Lemma 5.3. *If $x \log a(x) \to 0$ as $x \to 0$ and $a(x) \to \infty$ as $|x| \to \infty$, then $\lambda_0(\tau)/\log \tau \to \infty$ as $\tau \to \infty$.*

Proof. The proof of (5.2) did not utilize the hypothesis that a was nondecreasing. Given merely that $\lim_{x\to 0} x \log a(x) = 0$, then for each $\delta > 0$ there exists $c_\delta > 0$ such that $a(x) \ge c_\delta \exp(-\delta/2|x|)$ as $x \to 0$. When $y = \delta/\log\tau$, the factor on the right in (5.2) becomes

$$y + \tau^{-1}[\min_{|x|\ge y} a(x)]^{-1} \le \delta/\log\tau + C_\delta \tau^{-1}\exp(\delta\log\tau/2\delta) \le C\delta/\log\tau$$

for all sufficiently large τ, with C' independent of δ, τ. Given any τ so large, applying (5.2) with these choices of δ and of y yields $\lambda_0(\tau) \ge c\delta^{-1}\log\tau$. $\square$

Lemma 5.4. *For any coefficient $a \in C^\infty$, if L_2 is hypoelliptic in some neighborhood of 0, then $\lambda_0(\tau)/\log\tau \to \infty$ as $\tau \to +\infty$.*

Proof. The coefficient $a(x)$ may be assumed to tend to $+\infty$ as $|x| \to \infty$. Then each operator $\mathcal{L}_\tau$ is an essentially self-adjoint operator on $L^2(\mathbb{R})$, with a discrete spectrum tending to $+\infty$. Its lowest eigenvalue is $\lambda_0^2(\tau)$. There exists an associated eigenfunction f_τ that is strictly positive everywhere, is even, and assumes its maximum value at $x = 0$. Normalize it so that $f_\tau(0) = 1$. Define a solution to $LF_\tau \equiv 0$ by

$$F_\tau(x, y, t) = e^{i\tau t} e^{\lambda_0(\tau) y} f_\tau(x).$$

Suppose L_2 to be hypoelliptic in some small bounded neighborhood V of 0. Then by the Baire category theorem, for each positive integer k there must exist $C < \infty$ such that for every $F \in C^0(V)$ satisfying $L_2 F \equiv 0$ in V,

$$|\partial_t^k F(0)| \le C\|F\|_{L^\infty(V)}. \tag{5.3}$$

Considering any large $\tau \in \mathbb{R}^+$ and taking $F = F_\tau$, the right hand side of (5.3) is bounded by $C\exp(\lambda_0(\tau))$ since $|f_\tau| \le 1$. The left hand side equals $|\tau^k f_\tau(0)| = \tau^k$. Therefore for any k there must exist $C_k < \infty$ such that $\tau^k \le C_k e^{\lambda_0(\tau)}$. Thus $\lambda_0(\tau)/\log\tau \ge k - \log C_k/\log\tau$, whence $\liminf_{\tau\to\infty}(\lambda_0(\tau)/\log\tau) \ge k$. $\square$

This argument is parallel to that used by many authors to disprove analytic or Gevrey class hypoellipticity for various classes of examples.

Proof of Proposition 1.2. Combining Lemmas 5.2 and 5.4, we find that if L_2 is hypoelliptic then $x\log a(x)$ tends to 0 as $x \to 0$.

Let $(x, t; \xi, \tau)$ be coordinates in $T^*\mathbb{R}^2$. If $x\log a(x) \to 0$, then $\lambda_0(\tau)/\log\tau \to \infty$. To relate this information to $\iint \log^2\langle(\xi,\tau)\rangle |\hat{u}(\xi,\tau)|^2\,d\xi d\tau$, observe that L_2 is elliptic except where $\xi \ne 0$. Therfore by a microlocalization it suffices to majorize the integral over the region $|\tau| \ge |\xi|$. Replacing $\log\langle(\xi,\tau)\rangle$ by $\log\langle\tau\rangle$,

$$\iint \log^2\langle\tau\rangle |\hat{u}(\xi,\tau)|^2\,d\xi d\tau = c\iint \log^2\langle\tau\rangle |\tilde{u}(x,\tau)|^2\,dx d\tau$$

where $\tilde{u}$ denotes the partial Fourier transform with respect to the second coordinate. For each τ,

$$\log^2\langle\tau\rangle \int |\tilde{u}(x,\tau)|^2\,dx \le \log^2\langle\tau\rangle \lambda_0(\tau)^{-2} \int |(X_1 u(x,\tau))\tilde{}\,|^2 + |(X_2 u(x,\tau))\tilde{}\,|^2\,dx.$$

Because $\log \tau/\lambda_0(\tau) \to 0$, this leads to the superlogarithmic inequality (2.4) by splitting the analysis into two cases $|\tau| \ge A$ and $|\tau| < A$, and choosing $A = A(\delta)$ to tend to ∞ sufficiently rapidly as $\delta \to 0$.

Lastly (2.4) implies that L_2 is hypoelliptic, by the general Theorem 2.1. □

The reasoning used above to estimate $\lambda_0(\tau)$ can be modified to prove that if $a \in C^\infty$ vanishes so rapidly at 0 that for any $A < \infty$ there exists C_A such that $|a(x)| \le C_A \exp(-A|x|^{-1})$ as $x \to 0$, then $\lambda_0(\tau)/\log\tau \to \infty$, and consequently L_2 fails to be hypoelliptic.

Proof of Proposition 1.1. Write $L_1 = -\partial_{x_1}^2 - a^2(x_1)\partial_{x_2}^2$ in coordinates $x = (x_1, x_2)$, and let $\xi = (\xi_1, \xi_2)$ be dual coordinates. Set $X_1 = \partial_{x_1}$, $X_2 = a(x_1)\partial_{x_2}$. Let $R = \{(x,\xi) : x = 0,\ \xi_1 = 0,\ \xi_2 > 0\}$. The principal symbol of L_1 vanishes precisely on the symplectic manifold where $x_1 = \xi_1 = 0$. Since L_1 is invariant under translation with respect to x_2, and reflection about the origin, it suffices to prove that $Lu \in H^s(R) \Rightarrow u \in H^s(R)$.

In order to apply Theorem 2.3 we set $p \equiv 0$ and seek to construct ψ having favorable commutation properties. Given any number $\rho > 0$, there exists $\psi \in C^\infty(T^*V)$ that is homogeneous of degree zero with respect to ξ, is $\equiv 1$ where $|(x_1, x_2, \xi_1/\xi_2)| \ge 3\rho$, is $\equiv 0$ where $|(x_1, x_2, \xi_1/\xi_2)| \le \rho$, and depends only on x_2 where $|(x_1, \xi_1/\xi_2)| \le 2\rho$. In order to apply Theorem 2.3, we must verify that $\mathrm{Op}[\log\langle\xi\rangle\{\psi, \sigma(X_i)\}]$ is controlled by X_1, X_2 in the sense (2.7). Microlocally where $(x_1, \xi_1) \ne 0$, $L = X_1^2 + X_2^2$ is elliptic, so we have better control than is needed. Therefore it suffices to work microlocally where $|(x_1, \xi_1/\xi_2)| < \rho$.

In this region $\{\psi, \sigma(X_1)\} \equiv 0$, while $\{\psi, \sigma(X_2)\}(x,\xi) \equiv ia(x_1)\partial\psi/\partial x_2$. Since $\partial\psi/\partial x_2 \in S^0_{1,0}$ and because of the microlocalization to $|\xi_1| \ll \xi_2$, it suffices to verify that the quantity $\|\mathrm{Op}[\log\langle\xi_2\rangle a(x_1)]u\|$ is majorized by $\delta\|X_2 u\| + C_\delta\|u\|$, for all u supported where $|x| < 3\rho$. This majorization follows from an application of the partial Fourier transform with respect to x_2, since $\log\langle\xi_2\rangle/\xi_2 \to 0$ as $\xi_2 \to +\infty$. □

Proof of Proposition 1.3. Let $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$ and $L_3 = -\partial_{x_1}^2 - b^2(x_1)\partial_{x_2}^2 - a^2(x_1)\partial_{x_3}^2 = -X_1^2 - X_2^2 - X_3^2$. Now the principal symbol of L_3 vanishes only where $x_1 = \xi_1 = 0$. As in the proof of Proposition 1.1, it suffices to work near where $x_1 = x_2 = \xi_1 = 0$. Again let $p \equiv 0$. Given any small number $\rho > 0$ we may construct ψ so as to have all properties required of it in Theorem 2.3, and in addition to be independent of x_1, ξ_1 where $|(x_1, \xi_1/|\xi|)| < \rho$. Microlocalizing to a small conic neighborhood of the nonelliptic region for L_3 as in the proof of Proposition 1.1, we have $\{\psi, \sigma(X_1)\} \equiv 0$, $\{\psi, \sigma(X_2)\} \equiv ib(x_1)\partial\psi/\partial x_2$, and $\{\psi, \sigma(X_3)\} \equiv ia(x_1)\partial\psi/\partial x_3$. Because $b \ge a$, the analysis is straightforward where $|\xi_2| \ge |\xi_3|$, using the partial Fourier transform with respect to (x_2, x_3).

Where $|\xi_3| \ge |\xi_2|$, we must majorize in terms of $\partial\tilde{u}/\partial x_1$ and $a(x_1)\xi_3\tilde{u}(x_1, \xi_2, \xi_3)$ the $L^2(dx_1)$ norm of the partial Fourier transform $\log\langle\xi_3\rangle b(x_1)\tilde{u}(x_1, \xi_2, \xi_3)$; the contribution of $X_2 u$ is of no use since ξ_2 could vanish.

It now suffices to show that under the hypothesis $b(x)x\log a(x) \to 0$, for all $\tau \in \mathbb{R}^+$

$$\log\tau\|b\varphi\| \le \varepsilon(\tau)\|\varphi'\| + \varepsilon(\tau)\|a\tau\varphi\| \qquad \text{for all } \varphi \in C_0^1(\mathbb{R}^1), \tag{5.4}$$

where $\varepsilon(\tau) \to 0$ as $\tau \to +\infty$ and each norm is that of $L^2(\mathbb{R}^1)$.

For the remainder of the discussion, x denotes an element of $\mathbb{R}^1$. In the course of the proof of Lemma 5.1 it was shown that for any smooth even coefficient a vanishing only at $x = 0$ and nondecreasing on $\mathbb{R}^+$,

$$\|\partial_x u\|^2 + \|\tau a \cdot u\|^2 \geq c \int \tau^2 \max(a^2(x), a^2(r))|u^2(x)|\,dx \qquad \text{for all } u \in C_0^1(\mathbb{R}),$$

where $r = r(\tau)$ is chosen so that $r^{-1} = \tau a(r)$. Therefore (5.4) would be a consequence of the majorization

$$b(x)\log\tau \leq \varepsilon(\tau)\tau\max(a(x), a(r(\tau))) \tag{5.5}$$

uniformly in x, where $\varepsilon(\tau) \to 0$ as $\tau \to \infty$.

The definition of r implies that $\log\tau = \log r^{-1} + \log a^{-1}(r) \sim \log a^{-1}(r)$ and $\tau a(r) = r^{-1}$, so that when $x = r$, $[b(r)\log\tau]/[\tau a(r)] \sim b(r)r|\log a(r)| = \varepsilon(\tau)$, which indeed tends to zero as $\tau \to \infty$ by hypothesis. Since b is nondecreasing on $\mathbb{R}^+$, this implies that $b(x)\log\tau \leq \varepsilon(\tau)\tau a(r)$ for all $|x| \leq r$. Write $\delta(x) = b(x)x|\log a(x)|$. Then for $x > r$,

$$\frac{b(x)\log\tau}{\tau a(x)} \leq \frac{\delta(x)\log\tau}{x|\log a(x)|\tau a(x)} \sim \delta(x)\cdot\frac{ra(r)|\log a(r)|}{xa(x)|\log a(x)|} \leq C\delta(x)\frac{r}{x}.$$

If $x \geq r^{1/2}$ this is bounded by $Cr^{1/2}$, which tends to zero as $\tau \to \infty$. If $x \leq r^{1/2}$ it is bounded by $\min_{0<t<r^{1/2}}\delta(t)$, which also tends to zero as $\tau \to \infty$. Thus (5.5) is indeed valid.

That hypoellipticity of L_3 implies (1.2) when b vanishes to finite order follows from reasoning similar to that underlying Lemmas 5.1 and 5.2. The details are left to the reader. ∎

The above analysis shows that the nonhypoelliptic example $L_2 = -\partial_x^2 - a^2(x)\partial_t^2 - \partial_y^2$, with coefficient $a(x) = \exp(-1/|x|)$, does satisfy the inequality (2.2) with the borderline weight $w(\xi) = \log\langle\xi\rangle$.

6. Proofs of Theorem 3.5 and Lemma 3.1

Proof of Lemma 3.1 . Since the bracket hypothesis is satisfied on the complement of M, it is no loss of generality to assume U to be a small neighborhood of a point $x_0 \in M$, and to assume that $\beta(x) \neq 0$ for all $x \in \overline{U}\backslash M$.

Let $A = (I - \Delta)^{1/2}$. Fix U and consider any $u \in C_0^\infty(U)$. It is known [32] that for any k and any $Z \in \mathfrak{g}_k$,

$$\|ZA^{-1+2^{-k}}u\|^2 \leq C\sum\|X_j u\|^2 + C\|u\|^2. \tag{6.1}$$

This is a key step in one proof of hypoellipticity of sums of squares operators satisfying Hörmander's bracket hypothesis [32], but its proof does not require that hypothesis.

Consider any smooth vector field Y. By Cramer's rule we may express $Y = \sum_{i,\alpha} c_{i,\alpha}Z_{i,\alpha}$ where the coefficients c are smooth on $V\backslash M$ and are $O(\beta^{-1/2})$ as $x \to M$. Therefore

$\beta(x)\nabla_x$ may be expressed as $\sum_{i,\alpha} \tilde{c}_{i,\alpha} Z_{i,\alpha}$ where the vector valued coefficients $\tilde{c}(x)$ are uniformly bounded on $\overline{U}$. Thus there exists $\delta > 0$ such that

$$\|\beta \nabla_x A^{-1+\delta} u\|^2 \le C \sum \|X_j u\|^2 + C\|u\|^2.$$

Since $\beta \in C^\infty$, this implies that

$$\|\beta A^\delta u\|^2 \le C \sum \|X_j u\|^2 + C\|u\|^2. \tag{6.2}$$

Fix coordinates $(s, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ in which $M = \{s = 0\}$, and in which one of the vector fields X_j equals ∂_s. In order to deduce the superlogarithmic gain estimate (3.3), it now suffices[11] to show that under the hypothesis (3.2),

$$\int \log^2\langle\xi\rangle \, |\hat{u}(\xi)|^2 \, d\xi \le \varepsilon \left(\|\partial_s u\|^2 + \|\beta A^\delta u\|^2 \right) + C_\varepsilon \|u\|^2 \tag{6.3}$$

for all ε, u.

Fix an auxiliary function $h \in C_0^\infty(\mathbb{R}^+)$ that is supported on $[1/2, 4]$ and is identically equal to 1 on $[1, 2]$. Fix a second such function $\tilde{h} \in C_0^\infty(\mathbb{R}^+)$ that is identically equal to one on the support of h. Fix $\eta \in C_0^\infty(\mathbb{R}^n)$ supported in a small neighborhood of the closure of U, and $\equiv 1$ in a smaller neighborhood of its closure. Let $P_t, \tilde{P}_t$ be the pseudodifferential operators with symbols $\eta(x)h(|\xi|/t), \eta(x)\tilde{h}(|\xi|/t)$, respectively. We seek a bound for $\log(t)\|P_t u\|$, for $t \ge e$.

The hypothesis (3.2) plus Lemma 5.3 with $\tau = t^\delta$ give

$$\log^2(t) \int |g(s)|^2 \, ds \le \varepsilon(t) \int |\partial_s g|^2(s) \, ds + \varepsilon(t) t^{2\delta} \int \beta^2(s, y) |g|^2(s) \, ds$$

for all g supported in a sufficiently small neighborhood of $s = 0$, where $\varepsilon(t) \to 0$ as $t \to \infty$ and ε is a nonincreasing function of t. This inequality is uniform in y. Setting $g(s) = P_t u(s, y)$ and integrating both sides of the preceding inequality with respect to y yields

$$\log^2(t)\|P_t u\|^2 \le \varepsilon(t)\|\partial_s P_t u\|^2 + \varepsilon(t) t^{2\delta} \|\beta P_t u\|^2 + C_N t^{-N} \|u\|^2$$

for all N; this last term arises because $P_t u$ will not have compact support and hence must be truncated. Similar terms in estimates below arise in the same way. By pseudodifferential calculus,

$$\|\partial_s P_t u\|^2 + t^{2\delta} \|\beta P_t u\|^2 \le \|\tilde{P}_t \partial_s u\|^2 + C\|\tilde{P}_t \beta A^\delta u\|^2 + C_N t^{-N} \|u\|^2.$$

[11] The order of magnitude of the quantity $\int \log^2\langle\xi\rangle \, |\hat{u}(\xi)|^2 \, d\xi$ is invariant under changes of coordinates. This is a consequence of diffeomorphism invariance of the classes $\mathrm{Op}(S^m_{\rho,\delta})$ for (ρ, δ) sufficiently close to $(1, 0)$; see Theorem 5.1, Chapter II of [31].

Therefore choosing any large parameter T,

$$\begin{aligned}\int \log^2\langle\xi\rangle\,|\hat{u}(\xi)|^2\,d\xi &\sim \int_e^\infty \log^2(t)\|P_t u\|^2\,\tfrac{dt}{t} + \|u\|^2\\ &\le \int_e^\infty \varepsilon(t)\|\tilde{P}_t\partial_s u\|^2\,\tfrac{dt}{t} + \int_e^\infty \varepsilon(t)\|\tilde{P}_t\beta A^\delta u\|^2\,\tfrac{dt}{t} + C\|u\|^2\\ &\le \varepsilon(T)\int_T^\infty \Big(\|\tilde{P}_t\partial_s u\|^2 + \|\tilde{P}_t\beta A^\delta u\|^2\Big)\,\tfrac{dt}{t} + C_T\|u\|^2\\ &\le C\varepsilon(T)\Big(\|\partial_s u\|^2 + \|\beta A^\delta u\|^2\Big) + C_T\|u\|^2\\ &\le C\varepsilon(T)\sum_j \|X_j u\|^2 + C_T\|u\|^2.\end{aligned}$$

Choosing T so that $C\varepsilon(T)\le\delta$ yields the desired bound $\delta\sum_j\|X_ju\|^2 + C_\delta\|u\|^2$. □

Coupling the next lemma with Theorem 3.5 directly implies Corollary 3.6. The symbol $\|\cdot\|$ with no subscript will always denote the L^2 norm.

Lemma 6.1. *Suppose that a finite collection $\{X_j\}$ of C^∞ real vector fields satisfies the bracket condition at every point of $V\backslash\{x_0\}$, and that $X_1(x_0)\ne 0$. Let $U \Subset V$ be relatively compact. Then for every $\delta>0$, for all $u\in H^1$ supported in U,*

$$\|u\| \le \delta\sum\|X_j u\| + C_\delta\|u\|_{H^{-1}}. \tag{6.4}$$

Moreover, C_δ may be chosen to be a nondecreasing, continuous function of δ^{-1}.

Proof. Choose coordinates in which $x_0=0$, and let $B_r=\{x:|x|<r\}$. Since $X_1(0)\ne 0$,

$$\|u\|_{L^2(B_r)} \le Cr\|X_1u\| + C\|u\|_{L^2(B_{2r}\backslash B_r)} \qquad \text{for all } r>0,$$

by the fundamental theorem of calculus. For any r and any $\varepsilon>0$,

$$\|u\|_{L^2(U\backslash B_r)} \le \varepsilon\sum\|X_ju\| + C_{\varepsilon,r}\|u\|_{H^{-1}},$$

by the bracket condition and hypoellipticity. Putting these ingredients together and choosing $r=\delta$ gives

$$\|u\| \le \|u\|_{L^2(B_\delta)} + \|u\|_{L^2(U\backslash B_\delta)} \le C\delta\|X_1u\| + C_\delta\varepsilon\sum\|X_ju\| + C_{\varepsilon,\delta}\|u\|_{H^{-1}}.$$

Choosing ε as a function of δ so that $\varepsilon C_\delta\le\delta$ completes the proof. □

Lemma 6.2. *Suppose that a finite collection $\{X_j\}$ of C^∞ real vector fields satisfies* (6.4) *in some open set $U\subset\mathbb{R}^d$. Then there exists $w:\mathbb{R}^d\mapsto\mathbb{R}^+$ such that $w(\xi)\to\infty$ as $|\xi|\to\infty$, and such that for every $u\in H^1$ supported in U,*

$$\int_{\mathbb{R}^d} w^2(\xi)|\hat{u}(\xi)|^2\,d\xi \le C\sum_j\|X_ju\|^2 + C\|u\|^2. \tag{6.5}$$

Conversely, (6.5) easily implies (6.4), though this fact will not be needed here.

Proof. Let $\{P_t, \tilde{P}_t : t \in \mathbb{R}^+\}$ be the collection of operators employed in the proof of Lemma 3.1. Let $t \mapsto \delta(t)$ be a nonincreasing function to be chosen below, tending to zero as $t \to \infty$. Uniformly for all sufficiently large t,

$$\begin{aligned}\|P_t u\| &\le \delta(t) \sum_j \|X_j P_t u\| + C_{\delta(t)} \|P_t u\|_{H^{-1}} \\ &\le \delta(t) \Big(\sum \|P_t X_j u\| + C\|\tilde{P}_t u\| + C t^{-1} \|u\| \Big) + C_{\delta(t)} t^{-1} \|P_t u\|.\end{aligned}$$

Here $C_\delta < \infty$ depends continuously on δ.

Choose $\delta(t)$ to be a nonincreasing continuous function of t tending to zero slowly enough as $t \to \infty$ that $t^{-1} \cdot C_{\delta(t)} \to 0$ and $\delta(t) \ge t^{-1/2}$. Then the final term in the preceding inequality may be absorbed into the left hand side for large t, yielding

$$\|P_t u\| \le \delta(t) \sum \|P_t X_j u\| + C\delta(t) \|\tilde{P}_t u\| + C t^{-1} \|u\|,$$

whence

$$\|\delta(t)^{-1} P_t u\| \le \sum \|P_t X_j u\| + C\|\tilde{P}_t u\| + C t^{-1} \delta(t)^{-1} \|u\|,$$

and moreover the last term is $\le C t^{-1/2} \|u\|$. Squaring both sides and integrating yields

$$\int_1^\infty \|\delta(t)^{-1} P_t u\|^2 \, \tfrac{dt}{t} \le C \sum \|X_j u\|^2 + C\|u\|^2.$$

Because δ^{-1} is a continuous function of t which tends to ∞ as $t \to \infty$, the left hand side is $\ge \int_{\mathbb{R}^d} w^2(\xi) |\hat{u}|^2(\xi) \, d\xi - C\|u\|^2$ for some continuous function w tending to ∞ as $|\xi| \to \infty$, by Plancherel's theorem and the definition of $\{P_t\}$. $\square$

Proof of Theorem 3.5. Suppose that $\{X_j\}$ satisfies the bracket hypothesis at every point of $V \setminus \{x_0\}$, and that $X_j(x_0) \ne 0$ for some j. Let $U \Subset V$ be any sufficiently small relatively compact neighborhood of x_0. Fix a nonnegative function $\Psi \in C^\infty(U)$ such that $\Psi \equiv 0$ in some small neighborhood of x_0, yet $\Psi \equiv 1$ except in a relatively compact subset of U. Let $\eta' \in C^\infty(U)$ be any function supported in the region where $\Psi \equiv 1$, such that $\eta' \ge c > 0$ except on a compact subset of U. Define the function $\psi(x, \xi)$ to be $\Psi(x)$.

For any $\delta > 0$ there exists C_δ such that for any $u \in C_0^\infty(U)$, for all i, the principal hypothesis

$$\| \operatorname{Op}[\log\langle\xi\rangle \{\psi, \sigma(X_i)\}] u\| \le \delta \sum_j \|X_j u\|^2 + C_\delta \|u\|^2$$

of Theorem 2.4 holds because $\{X_j\}$ satisfies a subelliptic estimate on a neighborhood of the set of all x for which there exists ξ such that $\{\psi, \sigma(X_i)\}(x, \xi) \ne 0$, and because of the pseudolocality of operators in the class $\operatorname{Op}(S^m_{1-\varepsilon,\varepsilon})$. Theorem 2.4 therefore applies.

The proof of the second part of the theorem proceeds in the same manner but relies instead on Theorem 2.3. The details are left to the reader, as is the proof of Theorem 3.7. $\square$

The relevant literature is extensive. The following is a list of works cited in the text, plus a few others that are particularly relevant to C^∞ hypoellipticity. I apologize to those authors whose works are relevant but are not listed.

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