Cauchy-Riemann Equations Magnetic Schrödinger Operators and Long-range Correlations of Random Zeros

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Subject:
Two problems dealing with $\bar\partial$ and Szegö kernels in SCV where "magnetic" effects play significant roles.

One result is joint with Siqi Fu.

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Schrödinger operators with both electric and magnetic potentials arise in connection with CR manifolds with S^1 symmetry. This situation arises in two ways:

(1) If $\Omega \subset \mathbb{C}^{n+1}$ is a Hartogs domain

$$|z_0|^2 + \varphi(z_1, \ldots, z_n) < 1$$

where $\Delta\varphi \geq 0$ ($\Leftrightarrow$ pseudoconvexity), then $\partial\Omega$ is such a CR manifold.

(2) If L is a holomorphic line bundle over a compact complex manifold M, equipped with a Hermitian metric h, and if X is the unit circle bundle in L^*, then X is such a CR manifold.

In local coordinates $(z, \theta) \in \mathbb{C}^n \times (\mathbb{R}/2\pi\mathbb{Z})$, Cauchy-Riemann operators take the form

$$\overline{L}_j = (\partial_{x_j} + \phi_{y_j}\partial_\theta) + i(\partial_{y_j} - \phi_{x_j}\partial_\theta)$$

Taking $n = 1$ for simplicity of notation, and performing Fourier decomposition in θ, associated second-order operators are

$$-(\partial_x + i\lambda\phi_y)^2 - (\partial_y - i\lambda\phi_x)^2 + \lambda\Delta\phi$$

where Δ is the Laplacian in $z = x + iy$.
$\lambda \in \mathbb{Z}^+$ is Fourier variable dual to θ.

Electric potential $= \lambda\Delta\phi$.
Magnetic potential $= (\lambda\phi_y, -\lambda\phi_x)$.

Always assume: $\Delta\varphi \geq 0$.

First motivating problem

Long-range correlations of random zeros

Bleher, Shiffman, and Zelditch have developed a theory of random zeros (and zero varieties) of holomorphic sections of high powers L^λ of *positive* holomorphic line bundles over compact complex manifolds.

(Example 0: Homogeneous complex polynomials of degree λ, as $\lambda \to \infty$.)

This is a variant of the theory of random matrices (random eigenvalues).

Under natural hypotheses (see below), the vector space $\mathcal{H}_\lambda$ of all holomorphic sections of L^λ has finite dimension $c\lambda^n + O(\lambda^{n-1})$, where n is the dimension of M.

This vector space inherits from $L^2(M, L^\lambda)$ an inner product structure.

This endows the unit sphere in $\mathcal{H}_\lambda$ with a measure. Normalizing this to be a probability measure, one has a notion of a random section (of norm 1), and hence of random zero varieties.

The same may be done for random k-tuples, for any k. In particular, for $k = n$, the zero varieties are generically discrete.

• Bleher et. al. have shown that statistical properties of interest — e.g. **correlations** of random zeros — can be expressed entirely in terms of associated Szegö kernels Π_λ (and their first and second covariant derivatives).

Two-point correlations:

$$\frac{\text{Prob}\,(\text{zero at } z \text{ and zero at } z')}{\text{Prob}\,(\text{zero at } z) \times \text{Prob}\,(\text{zero at } z')} - 1$$

where Prob $(\star)$ denotes probability density.

• Estimates for such Szegö kernels, asymptotically as $\lambda \to \infty$, thus have implications for this theory.

• Topic of much physical interest: *universality*.

Setup

- L line bundle over complex manifold M
- $h =$ positive C^∞ Hermitian metric on L
- M equipped with associated Kähler metric
- $X =$ unit circle bundle in L^*
- h is positive $\Leftrightarrow$ X is strictly pseudoconvex.

- $S(z, \theta; z', \theta')$ Szegö kernel on $X \times X$.
- $\mathcal{H}_\lambda =$ space of all CR functions on X satisfying $f(z, \theta + \beta) \equiv e^{i\lambda\beta} f(z, \theta)$.
- $\mathcal{H}_\lambda$ is in natural isometric correspondence with the Hilbert space of all sections of L^λ.
- Π_λ orthogonal projection of $L^2(X)$ onto $\mathcal{H}_\lambda$.

In local coordinates on M

$$\Pi_\lambda(z, \theta, z', \theta') \equiv e^{i\lambda(\theta - \theta')} \Pi_\lambda(z, z')$$

$$\Pi_\lambda(z, z') = (2\pi)^{-1} \int S(z, \theta; z', 0) e^{-i\lambda\theta} \, d\theta.$$

Question (Zelditch) How rapidly does $\Pi_\lambda(z, z')$ decay as $\lambda \to +\infty$, for distance$(z, z') \geq \delta > 0$? Is it $O(e^{-c\lambda})$?

For C^ω metrics h, YES. (Consequence of analytic hypoellipticity theory + exponential decay of Fourier coefficients of C^ω functions.)

This is question about **partial** regularity (with respect to θ only). Data is C^ω WRT θ.

Boutet de Monvel–Sjöstrand have given a complete asymptotic expansion of S (in sense of ψdo/FIO theory). Every term in that expansion is C^ω as function of θ, off of diagonal in $X \times X$.

Theory of Schrödinger operators with electric (only) potentials suggests $O(e^{-c\sqrt{\lambda}})$ and no better (Agmon metric ideas).

"Electric" effect does always imply $O(e^{-c\sqrt{\lambda}})$, even for C^2 metrics.

Theorem For any C^∞ positive metric h,
• $\Pi_\lambda(z,z') = O(e^{-A\sqrt{\lambda}\sqrt{\log\lambda}})\ \forall A < \infty$
away from the diagonal $z = z'$
• and no better, in general.

Theorem (universality)
Π_λ exhibits universal behavior on scale $c\sqrt{\frac{\log\lambda}{\lambda}}$, in the sense that

$$\frac{\lambda^{-n}\Pi_\lambda(z' + \lambda^{-1/2}\zeta, z',)}{\Pi_*(\zeta,0)} \to 1$$

as $\lambda \to +\infty$, uniformly for $|\zeta| \leq C\sqrt{\log\lambda}$, in appropriate coordinates around any $z' \in M$.

Here the universal limiting situation is a trivial bundle over $\mathbb{C}^n$, with metric $e^{-|z|^2}$, and

$$\Pi_*(\zeta,\zeta') = \pi^{-n} e^{\zeta\cdot\bar{\zeta}'} e^{-\frac{1}{2}|\zeta|^2 - \frac{1}{2}|\zeta'|^2}$$

Corollary/ small application.
• Correlations between zeros at z, z' are $O(e^{-A\sqrt{\lambda \log \lambda}})$, provided distance$(z, z') \geq \delta$.
• Correlations appear to be much stronger than $O(\exp(-c\lambda))$.

However: Last part is not proved. Correlations are expressed as particular homogeneous expressions in terms of $\Pi_\lambda(z, z')$, $\Pi_\lambda(z, z)$, and $\Pi_\lambda(z', z')$, and their first and second covariant derivatives. I have not yet verified that there isn't some miraculous cancellation in these particular expressions (but believe such cancellation to be highly unlikely).

The universality proposition extends result of Bleher, Shiffman, and Zelditch, who showed this for the natural scale $\lambda^{-1/2}$.

I believe that $\sqrt{(\log \lambda)/\lambda}$ is best possible universality scale for C^∞ metrics. Certainly λ^δ is impossible for $\delta > -1/2$.

Second motivating problem: Compactness of $\bar{\partial}$–Neumann problem for Hartogs domains.

- Global regularity problem: Under what circumstances does Bergman projection $L^2(\Omega) \mapsto H^2(\Omega)$ preserve $C^\infty(\overline{\Omega})$?
- Not always, for C^∞ pseudoconvex domains.
- *Compactness* of the quadratic form

$$\int_\Omega |\bar{\partial} u|^2 + |\bar{\partial}^* u|^2$$

(for u satisfying certain boundary condition) implies this type of global regularity, but is a *local* property of $\partial\Omega$ and hence should be stabler/more robust.
- Matheos (1998) gave an example of a pseudoconvex Hartogs domain in $\mathbb{C}^2$, whose boundary contains no *complex disks*, for which compactness nonetheless fails.
- His analysis based on making electric field $\Delta\phi \equiv 0$ on "thick" set with no interior (but nonempty "fine interior" in language of potential theory).

Theorem/Example (joint with Siqi Fu): The magnetic field can be a source of compactness when the electric field is too weak.

Thus compactness is an even subtler issue than previous examples had shown.

But we have not succeeded in making a C^∞ example. I am very skeptical.

For a domain $V_0 \subset \mathbb{C}^1$, with Dirichlet boundary condition: Assume $\Delta\psi \geq 0$.

$$\mathcal{D}_\psi = (\partial_x + i\psi_y,\, \partial_y - i\psi_x)$$
$$\lambda_\psi = \inf_{\|u\|_{L^2}=1} \int_{V_0} \left[|\mathcal{D}u|^2 + \Delta\psi|u|^2\right]$$
$$\tilde{\lambda}_\psi = \inf_{\|u\|_{L^2}=1} \int_{V_0} \left[|\nabla u|^2 + \Delta\psi|u|^2\right]$$

Theorem/Example (restated):
There exists subharmonic φ such that

- $\tilde{\lambda}_{n\varphi}$ remains bounded as $n \to +\infty$
- $\lambda_{n\varphi} \to \infty$ as $n \to +\infty$

and $\varphi \in C^0$, $\nabla\varphi \in L^2$.

To make $\tilde{\lambda}_{n\varphi}$ bounded, one essentially needs a nontrivial function with finite Dirichlet norm, supported on $\{z : \Delta\varphi(z) = 0\}$. Thus this set must be "thick".

To make $\lambda_{n\varphi} \to +\infty$, one needs to show that the magnetic Dirichlet form $\int |\mathcal{D}_{n\varphi} u|^2$ is very strong as $n \to +\infty$, for functions u supported where $\Delta\varphi = 0$.

Major complication: this expression is not at all *monotonic* in n.

Proofs

Three proofs:

• $\Pi_\lambda(z, z') = O(e^{-A\sqrt{\lambda \log \lambda}})$ away from diagonal.

• If $k(\lambda) \gg \sqrt{\lambda \log \lambda}$ then there exists an example in which Π_λ is not $O(e^{-k(\lambda)})$.

• Construction and analysis of example such that $\Delta\varphi = 0$ on a thick set, yet magnetic effect causes $\lambda_{n\varphi} \to \infty$.

These three elements are largely unrelated to one another.

I. Proof that $\Pi_\lambda(z,z') = O(\exp(-A\sqrt{\lambda \log \lambda}))$ off the diagonal:

• Suffices to estimate fundamental solution $G(z,z')$ for $\Box_\lambda = \overline{D}_\lambda \overline{D}_\lambda^*$, because

$$I - \Pi_\lambda = \overline{D}_\lambda^* \Box_\lambda^{-1} \overline{D}_\lambda.$$

• Choice of coordinates (origin at $z' = 0$) plus gauge transformation effects reduction to $\overline{D}_\lambda = e^{-\lambda\phi}\bar{\partial}e^{\lambda\phi}$
where $\phi(z) = |z|^2 + O(|z|^3)$.

• Decompose $\phi = Q + R$ where
$Q =$ polynomial, $R = O(|z|^N)$.

• Apply analytic hypoellipticity theory to fundamental solution for $\Box_{\lambda,Q}$ to deduce that its fundamental solution (with pole at 0) is

$$G_{\lambda,Q}(z,0) = O(e^{-c\lambda|z|^2})$$

(ignoring singularity at $z = 0$).

• *Uniformity*: Bound is independent of Q, provided that λ is sufficiently large, depending on order of Q = Taylor polynomial.

• Smoothly truncate resulting kernel $G_{\lambda,Q}(z,z')$ to region where distance$(z,z') < r$, where $r = r(\lambda)$. Get approximate inverse P_λ for $\Box_\lambda$.

• How small is $\Box_\lambda \circ P_\lambda - I$?
Two sources of error:
(1) Derivatives of cutoff function: $O(e^{-c\lambda r^2})$.
(2) Contribution of $R = \phi - Q$: $O(r^N)$.
Optimize: these match at $r = A\sqrt{\log\lambda/\lambda}$.
Resulting error: $O(\lambda^{-cA})$.

• Get better parametrix via Neumann series; iterate $\Box_\lambda \circ P_\lambda - I$ about $cr^{-1} = cA^{-1}\Big(\lambda/\log\lambda\Big)^{1/2}$ times. Do bookkeeping.

II. Proof that Π_λ decays "slowly" in some cases

Let dimension $n = 1$, coordinates $z = x + iy$.
Let

$$\overline{D}_\lambda = \partial_x + i\partial_y + \lambda a'(x)$$

where $a'' > 0$.
Let $k(\lambda)$ grow faster than $\sqrt{\lambda \log \lambda}$.

Because
• $I - \Pi_\lambda = \overline{D}^*_\lambda \left(\overline{D}_\lambda \overline{D}^*_\lambda\right)^{-1} \overline{D}_\lambda$
and
• $\overline{D}_\lambda$ does have a right inverse which is $O(\exp(-c\,\text{distance}(z, z')))$ (easily constructed),
it suffices to construct example where

Green's kernel for $\overline{D}_\lambda \overline{D}^*_\lambda$
is not $O(e^{-k(\lambda)})$ off diagonal.

Thus want u_λ satisfying:

$\overline{D}_\lambda \overline{D}^*_\lambda u_\lambda \equiv 0$

$u_\lambda = O(e^{\varepsilon k(\lambda)})$ for all $\varepsilon > 0$
and same for $\overline{D}^*_\lambda u_\lambda$.

$|\overline{D}^*_\lambda u_\lambda(0)| \geq c_0 > 0.$

First two bounds hold in a fixed neighborhood of 0.

All this for a sequence of values $\{\lambda_j\} \to +\infty$.

Nonlinear Eigenvalue argument

- Acting on functions $u_\lambda(x) = e^{i\xi y} f(x)$,

$$\overline{D}_\lambda \overline{D}_\lambda^* \leftrightarrow (\partial_x - \xi + \lambda a'(x)) \circ (-\partial_x - \xi + \lambda a'(x)).$$

- Nontrivial solution:

$$f(x) = e^{-\xi x + \lambda a(x)} \int_{-\infty}^{x} e^{2[\xi t - \lambda a(t)]}\, dt.$$

- For $\xi \in \mathbb{R}$, this function is much larger for $x \sim \pm 1$ than for $x = 0$, by an unacceptable factor $\sim e^{c\lambda}$.

- Introduce

$$\mathcal{N}_\lambda(\xi) = \int_{\mathbb{R}} e^{2[\xi t - \lambda a(t)]}\, dt.$$

If $\xi \in \mathbb{C}$ satisfies the nonlinear eigenvalue equation

$$\mathcal{N}_\lambda(\xi) = 0$$

(and if Re(ξ) is appropriate) then f can be essentially maximal near $x = 0$.

• We lose a factor of $e^{c|\operatorname{Im}(\xi)|}$ here from nonunimodularity of $e^{i\xi y}$.
• Thus it is essential to have a zero in the strip $|\operatorname{Im}(\xi)| \leq k(\lambda)$.

Counterexample boils down to:
Construct $a(x) \in C^\infty$, a small perturbation of x^2, such that there exists a sequence $\lambda_\nu \to +\infty$, $\xi_\nu \in \mathbb{C}$ such that

$$0 = \mathcal{N}_{\lambda_\nu}(\xi_\nu) \text{ and } |\operatorname{Im}(\xi_\nu)| \leq k(\lambda_\nu).$$

• Ordinarily one would analyze

$$\mathcal{N}_\lambda(\xi) = \int_{\mathbb{R}} e^{2[\xi t - \lambda a(t)]}\, dt$$

by stationary phase and deformation of contour of integration into complex domain — but here $a(t)$ is not analytic. We use this to our advantage.

More details than you want to see:

- $a(x) = x^2 + \sum_j a_j(x)$

- $\{\lambda_j\}$ very rapidly increasing to $+\infty$.

- $\xi_j = \lambda_j^{1/2} + ik_j z$ ($z = z_j$ is TBA)

- $a_j(\frac{1}{2}\lambda_j^{-1/2} + {k_j}^{-1}t) = \frac{\beta_j}{2\lambda_j}\alpha(t)$

- $\alpha \in C^\infty$ with compact support is fixed.

- $\beta_j = O(\lambda_j^{-N})$ for all N.

- (Consequently) the real part of the phase $\xi_j t - \lambda_j a(t)$ has critical point in the support of a_j.

- Those a_i with $i \neq j$ contribute very little to $\mathcal{N}_{\lambda_j}(\xi_j)$, provided $z_j = O(1)$.

Modulo less significant terms,

$$\mathcal{N}_{\lambda_j}(\xi_j) \approx \int_{\mathbb{R}} e^{-2s^2+2i\lambda_j^{-1/2}k_j z}\, ds \\ - \beta_j \lambda_j^{1/2} k_j^{-1} \Big(\widehat{\alpha}(-2z) + O(\lambda_j k_j^{-2}) \Big).$$

• For $z = O(1)$, the "main" (first) term is $O(e^{-ck_j^2\lambda_j^{-1}})$.

• The second term is larger, if $\beta_j \gg e^{-ck_j^2\lambda_j^{-1}}$.

• This is consistent with the requirement $\beta_j = O(\lambda_j^{-N})$ for all N ($\Leftrightarrow$ smoothness of a) provided $k(\lambda)^2 \gg \lambda \log \lambda$.

• Apply Rouché's theorem near a fixed zero of the Fourier transform $\widehat{\alpha}$ to obtain a zero $z = z_j$. □

Remarks.

There certainly exist nonanalytic metrics for which Π_λ does decay exponentially off of the diagonal. Don't know any general statements.

One can formulate similar results in other function spaces than C^∞. Rate of decay varies with regularity assumption.

In this very special setup, we are deforming the problem in the complexified cotangent bundle (as is to be expected). What that means, in general, is wholly unclear since our operator has nonanalytic coefficients.

This nonlinear eigenvalue construction has been used to produce various examples of nonhypoellipticity, in C^ω and C^∞, in various situations where there is some sort of "geometric" degeneracy. The novelty here is its application to a different kind of degeneracy (lack of smoothness).

Third proof.
Construction of an example for which $\lambda_{n\varphi} \to \infty$ as $n \to +\infty$, yet $\tilde{\lambda}_{n\varphi}$ remains bounded.

• (Kato:) For any real-valued ψ,

$$|\mathcal{D}_\psi u| \geq \Big|\, \nabla |u| \,\Big|$$

a.e. for $u \in C^1$.

• A situation in which magnetic field has strong effect:
Let A be an annulus centered at $z = 0$, and D be the disk inside the inner radius of A. Suppose that $\Delta\psi \equiv 0$ on A. Then

$$\int_A |\mathcal{D}_\psi u|^2 \geq c w^2 \int_A |u(z)|^2 |z|^{-2}$$

where

$$w = \int_D \Delta\psi \quad \underline{\underline{\text{modulo } 2\pi\mathbb{Z}}}.$$

• If the outer radius of A is small, the factor $|z|^{-2}$ represents a strong magnetic effect.

• Situation is highly nonmonotonic as function of $\Delta\psi$. If $w \in 2\pi\mathbb{Z}$ then after a gauge transformation, $\mathcal{D}_\psi$ is unitarily equivalent to ∇ in A; no magnetic effect.

Form of example:

• Start with $\Omega_0 =$ disk.

• Construct $\Omega = \Omega_0 \setminus \cup_{k=1}^{\infty} \cup_j D_j^k$ where $D_j^k = B(z_j^k, \rho_k)$ are many small disks.

• For each k, the centers z_j^k form an ε_k-net in $\Omega_0 \setminus \cup_{m=1}^{k-1} \cup_j D_j^m$.

• $\varepsilon_k = 2^{-k}$ and $\rho_k = e^{-k^4 2^{2k}}$. (Tiny disks $\Rightarrow$ nonempty fine interior.)

• $\Delta\varphi =$ sum of pieces supported on the disks D_j^k.

• Let $N_k = e^{c2^k}$. **Key:** Arrange that for each $n \in [N_k, N_{k+1})$, there exist lots of "good" disks D_j^k such that

$$n \int_{D_j^k} \Delta\varphi \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \quad \text{modulo } 2\pi\mathbb{Z}.$$

• More specifically:
Each $\varepsilon_k^{1/2}$–ball in $\Omega_0 \setminus \cup_{m<k} \cup_j D_j^k$ contains at least one such good disk D_j^k.

Combine:
• Favorable magnetic inequality in a small "annulus" about each good disk D_j^k
• Global control of ∇
• High "density" of good disks.