Math 202B— UCB, Spring 2014 — M. Christ
Problem Set 8, due Wednesday March 19

(8.1) Folland problem 6.2. □

(8.2) Folland problem 6.7. □

(8.3) Folland problem 6.12. □

(8.4) Folland problem 6.19. □

(8.5) Folland problem 6.20. □

(8.6) Define the mappings $\pi_n : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\pi_1(x) = (x_2, x_3), \qquad \pi_2(x) = (x_1, x_3), \qquad \pi_3(x) = (x_1, x_2)$$

where $x = (x_1, x_2, x_3)$. Prove that

$$\int_{\mathbb{R}^3} \prod_{n=1}^{3} f_n(\pi_n(x))\, dm(x) \le \prod_{n=1}^{3} \|f_n\|_{L^2}$$

for all nonnegative measurable functions $f_n : \mathbb{R}^2 \to [0, \infty]$. (This is the *Loomis-Whitney inequality* for $\mathbb{R}^3$; there is an extension to higher dimensions.) □

(8.7) For L^2 there is the useful parallelogram identity. In this problem, we establish a substitute for L^p, for $2 < p < \infty$. All functions are assumed (for the sake of simplicity) to be real-valued in this problem. Let $(X, \mathcal{A}, \mu)$ be a measure space; L^p refers to $L^p(X, \mathcal{A}, \mu)$.

Let $p \in [2, \infty)$. One of Clarkson's inequalities states: For any $f, g \in L^p$,

$$\|f + g\|_p^p + \|f - g\|_p^p \le 2^{p-1}\|f\|_p^p + 2^{p-1}\|g\|_p^p.$$

(8.7)(a) Prove this inequality.
(8.7)(b) Show that as a consequence, if $\|f\|_p = \|g\|_p = 1$, then $\|f + g\|_p \le 2(1 - 2^{-p}\|f - g\|_p^p)^{1/p}$.
(8.7)(c) Show that if $f, g \in L^p$ satisfy $\|f+g\|_p = \|f\|_p + \|g\|_p$ and $\|f\|_p = \|g\|_p$ then one function is a nonnegative constant multiple of the other, almost everywhere. (This is true without the assumption of equal norms; but that requires a different argument.)
(8.7)(d) Let $2 < p < \infty$. Let V be a closed subspace of L^p. Let $f \in L^p$. Define $\operatorname{dist}(f, V) = \inf_{h \in V} \|f - h\|_p$. Show that there exists $g \in V$ such that $\|f - g\|_p = \operatorname{dist}(f, V)$. Show that g is unique. □

Hints

□

(8.6) Apply Cauchy-Schwarz with respect to the x_1 variable, keeping the other two coordinates fixed. Then apply Cauchy-Schwarz again. □

(8.7)(a) Show that it suffices to prove that $(1+t)^p + (1-t)^p \leq 2^{p-1} + 2^{p-1}t^p$ for all $t \in [0,1]$. Then prove this. □

(8.7)(d) Mimic the proof that we gave for the corresponding result concerning Hilbert spaces. □