Math 202B— UCB, Spring 2014 — M. Christ
Problem Set 5, due Wednesday February 26

$\mathbb{F}$ denotes either of the fields $\mathbb{R}, \mathbb{C}$. X denotes a normed vector space over $\mathbb{F}$.

(5.1) (Folland 5.28) Show the analogue of Baire's Theorem for locally compact Hausdorff spaces X. (If $\mathcal{O}_n$ are dense open sets then $\cap_n \mathcal{O}_n$ is dense, and is not a countable union of nowhere dense sets.)

(5.2) Folland 5.29. This problem establishes a couple of basic examples, using the important Banach space $\ell^1 = L^1(\mathbb{N})$.

(5.3) Folland problem 5.30. This problem is concerned with a fundamental real-life example of a closed unbounded operator.

(5.4) (Folland 5.32) Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on a vector space X. Suppose that $\|x\|_2 \le \|x\|_1$ for all $x \in X$. Show if X is complete with respect to both of these norms, then they are equivalent. (The original statement of this problem omitted the hypothesis $\|\cdot\|_2 \le \|\cdot\|_1$. Without some such hypothesis, the conclusion is false in general. An example is given on the following page.)

(5.5) (Folland 5.34 part (a) only) Use the Closed Graph Theorem to show that the natural inclusion map $L^1_k([0,1]) \to C^{k-1}([0,1])$ is continuous. (You may assume the results of problems 5.9 and 5.10 of our text.) (This problem provides some familiarity with two basic Banach spaces, and illustrates the utility of the CGT. For full impact one should do part (b), which asks for an alternative proof that doesn't use the CGT or equivalent. I don't insist that you do this, because I want to keep this problem set short, but please contemplate how you might proceed. This is typical of many applications of the CGT: There is a perfectly good workaround, but the CGT saves time and effort and allows one to focus on what is essential.)

(5.6) (Folland 5.38) Let X be a Banach space, let Y be a normed vector space, and let $T_n \in \mathcal{L}(X,Y)$. Suppose that $\lim_{n\to\infty} T_n(x)$ exists for every $x \in X$. Define $T : X \to Y$ by $T(x) = \lim_{n\to\infty} T_n(x)$. This mapping is obviously linear. Prove that T is bounded.

(5.7) (Folland 5.41) Let X be a vector space of countably infinite dimension. That is, there exists a linearly independent subset $\{v_n : n \in \mathbb{N}\} \subset X$ such that every element $x \in X$ can be expressed as a finite linear combination $\sum_{n=1}^N t_n v_n$, with $N = N(x) < \infty$ and $t_n \in \mathbb{F}$; and such a linear combination vanishes only if all coefficients t_n equal 0. Show that there exists no norm on X with respect to which X is complete. (Our author gives a good hint.)[1]

(5.8) (Folland 5.42) Let $X = C([0,1])$. Show that the set of all functions $f \in X$ that are nowhere differentiable on $[0,1]$ is residual[2] in X.

I promised a shorter problem set this week, so that's it. A couple of these problems have short solutions.

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[1] For a bigger challenge (not part of the problem set), prove this without using Baire's Theorem.

[2] This is a typical illustration of how one can prove that something (a nowhere differentiable continuous function) exists by proving that such things are prevalent, without actually constructing an example. Such proofs often have the advantage of clarifying on a conceptual level the reasons for existence. If you compare this solution to some constructions of nowhere differentiable functions then you may find common elements, but the details may seem simpler for this solution. In particular, this solution involves no infinite series. But if we recall that a normed vector space is complete if and only if every absolutely convergent series of elements converges, and that the proof of Baire's theorem included construction of a set of points satisfying $\sum_n \rho(x_n, x_{n-1}) < \infty$, then we realize that infinite series are still used, but have been incorporated into the general principles underlying this solution.

Hints

(5.1) Simply mimic the proof we learned for complete metric spaces. Use the Finite Intersection Property of compact sets, Prop 4.21.

Example. Here is an example of inequivalent norms $\|\cdot\|_1$, $\|\cdot\|_2$ on a vector space X, with X complete with respect to both norms. Let $(X, \|\cdot\|_1)$ be any Banach space with X infinite-dimensional. Let $\phi : X \to X$ be any invertible linear transformation. Define

$$\|x\|_2 = \|\phi(x)\|_1 \text{ for all } x \in X.$$

It is trivial to verify that this defines a norm on X.

A sequence (x_n) in X converges to a limit x in X with respect to $\|\cdot\|_1$ if and only if $\phi(x_n) \to \phi(x)$ with respect to $\|\cdot\|_2$. A sequence (x_n) is Cauchy with respect to $\|\cdot\|_1$, if and only if $(\phi(x_n))$ is Cauchy with respect to $\|\cdot\|_2$. From these facts, it follows that $(X, \|\cdot\|_2)$ is complete.

These two norms are equivalent if and only if ϕ is bounded. (Indeed, if ϕ is bounded then one can prove that ϕ^{-1} is bounded by applying problem (5.4) to $\|\cdot\|_3 = \|\cdot\|_2/\|\phi\|_{\mathcal{L}(X,X)}$.)

So it suffices to show that if $(X, \|\cdot\|_1)$ is any normed vector space with X infinite-dimensional, then there exists an unbounded invertible linear transformation $\phi : X \to X$. By means of Zorn's Lemma, one can show that X has a Hamel basis, a collection of elements $\{v_\alpha : \alpha \in A\}$ such that any element of X can be represented as a finite linear combination of some of these elements, and such a representation is unique.

Let $\{c_\alpha : \alpha \in A\}$ be a family of nonzero scalars. Define

$$\phi(\sum_\alpha t_\alpha v_\alpha) = \sum_\alpha c_\alpha t_\alpha v_\alpha;$$

remember that all these sums are essentially finite sums, in the sense that $t_\alpha = 0$ for all but finitely many indices α, so there is no issue with convergence. ϕ is an invertible linear transformation.

Since $\phi(v_\alpha) = c_\alpha v_\alpha$, $\|\phi\|_{\mathcal{L}(X,X)} \geq \sup_\alpha |c_\alpha|$. Choosing any unbounded family of nonzero coefficients c_α therefore produces an unbounded linear operator. $\square$