**Math 202B— UCB, Spring 2014 — M. Christ**
**Problem Set 2, due Wednesday February 5**

**(2.1)** Complete the proof of Theorem 2.41 of our text: Let $n \ge 1$. Let $m$ be Lebesgue measure in $\mathbb{R}^n$. Let $f \in L^1(\mathbb{R}^n, \mathcal{L}^n, m)$ and let $\varepsilon > 0$.
(i) There exists a simple function $g = \sum_{j=1}^N c_j \mathbf{1}_{R_j}$, where $R_j$ are genuine rectangles, such that $\|f - g\|_{L^1} < \varepsilon$.
(ii) There exists a continuous function $\varphi$ that vanishes outside some bounded set and satisfies $\|f - \phi\|_{L^1} < \varepsilon$.

**(2.2)** Which conclusions of Theorem 2.44 of our text remain valid if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear but not invertible? Justify your answer.

**(2.3)** The Gamma function is defined to be $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt$, for $z \in \mathbb{C}$ with $\mathrm{Re}\,(z) > 0$. Use Fubini's Theorem to prove that $\Gamma(x)\Gamma(y) = \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ whenever $x, y, x+y$ all have real parts $> -1$. (Here all exponentials with complex exponents are defined using the principal branch of the logarithm function; $t^w = e^{w \ln(t)}$ for $t \in \mathbb{R}^+$.)

**(2.4)** The measure $\sigma$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is defined in §2.7 of our text. Show that $\sigma$ is invariant under rotations. That is, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear transformation, then for all Borel sets $E \subset S^{n-1}$, $T(E) \subset S^{n-1}$ is a Borel set, and $\sigma(T(E)) = \sigma(E)$.

(A complication: There are two natural ways to define the Borel subsets of $S^{n-1}$: (a) All sets which are Borel subsets of $\mathbb{R}^n$ and are contained in $S^{n-1}$. (b) $S^{n-1}$ is a topological space, under the relative topology that it inherits from its inclusion as a subset of $\mathbb{R}^n$. Form the smallest $\sigma$–algebra of subsets of $S^{n-1}$ that contains all sets that are open with respect to this topology. These two candidates are in fact equal (an easy exercise). You need not prove this.)

**(2.5)** Let $a, b \in \mathbb{R}$ and consider $f(x) = |x|^a \cdot |\ln(|x|)|^b$ for $0 \ne x \in \mathbb{R}^n$. For which values of $a, b$ is $f \in L^1(\{x : |x| \le \frac{1}{2}\})$? What about $L^1(\{x : |x| \ge 2\})$? (You may want to use polar coordinates in $\mathbb{R}^n$. You may use the material in §2.7 of our text for this purpose.)

**(2.6)** Let $f : \mathbb{R}^n \to \mathbb{C}$, and define $F : \mathbb{R}^{2n} \to \mathbb{C}$ by $F(x, y) = f(x - y)$.
(a) Show that if $f$ is Borel measurable, then so is $F$.
(b) Show that if $f$ is Lebesgue measurable, then so is $F$.

**(2.7)** [Folland (2.63)] Let $n \ge 2$. Let $f(x) = \prod_{j=1}^n x_j^{\alpha_j}$ where each exponent $\alpha_j$ belongs to $\{0, 1, 2, \ldots\}$. Show that if all $\alpha_j$ are even then

$$\int_{S^{n-1}} f\,d\sigma = 2\,\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1 + \cdots + \beta_n)}$$

where $\beta_j = \frac{1}{2}(\alpha_j + 1)$. Show that if any $\alpha_j$ is odd then the integral vanishes. (Our text provides a useful hint.)

The final two problems are remarks that were made in class on Wednesday 1/29.

**(2.8)** Show that there exists a compact set $E \subset \mathbb{R}$ such that $m(E) > 0$, but $E$ contains no interval of positive length. (Hint below.)

**(2.9)** Let $E = \mathbb{Q} \cap [0, 1]$. Show that $m(E) = 0$. Show that if $\{I_j\}$ is a *finite* collection of intervals such that $E \subset \cup_j I_j$, then $\sum_j m(I_j) \ge 1$.

## Hints

**(2.8)** Recursively construct a sequence of compact sets $[0,1] = E_0 \supset E_1 \supset E_2 \supset \dots$ as follows. Choose a sequence of positive numbers $r_j \in (0,1)$. To construct $E_1$ from $E_0$, delete from $E_0$ an open interval of length $r_1$ centered at $\frac{1}{2}$. $E_1$ is a union of two closed intervals $I_{1,j}$ for $j = 1,2$, each of which has length $\rho = \frac{1}{2}(1 - r_1)$. From each of $I_{1,j}$ delete an open interval of length $r_2\rho_1$ centered at the center of $I_{1,j}$. This leaves $2^2$ pairwise disjoint closed intervals $I_{2,j}$, $1 \le j \le 4$, each of length $\rho_2 = \frac{1}{2}(\rho_1 - r_2\rho_1) = \frac{1}{2}\rho_1(1-r_2) = \frac{1}{4}(1-r_1)(1-r_2)$. Their union is $E_2$. From each of these $2^2$ intervals delete a centered open interval of length $r_3\rho_2$, to obtain $E_3$, a union of $2^3$ disjoint closed intervals. And so forth. Define $E = \cap_{k=0}^{\infty} E_k$. Show that the parameters $r_k$ can be chosen so that $E$ has the required properties. $\square$