Math 202B— UCB, Spring 2014 — M. Christ
Problem Set 10, due Wednesday April 9

(10.1) (Folland problem 7.17) (a) If μ is a positive Radon measure on X satisfying $\mu(X) = \infty$, there exists $0 \le f \in C_0(X)$ satisfying $\int f\,d\mu = \infty$. (b) Show as a consequence that if I is a positive linear functional on $C_0(X)$, then I is necessarily bounded. (Warning: Part (b) is a bit tricky, not an immediate consequence of part (a).) □

(10.2) Folland problem 7.20(b). □

(10.3) Folland problem 7.21. □

(10.4) Folland problem 7.22. □

(10.5) Folland problem 7.24. (Typo in text: In part (b), $\mu = 0$. That is, $\mu_n \to 0$ vaguely, but there exists f bounded and measurable with compact support such that $\int f\,d\mu_n$ does not tend to zero.) □

(10.6) Folland problem 7.25 (assume X is first countable). □

(10.7) Folland problem 7.27. □

(10.8) Folland problem 8.3. □

(10.9) Folland problem 8.4. □

Hints

□

(10.1) (a) As a warmup, consider the case in which there are pairwise disjoint compact sets K_n such that $\sum_n \mu(K_n) = \infty$.

(b) With (a) in hand, we need to show that if I is positive, if μ is a finite Radon measure, and if $I(f) = \int f\,d\mu$ for all $f \in C_c(X)$, then $I(f) = \int f\,d\mu$ for all $f \in C_0(X)$. This is related to the proof that any positive linear functional on $C_c(X)$ is locally bounded. □

(10.4) The UBP may be helpful in proving that $\|f_n\|_u$ must be uniformly bounded. □

(10.7) The main idea is to relate the space $C^k([0,1])$ to $C^0(Y)$ for some LCH space Y, so that the Riesz Representation Theorem can be used. The quantity $\|f^{(k)}\|_{C^0([0,1])} + \max_{0\le n<k} |f^{(n)}(0)|$ defines an equivalent norm for $C^k([0,1])$. □

(10.9) Here $A_r f(x) = (2r)^{-1} \int_{[x-r,x+r]} f$. Show that if $f \in L^\infty$ then A_r is continuous for every $r > 0$. Show that if $r_n \to 0^+$ then the sequence $(A_{r_n} f)$ is Cauchy in the uniform norm. Show that the resulting limit function equals f almost everywhere. (You do not need Theorem 3.18 for this.) □