Math 202B — UCB, Spring 2014 — M. Christ
On convergence of Fourier series

1 Clarification

I want to reemphasize that the Fourier inversion formula

$$f = (\widehat{f})^\vee$$

holds for all $f \in L^2(\mathbb{R}^d)$, *in the following sense*: Let T be the unique bounded linear operator $L^2(\mathbb{R}^d) \to \mathbb{R}^2(\mathbb{R}^d)$ that satisfies $T(f) = \widehat{f}$ for all $f \in L^1 \cap L^2$, where $\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi}\, dx$. (Here $f, T(f)$ are equivalence classes of functions.) Likewise let S be bounded and linear and satisfy $S(f) = f^\vee$ for $f \in L^1 \cap L^2$. Then $S \circ T : L^2 \to L^2$ is well-defined, and $S(T(f)) = f$ (as equivalence classes of functions) for all $f \in L^2$.

Proof. $\mathcal{S}$ is contained in $L^1 \cap L^2$, and is dense in L^2. If $f \in \mathcal{S}$ then $\widehat{f} \in \mathcal{S}$. In particular, both $f, \widehat{f}$ belong to L^1. In particular, $(\widehat{f})^\vee$ is well-defined by the formula $(\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi$. We have proved that whenever $f, \widehat{f} \in L^1$, $f = (\widehat{f})^\vee$. (The integral defines a continuous function, which agrees almost everywhere with any representative of the equivalence class of f.)

Therefore $(S \circ T)(f) = f$ for all $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in L^2, and since $S \circ T$ is continuous, this identity holds for all $f \in L^2$. □

2 Motivation

Around 1820, Joseph Fourier studied the propagation of heat. Consider a wire made of metal or another material that conducts heat. Suppose that different points at the wire are initially at different temperatures, that all sources of heating or cooling are removed so that heat neither enters nor leaves the wire. Then as time moves forward, heat will flow from hotter regions to cooler ones, until an equilibrium is reached.

The mathematical description: Parametrize the wire by $x \in \mathbb{R}$. Let t be time. Let $u(t, x)$ be the temperature at points x at time T. This system is modeled by the partial differential equation

$$\partial_t u = \partial_x^2 u,$$

called the *heat equation*. (Integrating this with respect to $x \in [a, b]$, it says that heat energy flows from warmer regions to cooler regions at a rate proportional to the difference in their temperatures.)

The problem Fourier sought to solve was: If u is know at time $t = 0$, how can u be calculated for future times? Mathematically, the problem is to solve the above partial differential equation for $t > 0$, with the initial condition $u(0, x) = f(x)$, a known function of x.

Let us consider a circular piece of wire, so that the complication of endpoints is eliminated (Fourier did deal with the endpoints). Then we can regard f, u as periodic functions of $x \in \mathbb{R}^1$. Assuming the period to be 1, we can equivalently regard them as functions from $\mathbb{T}$ to $\mathbb{R}$. Recall the functions $e_n(x) = e^{2\pi i n x}$, for $n \in \mathbb{Z}$.

Fourier knew that $\partial^2(e_n) \equiv (-2\pi i n)^2 e_n$; these are eigenfunctions for ∂_x^2. That is no surprise if one recognizes that

$$\partial_x f = \lim_{y \to 0} \frac{\tau_{-y} f - f}{y}$$

and recalls that these are eigenfunctions for τ_y for all y. Thus it is reasonable to look for solutions of the heat equation of the form $a_n(t) e_n(x)$, and one immediately finds that

$$e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

is a solution for every $n \in \mathbb{N}$, which equals $e_n(x)$ when $t = 0$.

Fourier also knew that the heat equation is linear; adding two solutions, or multiplying a solution by a scalar, gives a new solution. So any finite linear combination

$$\sum_n a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x},$$

where the coefficients a_n are constants, is a solution. (In order to have a real solution, one needs $a_{-n} = \overline{a_n}$ for all n. But let's ignore the fact that temperatures ought to be real.)

Now Fourier took several bold steps. (i) He considered infinite series

$$\sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

and claimed that these are also solutions;
(ii) he claimed that any function $f : \mathbb{T} \to \mathbb{C}$ could be represented as the sum of a series $\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ for some sequence of coefficients a_n, and
(iii) he put these two claims together to assert a method for solving the heat equation with initial datum $u(0, x) = f(x)$, for any f.

This raised all sorts of questions, which were beyond the scope of Mathematics at that time. Can any f be so represented? Can a sum of continuous functions $a_n e_n$ equal a discontinuous function f? Does the sum of the infinite series actually satisfy the heat equation? In what sense is $\lim_{t \to 0^+} u(t, x)$ equal to $f(x)$? In what sense, and for which f, is $f(x) = \sum_{n=-\infty}^{\infty} a_n e_n(x)$? A big chunk of modern analysis originated in the effort to resolve these and related issues.

3 Divergence of Fourier series

We know some facts relevant to these questions. Define

$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}.$$

- Given $f \in L^2$, if we define $a_n = \widehat{f}(n)$ then the series $\sum_n a_n e_n$ converges to f, in the sense that
$$\|f - S_N(f)\|_{L^2(\mathbb{T})} \to 0 \quad \text{as } N \to \infty.$$
- If $f \in C^0(\mathbb{T})$ and $\widehat{f} \in \ell^1$ then $S_N(f) \to f$ uniformly on $\mathbb{T}$.

The second fact is proved by defining $g = \sum_n \widehat{f}(n)e_n$. The series converges uniformly, so $g \in C^0$. One can justify the formal calculation

$$\widehat{g}(m) = \int e^{-2\pi imx} \sum_n \widehat{f}(n)e^{2\pi inx}\,dx = \sum_n \widehat{f}(n) \int e^{2\pi i(n-m)x}\,dx = \widehat{f}(m)$$

using Fubini. Thus $\widehat{g} = \widehat{f}$, so $\widehat{g - f} \equiv 0$. Since $g - f \in L^2$, it follows that $g \equiv f$. □

But these facts fall far short of answering natural questions. In particular, if $f : \mathbb{T} \to \mathbb{C}$ is a general continuous function but no assumption is made on the nature of its Fourier coefficients, then does $S_N(f)$ converge to f *pointwise*? We will now investigate this question.

A key point is that there is a relatively simple expression for $S_N(f)$, much simpler for large N than the sum of $2N+1$ terms of a series. Firstly,

$$\begin{aligned}
S_N(f)(x) &= \sum_{|n|\le N} \widehat{f}(n)e^{2\pi inx} \\
&= \sum_{|n|\le N} \int_{\mathbb{T}} f(y)e^{-2\pi iny}\,dy e^{2\pi inx} \\
&= \int_{\mathbb{T}} f(y) \sum_{|n|\le N} e^{2\pi in(x-y)}\,dy \\
&= \int_{\mathbb{T}} D_N(x-y)f(y)\,dy \\
&= f * D_N(x)
\end{aligned}$$

where

$$D_N(y) = \sum_{|n|\le N} e^{2\pi iny}.$$

Thus

$$S_N(f) = f * D_N;$$

partial sums of the Fourier series are obtained by convolution with the functions D_N, which are called the Dirichlet kernel(s).

Secondly, the finite series defining D_N can be calculated:

$$\begin{aligned}\sum_{n=-N}^{N} e^{2\pi iny} &= e^{-2\pi iNy} \sum_{k=0}^{2} N e^{2\pi iky} \\ &= e^{-2\pi iNy} \frac{e^{2\pi i(2N+1)y} - 1}{e^{2\pi iy} - 1} \\ &= \frac{e^{\pi i(2N+1)y} - e^{-\pi i(2N+1)y}}{e^{\pi iy} - e^{-\pi iy}} \\ &= \frac{\sin(\pi(2N+1)y)}{\sin(\pi y)}\end{aligned}$$

for $y \neq 0$; obviously $D_N(0) = 2N + 1$ since every term in the series equals 1 when $y = 0$. So

$$D_N(y) = \frac{\sin(\pi(2N+1)y)}{\sin(\pi y)},$$

with the natural interpretation at $y = 0$. This ratio is a 1–periodic continuous function of $y \in \mathbb{R}$, or equivalently, a continuous function of $y \in \mathbb{T}$. Because $\sin(\pi y) = 0$ for $y \in [-\frac{1}{2}, \frac{1}{2}]$ only for $y = 0$, by comparing Taylor expansions at $y = 0$ one finds that D_N is an infinitely differentiable function on $\mathbb{T}$.

Individually, the functions D_N are tamely behaved, but not uniformly so.

Lemma. There exists $c > 0$ such that $\|D_N\|_{L^1(\mathbb{T})} \geq c\log(N)$ as $N \to \infty$. □

What we need to know below is simply that $\|D_N\|_{L^1} \to \infty$. This is implicit in the last problem of problem set 12, and is exercise 8.34 in our text. I will assume this result for now, leaving the verification to a problem set. This is a fundamental fact about Fourier series. □

This fact has a startling consequence.

Proposition. There exists a function $f \in C^0(\mathbb{T})$ such that the sequence of partial sums $S_N(f)(0)$ is unbounded, and in particular, fails to converge to $f(0)$. □

Since S_N is a convolution operator, $S_N(\tau_y f) = \tau_y(S_N(f))$ for any y. Therefore the sequence $(S_N(\tau_y f)(y) : N \in \mathbb{N})$ is unbounded; there is nothing special about the point 0 in this result. □

Proof. For each N, the mapping $\ell_N(f) = S_N(f)(0) = \int_{\mathbb{T}} f(y) D_N(-y)\,dy$ is a bounded linear functional from $C^0(\mathbb{T})$ to $\mathbb{C}$; indeed, $\ell_N(f) = \int f\,d\mu_N$ where $d\mu_N = D_N\,dx$ is a complex Radon measure on $C^0(\mathbb{T})$. We know from the easy part of the Riesz Representation Theorem that $\|\ell_N\|_{(C^0)^*} = \|\mu_N\|_{\mathcal{M}(\mathbb{T})}$; and this equals $\|D_N\|_{L^1}$.

If the sequence $S_N(f)(0) = \ell_N(f)$ were bounded for each $f \in C^0$, then by the Uniform Boundedness Principle, the functionals ℓ_N would be uniformly bounded, that is, there would exist $C < \infty$ such that $\|\ell_N\|_{(C^0)^*} \leq C$ for every N. But we have observed that this is not the case. □

I had promised you a genuine application of the Uniform Boundedness Principle before the end of the semester; here it is. □

You may have wondered whether the Fourier series of an L^2 function must converge to the function pointwise. We have now seen that even for a continuous function, this need not be the case. $\square$

Important philosophical point: It's not just that the UBP provides a slick way for showing existence of a function for which the series diverges, without our needing (or being able?) to actually construct such a function. The UBP focuses our attention on the essential point: we should examine $\|D_N\|_{L^1}$, rather than casting about trying to figure out how to guess or construct some counterexample. $\square$

The failure of convergence for continuous funtions is a matter of degree, in the sense that there is no problem for functions that are sufficiently continuous.

Proposition. If $f \in C^0(\mathbb{T})$ is Lipschitz continuous, then $S_N(f)(x)$ converges to $f(x)$ for every $x \in \mathbb{T}$ as $N \to \infty$. $\square$

Proof. Observe that $\int_{\mathbb{T}} D_N = \widehat{D_N}(0) = 1$ by the very definition of D_N. Therefore

$$f(x) - S_N(f)(x) = \int (f(x) - f(x-y)) D_N(y)\, dy = \int \frac{f(x) - f(x-y)}{\sin(\pi y)} \sin((2N+1)\pi y)\, dy.$$

The function $g(y) = \frac{f(x)-f(x-y)}{\sin(\pi y)}$ is bounded, hence is in L^1. By the (proof of the) Riemann-Lebesgue Lemma,

$$\int g(y) \sin((2N+1)\pi y)\, dy \to 0$$

as $N \to \infty$. $\square$

By repeating the proof of the Riemann-Lebesgue lemma, one can actually conclude that $S_N(f) \to f$ uniformly on $\mathbb{T}$. $\square$

4 Using the Fourier transform to solve the heat equation

In this section we follow Fourier to solve the d–dimensional heat equation

$$u_t = \Delta_x u.$$

We will first solve this problem for $\mathbb{R}^d$, then discuss $\mathbb{T}^d$ (which turns out to require an extra wrinkle). Here $\Delta_x = \Delta$ is the Laplace operator

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

We want $u(0, x) = f(x)$, where f is a given function.

4.1 The case of $\mathbb{R}^d$

According to the heuristics explained above, it is natural to define

$$u(t, x) = \int e^{-4\pi^2 |\xi|^2 t} e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\, d\xi.$$

Proceeding formally without bothering for now about convergence issues, we can write this in the alternative form

$$\begin{aligned} u(t,x) &= \int e^{-4\pi^2|\xi|^2 t} e^{2\pi i x\cdot\xi} \int f(y) e^{-2\pi i \xi\cdot y}\, dy\, d\xi \\ &= \int f(y) \Big(\int e^{-4\pi^2|\xi|^2 t} e^{2\pi i (x-y)\cdot\xi}\, d\xi \Big)\, dy \\ &= f * h_t(x) \end{aligned}$$

where

$$h_t(y) = \int e^{-4\pi^2|\xi|^2 t} e^{2\pi i y\cdot\xi}\, d\xi.$$

Recalling that

$$\int e^{-2\pi i y\cdot\xi} e^{-\pi s|\xi|^2}\, d\xi = s^{-d/2} e^{-\pi|y|^2/s},$$

by setting $s = 4\pi t$ we find that

$$h_t(y) = (4\pi t)^{-d/2} e^{-\pi|y|^2/4\pi t} = 2^{-d}\pi^{-d/2} t^{-d/2} e^{-|y|^2/4t}.$$

Lemma. The function $w(t,y) = h_t(y)$, defined for $(t,y) \in (0,\infty)\times\mathbb{R}^d$, is a solution of the heat equation $w_t = \Delta_y w$. □

This can be proved either by direct calculation using the expression $t^{-d/2}e^{-|y|^2/4t}$, or using the integral representation $\int e^{-4\pi^2|\xi|^2 t} e^{2\pi i y\cdot\xi}\, d\xi$. The latter has the advantage that if one passes $\partial_t - \Delta_y$ inside the integral then one sees immediately that the resulting integrand vanishes; but one needs to justify passing the differential operator inside the integral. □

Observe that for any $t > 0$, h_t is a Schwartz function. Therefore $f * h_t(x)$ is a well-defined C^∞ of x for all $t > 0$.

Lemma. If $f \in L^1(\mathbb{R}^d)$ or $f \in L^\infty(\mathbb{R}^d)$ then $u(t,x) = f * h_t$ is a C^∞ function of $(t,x) \in (0,\infty)\times\mathbb{R}^d$, which satisfies the heat equation $u_t = \Delta_x u$. □

Proof. Consider the expression $v(t,x) = \int f(y) e^{-|x-y|^2/4t}\, dy$. This is well-defined for $f \in L^1$ or L^∞ since the exponential factor belongs to $L^\infty \cap L^1$ as a function of y, for each $t > 0$.

The integrand is a C^∞ function of (x,t). To verify that $\partial_t v(t,x)$ exists, is continuous, and equals $\int f(y)(\frac{1}{4}|x-y|^2 t^{-2}) e^{-|x-y|^2/4t}\, dy$, consider

$$\frac{v(t+h,x) - v(t,x)}{h} = \int f(y) h^{-1} \Big(e^{-|x-y|^2/4(t+h)} - e^{-|x-y|^2/4t} \Big)\, dy.$$

As $h \to 0$, the integrand converges to $f(y)(\frac{1}{4}|x-y|^2 t^{-2}) e^{-|x-y|^2/4t}$. Provided that $|h| \le \frac{1}{2}t$, the Mean Value Theorem guarantees that the absolute value of the integrand does not exceed $|f(y)| \cdot (\frac{1}{4}|x-y|^2 4t^{-2}) e^{-|x-y|^2/2t}$, which is an L^1 function of y. Therefore the limit exists and equals $\frac{1}{4}t^{-2}\int f(y)|x-y|^2 e^{-|x-y|^2/4t}\, dy$. This integral represents $\frac{1}{4}t^{-2}$ times the convolution of f with the Schwartz function $|y|^2 e^{-|y|^2/4t}$, so is a continuous function of x.

This reasoning can be repeated to analyze partial derivatives of u of arbitrary order with respect to (t,x). □

Lemma. If f is bounded and continuous then $u(t,x) \to f(x)$ as $t \to 0^+$, for all $x \in \mathbb{R}^d$. If $f \in L^1$ then $u(t,\cdot) \to f$ in L^1 norm. More generally, for any exponent $p \in [1,\infty)$ and any $f \in L^p(\mathbb{R}^d)$, $u(t,\cdot) \to f$ in $L^p(\mathbb{R}^d)$ norm as $t \to 0^+$. □

Proof. $u(t,x) = (f * \varphi_s)(x)$ where $s = t^{1/2}$ and $\varphi(y) = (4\pi)^{-d/2} e^{-|y|^2/4}$. This is a Schwartz function, and we have already verified earlier in the course that it satisfies $\int \varphi = 1$. So we have a general theorem asserting that $u(t,\cdot) \to f$ in L^p whenever $f \in L^p$ and $1 \le p < \infty$.

If $f \in C_0$ then we already have a theorem asserting that $f * \varphi_s \to f$ uniformly as $s \to 0^+$. If it is merely assumed that f is continuous and bounded, then that result can't be applied, and its proof, which relied on the *uniform* continuity of f, is not valid. But the proof does work with a bit more care.

Let $z \in \mathbb{R}^d$; we want to prove $u(t,z) \to f(z)$. By replacing f by $\tau_z f$ we may assume $z = 0$. Then continuing to write $s = t^{1/2}$,

$$u(t,0) = \int f(-y)\varphi_s(y)\,dy.$$

Let $\varepsilon > 0$ and choose $\delta > 0$ so that $|f(y) - f(0)| < \varepsilon$ whenever $|y| < \delta$. Then

$$\begin{aligned}
|u(t,0) - f(0)| &= \Big| \int \big(f(-y) - f(0)\big)\varphi_s(y)\,dy \Big| \\
&\le \int \big|f(-y) - f(0)\big|\varphi_s(y)\,dy \\
&\le \int_{|y|\le\delta} \big|f(-y) - f(0)\big|\varphi_s(y)\,dy + \int_{|y|>\delta} \big|f(-y) - f(0)\big|\varphi_s(y)\,dy \\
&\le \varepsilon \int_{|y|\le\delta} \varphi_s(y)\,dy + 2\sup_{y\in\mathbb{R}^d}|f(y)| \int_{|y|>\delta} |\varphi_s(y)\,dy \\
&\le \varepsilon + 2\sup_{y\in\mathbb{R}^d}|f(y)| \int_{|y|>\delta} \varphi_s(y)\,dy.
\end{aligned}$$

Now

$$\int_{|y|>\delta} \varphi_s(y)\,dy = \int_{|y|>s^{-1}\delta} \varphi(y)\,dy \to 0 \quad \text{as } s \to 0^+.$$

Therefore

$$\limsup_{t\to 0^+} |u(t,0) - f(0)| \le \varepsilon$$

for all $\varepsilon > 0$. □

Notice that we never bothered to justify the steps of our initial derivation of the formula for u. The Fourier transform entered this discussion only as a way to guess a formula for u; we then verified directly that that formula worked. □

4.2 The Poisson Summation Formula

Theorem. Let $g \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\sum_{n\in\mathbb{Z}^d} \widehat{g}(n) = \sum_{n\in\mathbb{Z}^d} g(n).$$

□

This has a beautiful alternative formulation. Let λ be the infinite measure $\lambda = \sum_{n\in\mathbb{Z}^d} \delta_n$, a sum of Dirac masses, one at each element of $\mathbb{Z}^d$. It is possible to define the Fourier transform of any measure satisfying very mild conditions, although we won't do so yet. Once we do have a suitable definition, then the Theorem can be restated:

$$\widehat{\lambda} = \lambda.$$

Before discussing the derivation, here is a motivating application. To solve the heat equation on $(0,\infty) \times \mathbb{T}^d$ with $u(0,x) = f(x)$, we have been led to the formula

$$u(t,x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n) e^{-4\pi^2|n|^2 t} e^{2\pi i n\cdot x}.$$

As in the $\mathbb{R}^n$ case, we can at least formally rewrite this as $f * h_t(x)$ where now

$$h_t(y) = \sum_{n\in\mathbb{Z}^d} e^{-4\pi^2|n|^2 t} e^{2\pi i n\cdot x}.$$

The next step is to evaluate this sum in closed form. Our derivation for $\mathbb{R}$ used contour integration; but here we have an infinite series, not an integral! Contour integration doesn't look relevant here.

What to do? Use the Poisson Summation Formula, with $g(x) = e^{-4\pi^2 t|x|^2}$ to find that

$$h_t(y) = 2^{-d}\pi^{-d/2}t^{-d/2} \sum_{n\in\mathbb{Z}^d} e^{-|n|^2/4t}.$$

This is not as simple as in the case of $\mathbb{R}^d$, but it still contains quite a lot of useful information, and in particular can be used to derive analogues for $\mathbb{T}^d$ of all the facts shown above for solutions of the heat equation for $\mathbb{R}^d$. □

In our text you'll find the PSF stated for a larger class of functions g. You should understand that, like many formulas and principles that we have learned, the PSF is best regarded as a meta-identity or meta-theorem, which applies under various assumptions, or combinations of assumptions, on g and $\widehat{g}$. The class $\mathcal{S}$ is large enough to be dense in many function spaces, but small enough to ensure that the infinite sums on both sides of the equation are well-defined. □

The proof of the PSF revolves around the relationship between two different Fourier transforms; one for $\mathbb{R}^d$, one for $\mathbb{T}^d$. (In that respect it is like the Shannon Sampling Theorem.) Therefore we need suitable notation to distinguish these two transforms. For $h : \mathbb{T}^d \to \mathbb{C}$, temporarily define $\mathcal{F}(h)(k) = \int_{\mathbb{T}^d} h(x)e^{-2\pi i k\cdot x}\,dx$, reserving the notation $\widehat{f}$ for the $\mathbb{R}^d$–Fourier transform of $f : \mathbb{R}^d \to \mathbb{C}$.

To prove the PSF, let $g \in \mathcal{S}$ and consider the function

$$G(x) = \sum_{m\in\mathbb{Z}^d} g(x+m).$$

G is a periodic function, so can, and will, be regarded as a function of $x \in \mathbb{T}^d$. By the Fourier inversion formula for $\mathbb{T}^d$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} g(n) &= G(0)\\ &= \sum_{n\in\mathbb{Z}^d} \mathcal{F}(G)(n)\\ &= \sum_{n\in\mathbb{Z}^d} \int_{[0,1]^d} G(x)e^{-2\pi i n\cdot x}\,dx\\ &= \sum_{n\in\mathbb{Z}^d} \int_{[0,1]^d} \sum_{m\in\mathbb{Z}^d} g(x+m)e^{-2\pi i n\cdot x}\,dx\\ &= \sum_{n\in\mathbb{Z}^d} \int_{[0,1]^d} \sum_{m\in\mathbb{Z}^d} g(x+m)e^{-2\pi i n\cdot (x+m)}\,dx\\ &= \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} g(x)e^{-2\pi i n\cdot (x+m)}\,dx\\ &= \sum_{n\in\mathbb{Z}^d} \widehat{g}(n) \ !\end{aligned}$$

4.3 The heat equation for $\mathbb{T}^d$

Everything we've proved for $\mathbb{R}^d$ now goes through for $\mathbb{T}^d$. The proofs are slightly more complicated because for $\mathbb{T}^d$, the function denoted above by $h_t(x)$ is no longer of the usual form $s^{-d}h(s^{-1}x)$; indeed, dilation doesn't make sense for $\mathbb{T}^d$, so we can't expect that! But $h_t > 0$, $\int_{\mathbb{T}^d} h_t\,dx = \widehat{h_t}(0) = 1$, $h_t(y)$ is a C^∞ function on $(0,\infty)\times\mathbb{T}^d$, and $\int_{|y|\geq\delta} h_t(y)\,dy \to 0$ as $t \to 0^+$, if we identify $\mathbb{T}^d$ with $[-\frac{1}{2},\frac{1}{2}]^d$. These properties are sufficient to allow the proofs given above to be pushed through. $\square$

5 More on pointwise convergence

Consider a formal infinite series $\sum_{n=1}^{\infty} a_n$ of complex numbers and its partial sums $s_N = \sum n = 1^N a_n$. If the series converges, that is, if $s_n \to S$ as $n \to \infty$, then the averages $\sigma_N = N^{-1}\sum_{n=1}^N s_n$ of the successive partial sums also converge to S. But if the series is divergent, then the sequence σ_N may still converge. For instance, if $a_n = (-1)^{n-1}$ then the sequence s_n is $(1,0,1,0,1,0,\dots)$; no limit. Yet the sequence σ_N clearly converges, to $\frac{1}{2}$.

Consider now the Fourier series of some function $f \in L^1(\mathbb{T})$. Form

$$\begin{aligned}S_N(x) = S_N * (f)(x) &= \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi i n x}\sigma_N(x)\\ &= \sigma_N(f)(x) = (N+1)^{-1}\sum_{k=0}^{N} S_k(x).\end{aligned}$$

One calculates that

$$\sigma_N(f)(x) = \sum_{n=-N}^{N} \big(1 - (N+1)^{-1}|n|\big)\widehat{f}(n)e^{2\pi inx}.$$

This resembles

$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi inx},$$

with weights

$$\big(1 - (N+1)^{-1}|n|\big) \in [0,1]$$

inserted into the sum.

As in our discussion of S_N, D_N, $\sigma_N(f)$ can be rewritten as a convolution

$$\sigma_N(f)(x) = K_N * f(x)$$

where

$$\begin{aligned} K_N(y) &= \sum_{n=-N}^{N} \big(1 - (N+1)^{-1}|n|\big)e^{2\pi iny} \\ &= (N+1)^{-1}\sum_{k=0}^{N} S_k(y) \\ &= (N+1)^{-1}\sin(\pi y)^{-1}\sum_{k=0}^{N}\sin(\pi(2k+1)y) \\ &= (N+1)^{-1}\sin(\pi y)^{-1}(2i)^{-1}\sum_{k=0}^{N}\big(e^{i\pi(2k+1)y} - e^{-i\pi(2k+1)y}\big) \end{aligned}$$

and we're off to the races with two partial sums of geometric series. After invoking the formula twice and cleaning up, one ends up with

$$K_N(y) = (N+1)^{-1}\frac{\sin^2(\pi(N+1)y)}{\sin^2(\pi y)}.$$

Most basic properties of K_N:

- By definition of K_N,

$$\int_{\mathbb{T}} K_N(y)\,dy = (N+1)^{-1}\sum_{k=0}^{N}\int_{\mathbb{T}} D_k(y)\,dy = (N+1)^{-1}\sum_{k=0}^{N} 1 = 1.$$

- $K_N \ge 0$. In this respect, K_N is quite different from D_N.
- $\|K_N\|_{L^1(\mathbb{T})} = 1$ for all N. (Since $K_N \ge 0$, its L^1 norm equals its integral.) Thus the sequence of kernels (K_N) lacks the bad property of (D_N) on which our discussion of divergence of Fourier series of continuous functions relied.

- For any $\delta > 0$,

$$\int_{\delta \le |y| \le \frac{1}{2}} |K_N(y)|\, dy \to 0 \ \text{ as } N \to \infty.$$

(Of course, the absolute value signs on $|K_N|$ are redundant here.) Yet again, the Dirichlet kernels lack this property.

This property is easily verified using the explicit formula for K_N;

$$K_N(y) \le \min(N+1,\, (N+1)^{-1} \sin(\pi y)^{-2})$$

for all y.

Theorem. Let $f \in C^0(\mathbb{T})$. Then $\sigma_N(f) \to f$ uniformly on $\mathbb{T}$. □

This stands in stark contrast to the behavior of $S_N(f)$, for general f. □

Proof.

$$\sigma_N(f)(x) - f(x) = \int_{-1/2}^{1/2} \big(f(x-y) - f(x)\big) K_N(y)\, dy$$

and therefore

$$\big|\sigma_N(f)(x) - f(x)\big| \le \int_{-1/2}^{1/2} \big|f(x-y) - f(x)\big| K_N(y)\, dy.$$

We know what to do: Let $\varepsilon > 0$. Since $\mathbb{T}$ is compact and f is continuous, f is uniformly continuous. So there exists δ such that $|f(x-y) - f(x)| \le \varepsilon$ whenever $y \in [-\frac{1}{2}, \frac{1}{2}]$ satisfies $|y| \le \delta$. Now

$$\begin{aligned}
\int_{-1/2}^{1/2} \big|f(x-y) - f(x)\big| K_N(y)\, dy &\le \int_{|y|\le\delta} \varepsilon K_N(y)\, dy + 2\|f\|_u \int_{|y|>\delta} K_N(y)\, dy \\
&\le \varepsilon \int_{-1/2}^{1/2} K_N(y)\, dy + 2\|f\|_u \int_{|y|>\delta} K_N(y)\, dy \\
&\le \varepsilon + 2\|f\|_u \int_{|y|>\delta} K_N(y)\, dy.
\end{aligned}$$

Therefore by the final property of (K_N) listed above,

$$\limsup_{N\to\infty} \sup_x |\sigma_N(f)(x) - f(x)| \le \varepsilon,$$

for all $\varepsilon > 0$. □

Corollary. The set $\mathcal{P}$ of all trigonometric polynomials $P(x) = \sum_{|k|\le M} c_k e^{2\pi i k x}$ is dense in $L^2(\mathbb{T})$; $\{e^{2\pi i n x} : n \in \mathbb{N}\}$ is a complete orthonormal set in $L^2(\mathbb{T})$. □

Of course, we have already proved this using the Stone-Weierstrass Theorem. But the preceding theorem (whose proof did not rely on knowing this corollary) gives an alternative proof.

Proof. The set of C^0 of all continuous functions is dense in $L^2(\mathbb{T})$. By the Corollary, the set $\mathcal{P}$ is dense in C^0 in the uniform norm, hence is dense in C^0 in the L^2 norm. □