Math 202B — UCB, Spring 2014 — M. Christ
Topic 1: Product Measures
Wednesday January 22 through Monday January 27, 2014

Preliminaries

Definition. Premeasure on an algebra: $\mu \ge 0$, $\mu(\emptyset) = 0$, countably additive whenever disjoint union $\in \mathcal{A}$. (These imply $A \subset B \Rightarrow \mu(A) \le \mu(B)$.)

Definition. μ^* = outer measure associated to a premeasure μ: $\mu^*(E) = \inf \sum_j \mu(A_j)$, infimum over all coverings $E \subset \cup_j A_j$, $A_j \in \mathcal{A}$.

Theorem. Let μ be premeasure on $\mathcal{A}$ and $\mathcal{M}$ = σ–algebra generated by $\mathcal{A}$. Then
(i) $\mu^*|_{\mathcal{M}}$ is a measure, and coincides with μ on $\mathcal{A}$; that is, $\mu^*|_{\mathcal{M}}$ is an extension of μ to $\mathcal{M}$.
(ii) $\mu^*|_{\mathcal{M}}$ is the unique extension of μ to $\mathcal{M}$ if μ is σ–finite on $\mathcal{A}$.

Products

Products are one of the most basic constructions in mathematics. Given a notion of a structure on sets, and given two sets with this additional structure, it is virtually always useful to have a natural way to associate such a structure to the product set. One sees this in vector spaces; Abelian groups, metric spaces, topological spaces, manifolds, ...

Definition. Measurable rectangles $A \times B$ ...

Given measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, we seek to construct a measure, denoted by $\mu \times \nu$, on some σ-algebra $\mathcal{C}$ of subsets of $X \times Y$, with these rather minimal properties:

$$\text{Every measurable rectangle belongs to } \mathcal{C}, \tag{1}$$

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B) \text{ for all measurable rectangles.} \tag{2}$$

Definition. Given measure spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, $\mathcal{M} \otimes \mathcal{N}$ is the σ-algebra generated by the collection of all measurable rectangles.

I'll sometimes write $\mathcal{B}_n$ as shorthand for $\mathcal{B}_{\mathbb{R}^n}$, the Borel σ–algebra; this is the smallest σ–algebra that contains all open subsets of $\mathbb{R}^n$.

Example. Product of two copies of $(\mathbb{R}^1, \mathcal{B}_1)$. What is $\mathcal{B}_1 \otimes \mathcal{B}_1$, in more concrete terms? It contains all open sets — (countable unions of cubes with rational centers and sidelengths), so it contains $\mathcal{B}_2$. Anything else? (No; we'll return to this point in a subsequent lecture.) This example calls attention to the indirect character of the definition of $\mathcal{M} \otimes \mathcal{N}$.

Construction of $\mu \times \nu$

Our next step is to construct a product of two measures.

Definition. $\mathcal{A} = \{$finite disjoint unions of measurable rectangles$\}$.

Facts.
(i) $\mathcal{A}$ is an algebra.
(ii) $\mathcal{A}$ equals the collection of all (not necessarily pairwise disjoint) finite unions of measurable rectangles.

See Prop 1.7 in our text. I won't prove this in detail, but will indicate a couple of key ideas in the proof. (It can be helpful to draw schematic pictures, representing measurable rectangles by genuine rectangles in $\mathbb{R}^2$.)

Two identities are useful in the discussion:

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B')$$
$$(A \times B)^c = (A^c \times Y) \cup (X \times B^c).$$

Here's how to show that a union $E = \cup_{j=1}^3 (A_j \times B_j)$ of three measurable rectangles can be expressed as a finite union of pairwise disjoint ones: First consider X. Partition X into 8 pairwise disjoint measurable subsets, using a standard three set Venn diagram. Among these 8 sets are $A_1 \cap A_2 \cap A_3$; $A_1 \setminus (A_2 \cup A_3)$, et cetera. Denote these 8 sets as $A(S)$, where S ranges over the 8 subsets of $\{1, 2, 3\}$; $A(S)$ is the set of all $x \in X$ such that $x \in A_j$ if and only if $j \in S$. Likewise partition Y into 8 subsets $B(T)$, determined by the Venn diagram associated to B_1, B_2, B_3.

By taking products, obtain a partition of $X \times Y$ into a union of 64 pairwise disjoint measurable rectangles $C(S, T)$. Let $(x, y) \in X \times Y$. Define $S = \{i \in \{1, 2, 3\} : x \in A_i\}$ and $T = \{j : y \in B_j\}$. Then $(x, y) \in C(S, T)$.

If S, T are both nonempty then $C(S, T) \subset E = \cup_{j=1}^3 (A_j \times B_j)$. If $S = \emptyset$ then $x \notin \cup_j A_j$, and if $T = \emptyset$ then $y \notin \cup_j B_j$. Thus E is equal to a union of $7 \times 7 = 49$ of the sets $C(S, T)$. □

This proof works fine for unions of arbitrarily many measurable rectangles; just change 3 to N and 8 to 2^N ... □

Now let measures μ, ν on $(X, \mathcal{M})$, $(Y, \mathcal{N})$ respectively be given. For $E \in \mathcal{A}$ define

$$\rho(E) = \int_X \nu(E_x) \, d\mu(x).$$

A useful alternative formula is

$$\rho(E) = \int_X \Big(\int_Y \mathbf{1}_E(x, y) \, d\nu(y) \Big) \, d\mu(x).$$

In particular, $\rho(A \times B) \equiv \mu(A)\nu(B)$.

If $E \in \mathcal{A}$ is expressed as a disjoint union $E = \cup_j (A_j \times B_j)$, then the inner integral equals $\cup_j \mathbf{1}_{A_j} \nu(B_j)$, which is a measurable function of x; therefore the outer integral is defined. Thus $\rho(E)$ is well-defined, and obviously $= \sum_j \mu(A_j)\nu(B_j)$.

ρ is obviously additive on disjoint sets in $\mathcal{A}$ since $\mathbf{1}_{E_1 \cup E_2} = \mathbf{1}_{E_1} + \mathbf{1}_{E_2}$ and the integral of a sum equals the sum of the integrals ...

Fact. ρ is countably additive on $\mathcal{A}$, hence is a premeasure.[1]

Proof. We will show that if $E_j \in \mathcal{A}$, if $E \in \mathcal{A}$, and if $E_j \nearrow E$, then $\rho(E_j) \to \rho(E)$. This together with the finite additivity of ρ establishes countable additivity (easy exercise).

[1] "Countably additive" refers to countable disjoint unions *that happen to belong to* $\mathcal{A}$; it is not claimed that an arbitrary countable disjoint union of elements of $\mathcal{A}$ belongs to $\mathcal{A}$.

For each (x,y), $\mathbf{1}_{E_j}(x,y) \nearrow \mathbf{1}_E(x,y)$. Therefore by the Monotone Convergence Theorem,

$$f_j(x) = \int_Y \mathbf{1}_{E_j}(x,y)\,d\nu(y) \nearrow \int_Y \mathbf{1}_E(x,y)\,d\nu(y) = f(x)\ \forall x.$$

Moreover, f_j, f are obviously measurable; if $E = \cup_i (A_i \times B_i)$ then $f(x) = \sum_i \mathbf{1}_{A_i}(x)\nu(B_i)$; likewise for E_j, f_j. By monotone convergence once more,

$$\rho(E_j) = \int_X f_j\,d\mu \to \int_X f\,d\mu = \rho(E).$$

□

Definition. $\mu \times \nu$ is the measure obtained by restricting the outer measure ρ^* to $\mathcal{M} \otimes \mathcal{N}$. Theory of outer measures/premeasures above guarantees that the restriction of ρ^* to $\mathcal{A}$ agrees with ρ, so

$$(\mu \times \nu)(A \times B) \equiv \mu(A)\nu(B)\ \ \forall A \in \mathcal{M},\ B \in \mathcal{N}.$$

Recall that if $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are σ–finite, then $\mu \times \nu$ is the unique extension which agrees with ρ on $\mathcal{A}$. Every set $E \in \mathcal{A}$ can be expressed as a finite *disjoint* union of measurable rectangles. Therefore $(\mu \times \nu)(E)$ is uniquely determined by the requirement that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$. Thus for products of two σ-finite measure spaces, there can be at most one measure on $\mathcal{M} \otimes \mathcal{N}$ that satisfies (2). We have constructed it. The task remaining is to verify that it has the properties stated in the Tonelli/Fubini theorems.

One might debate whether this *construction* is natural, but in the σ–finite case at least, it produces the unique extension with the natural *property* that the measure of a product of two sets equals the product of their measures.

This is more or less the end of the Wednesday 1/22 lecture. Please beware that lecture notes are not carefully written and are barely proofread. They are not intended as a replacement for our excellent text.

Fubini's and Tonelli's Theorems

The main issue in the development of the foundation of the theory of product measures is not the construction of $\mathcal{M} \otimes \mathcal{N}$ and of $\mu \times \nu$, but rather, the proof that these have useful properties. Nearly everything one needs to know about product measures for most mathematical practice is contained in the theorems of Tonelli and Fubini (and the variant for complete measures).

Definition. E_x, E^y (These sets are called *slices.*); f_x, f^y ...

Tonelli's Theorem. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ–finite measure spaces. Let $E \in \mathcal{M} \otimes \mathcal{N}$, and let $f : X \times Y \to [0, \infty]$ be $\mathcal{M} \otimes \mathcal{N}$–measurable.
(i) $E^y \in \mathcal{M}$ for all y and $E_x \in \mathcal{N}$ for all x. Likewise f_x is $\mathcal{N}$–measurable for every x; f^y is $\mathcal{N}$–measurable for every y.
(ii) $\int_Y f_x\,d\nu(y)$ is an $\mathcal{M}$–measurable function of x; $\int_X f_y\,d\mu(x)$ is an $\mathcal{N}$–measurable function

of y.
(iii)

$$\int_X \Big(\int_Y f(x,y)\,d\nu(y) \Big)\,d\mu(x) = \int_Y \Big(\int_X f(x,y)\,d\mu(x) \Big)\,d\nu(y)$$
$$= \iint_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y). \quad (3)$$

□

In particular, this gives two ways to compute an integral with respect to $\mu\times\nu$ — and thus two ways to compute the measure of an $\mathcal{M}\otimes\mathcal{N}$–measurable set:

$$(\mu\times\nu)(E) = \int_X \Big(\int_Y \mathbf{1}_E(x,y)\,d\nu(y) \Big)\,d\mu(x).$$

Note that the first equality alone does not directly refer to product measure. Its proof does go through the product theory, though.

Fubini's Theorem. For two σ–finite measure spaces and any $f \in L^1(\mu\times\nu)$,
(i) f_x, f^y are measurable functions on Y, X for almost every x, y, respectively.
(ii) $f_x \in L^1(Y,\nu)$ for μ–almost every $x \in X$; $f^y \in L^1(X,\mu)$ for ν–almost every $y \in Y$.
(iii) $\int_Y f_x\,d\nu(y)$ defines a measurable function of x, which belongs to $L^1(X,\mu)$; likewise if the roles of the variables are interchanged.
(iv) Equations (3) hold. □

These are often used in tandem. If we want to apply Fubini to a function f to compute $\iint f\,d\mu\times\nu$, we need to verify that $f \in L^1(\mu\times\nu)$. For this, Tonelli's theorem can potentially be useful. Apply Tonelli to the nonnegative function $|f|$, and evaluate one of the two iterated integrals — whichever happens to be more convenient — to show that $\iint |f|\,d(\mu\times\nu) < \infty$. Thus $f \in L^1$, and now Fubini can be applied to f.

One example. Let $X = Y = \{0, 1, 2\ldots\}$, with counting measure and with all sets measurable. Define $f(x,y) = 1$ if $y = x$, $= -1$ if $y = x+1$, and $= 0$ otherwise. Then $\int_Y f_x\,d\nu = 0$ for every x, so $\int_X \int_Y f\,d\nu(y)\,d\mu(x) = 0$. On the other hand $\int_X f^y\,d\mu = 0$ for every $y \ge 1$, and $= 1$ for $y = 0$, so $\int_Y \int_X f\,d\mu(x)\,d\nu(y) = 1$. Put another way,

$$\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} f(x,y) \ne \sum_{y=0}^{\infty}\sum_{x=0}^{\infty} f(x,y).$$

Proof

Begin with proof of Tonelli's Theorem. **Recurrent theme:** All we know about $\mathcal{M}\otimes\mathcal{N}$ is that it is contained in any σ-algebra that contains $\mathcal{A}$. So apparently the only viable strategy for proving that every set in $\mathcal{M}\otimes\mathcal{N}$ has some property $\mathcal{P}$, is to prove that

Every set in $\mathcal{A}$ has $\mathcal{P}$,

The class of all sets with $\mathcal{P}$ contains some σ-algebra.

One way to prove that a class contains a σ-algebra, is to prove that it *is* a σ-algebra.

Put another way: Instead of proving that every element of a σ–algebra $\mathcal{D}$ has a desired property, we instead show that the collection of all sets having the desired property is a σ–algebra; then exploit the minimality of $\mathcal{D}$. This is a fundamental proof technique in this sub-subject. It only applies if $\mathcal{D}$ is minimal among σ–algebras containing some specified collection of sets.

Proof that $E_x \in \mathcal{N}$: Consider

$$\mathcal{C} = \{E \subset X \times Y : E_x \in \mathcal{N} \text{ for all } x\}.$$

We know $\mathcal{A} \subset \mathcal{C}$, so it suffices to prove that $\mathcal{C}$ contains a σ-algebra. We'll show that $\mathcal{C}$ *is* a σ-algebra.

Certainly $\mathcal{C}$ contains $\emptyset$. Since $Y \setminus E_x = (Y \setminus E)_x$, $\mathcal{C}$ is closed under complementation. If $E_j \subset X \times Y$ then (with the obvious notation)

$$\cup_j (E_j)_x = \big(\cup_j E_j\big)_x. \tag{4}$$

Consider any x. If $E_j \in \mathcal{C}$ then $(E_j)_x \in \mathcal{N}$ for all j, so since $\mathcal{N}$ is closed under countable unions, $\cup_j (E_j)_x \in \mathcal{N}$. So $\mathcal{C}$ is closed under countable unions. □

The corresponding statement for functions follows easily. First treat simple functions, then use the usual approximation-from-below scheme for nonnegative functions together with the fact that a pointwise limit of measurable functions is measurable. Then use $f = f^+ - f^-$. □

Proof that $x \mapsto \nu(E_x)$ defines an $\mathcal{M}$–measurable function of $x \in X$: Let $\mathcal{C}$ be the class of sets $E \subset X \times Y$ such that $x \mapsto \nu(E_x)$ is a measurable function. Obviously $\emptyset \in \mathcal{C}$. $\mathcal{C} \supset \mathcal{A}$, as observed above, so it suffices to prove that $\mathcal{C}$ is a σ-algebra (again!).

Since X, Y are σ-finite, an easy argument (omitted) reduces matters to the case where $\mu(X), \nu(Y) < \infty$. Since $(E^c)_x = (E_x)^c$,

$$\nu((E^c)_x) = \nu((E_x)^c) = \nu(Y) - \nu(E_x)$$

and the right-hand side is a measurable function because it is a difference of two *finite* measurable functions. So $E \in \mathcal{C} \Rightarrow E^c \in \mathcal{C}$.

$\mathcal{C}$ is obviously closed under *disjoint* unions: Given $E, E' \in \mathcal{C}$ disjoint, note that $(E \cup E')_x$ equals the disjoint union $E_x \cup E'_x$, so

$$\nu((E \cup E')_x) = \nu(E_x \cup E'_x) = \nu(E_x) + \nu(E'_x).$$

This is a sum of two measurable functions and hence is measurable.

And $\mathcal{C}$ is closed under countable ascending unions: If $E_j \in \mathcal{C}$ and $E_j \subset E_{j+1}$ for all j then $E = \cup_{j=1}^{\infty} E_j \in \mathcal{C}$; E_x equals the ascending union of the sequence $(E_j)_x$, and $\nu((E_j)_x) \to \nu(E_x)$ by a basic property of measures. Since any pointwise limit of a sequence of measurable functions is measurable, $x \mapsto \nu(E_x)$ is measurable, so $E \in \mathcal{C}$.

But what if $E, E' \in \mathcal{C}$ are not necessarily disjoint? Then $E \cup E'$ equals the disjoint union $E \cup (E' \setminus E)$. The second set in this union is the issue. Of course $E' \setminus E = E' \cap E^c$ is the intersection of two sets in $\mathcal{C}$. Can we treat intersections? There is no obvious formula for $\nu((E \cap E')_x)$ in terms of $\nu(E_x)$ and $\nu(E'_x)$. So we seem to be stuck. □

This difficulty forces a detour. I said earlier that there is only one viable strategy, but that's not accurate.

Detour: Monotone Classes, and the Monotone Class Lemma

Definition. Let Z be a set. A subset $\mathcal{C} \subset \mathcal{P}(Z)$ is a *monotone class* if $\mathcal{C}$ is closed under formation of countable ascending unions, and of countable descending intersections. (Nothing is said about complements or differences of sets in this definition.)

Any collection of sets in $\mathcal{P}(Z)$ is contained in a unique smallest monotone class, because the intersection of any collection of monotone classes in $\mathcal{P}(Z)$ is itself a monotone class. (Straightforward verification omitted.)

Lemma. [*Monotone Class Lemma*][2] If $\mathcal{A}$ is an algebra of sets, then the smallest monotone class $\mathcal{C}$ containing $\mathcal{A}$ equals the smallest σ-algebra containing $\mathcal{A}$. □

This is a statement in set theory alone, not measure theory. Complements/differences enter through the assumption that $\mathcal{A}$ is an algebra.

Example. Let X be any set with at least 3 points. Let $\mathcal{C} \subset \mathcal{P}(X)$ be the collection of all subsets of X that contain exactly 2 points. If $A \subset B$ are two sets in X, then necessarily $B = A$! Therefore all ascending chains, and all descending chains, are trivial and consequently $\mathcal{C}$ is a monotone class. But $\mathcal{C}$ is certainly not a σ–algebra.

This is more or less the end of the Friday 1/24 lecture.

Proof. Any σ-algebra is a monotone class, so the smallest σ-algebra $\mathcal{A}'$ containing $\mathcal{A}$ contains the smallest monotone class $\mathcal{C}$ containing $\mathcal{A}$. We need to show that $\mathcal{C}$ contains $\mathcal{A}'$. It suffices to prove that $\boxed{\mathcal{C} \text{ is a } \sigma\text{-algebra.}}$

Claim: $\mathcal{D} = \{A \subset Z : A^c \in \mathcal{C}\}$ is a monotone class. Proof: If for all j $A_j \subset A_{j+1}$ and $A_j^c \in \mathcal{C}$, then $(\cup_j A_j)^c = \cap_j A_j^c$ is a monotone intersection of elements of $\mathcal{C}$, so belongs to $\mathcal{C}$, so $\cup_j A_j \in \mathcal{D}$. In the same way it follows that $\mathcal{D}$ is closed under descending intersections.

Since $\mathcal{D} \supset \mathcal{A}$, $\mathcal{D} \supset \mathcal{C}$, which is the smallest monotone class containing $\mathcal{A}$. By definition of $\mathcal{D}$, we have shown that $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$.

This sets the pattern for all other steps of the proof of the Lemma, but there is a wrinkle. To prove closure under countable unions, we start with the simple matter of a union of two sets, *and* require one of the two to belong to $\mathcal{A}$.

Let $A \in \mathcal{A}$. Let $\mathcal{D}_A = \{B \subset Z : A \cup B \in \mathcal{C}\}$. Claim: $\mathcal{D}_A$ is a monotone class. If the claim is proved then we will have shown that

$$B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C} \quad \text{for all } A \in \mathcal{A}. \tag{5}$$

To prove the claim, let $B_1 \subset B_2 \subset \dots$ with $B_j \in \mathcal{D}_A$. $A \cup (\cup_j B_j) = \cup_j (A \cup B_j)$ is an ascending union of elements of $\mathcal{C}$, hence belongs to the monotone class $\mathcal{C}$. Descending intersections are handled in the same way, proving the claim.

[2]This slightly resembles *Dynkin's* $\pi - \lambda$ *Theorem*, a commonly used tool in probability theory, but it's not the same thing.

Now let $A \in \mathcal{C}$ and again consider $\mathcal{D}_A = \{B \subset Z : A \cup B \in \mathcal{C}\}$. Now

$$(5) \text{ says that } \mathcal{A} \subset \mathcal{D}_A.$$

Indeed, if $B \in \mathcal{A}$ then $A \cup B = B \cup A$ is the union of a set in $\mathcal{A}$ with a set in $\mathcal{C}$; we showed in the preceding step that any such union belongs to $\mathcal{C}$.

The reasoning that gave (5) can now be repeated to prove that $\mathcal{D}_A$ is a monotone class. We have proved this for every $A \in \mathcal{C}$, so

$$B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C} \;\text{ for all } A \in \mathcal{C},$$

the improvement being that $A \in \mathcal{C}$ rather than only $A \in \mathcal{A}$.

It remains only to prove that $\mathcal{C}$ is closed under countable unions. Closure under unions of two sets implies closure under arbitrary finite unions. The monotonicity hypothesis immediately leads to closure under countable unions. □

Proof of Tonelli and Fubini Theorems

Continue to assume that μ, ν are σ–finite measures. We need to show that if $E \in \mathcal{M} \otimes \mathcal{N}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{M}$–measurable function. Let $\mathcal{C}$ be the class of all $E \subset X \times Y$ such that $E_x \in \mathcal{N}$ for all $x \in X$ and $x \mapsto \nu(E_x)$ is measurable. We have shown that $\mathcal{C}$ is closed under ascending unions, and under complementation. Since the complement of a decreasing intersection is an ascending union of complements, it follows that $\mathcal{C}$ is closed under descending unions. Thus $\mathcal{C}$ is a monotone class. Obviously $\mathcal{C}$ contains the algebra $\mathcal{A}$ of all measurable rectangles. Therefore by the Monotone Class Lemma, $\mathcal{C}$ contains some σ–algebra which contains $\mathcal{A}$. □

Let's prove $\int_X \nu(E_x)\, d\mu(x) = (\mu \times \nu)(E)$ for all $E \in \mathcal{M} \otimes \mathcal{N}$. Equivalently,

$$\int_X \Big(\int_Y \mathbf{1}_E(x,y)\, d\nu(y) \Big)\, d\mu(x) = \iint_{X \times Y} \mathbf{1}_E(x,y)\, d(\mu \times \nu)(x,y).$$

This equation is already known to hold for all $E \in \mathcal{A}$.

The set of all E for which it holds is a monotone class. For ascending unions this is a direct consequence of the Monotone Convergence Theorem. For descending intersections it follows by passing to complements and invoking the result for ascending unions; this works because μ, ν and hence also $\mu \times \nu$ are finite measures ...

Therefore it holds for all E in the smallest monotone class containing $\mathcal{A}$, therefore for all E in the smallest σ-algebra containing $\mathcal{A}$, which by definition is $\mathcal{M} \otimes \mathcal{N}$.

The roles of the two variables can be interchanged, so the other identity in Tonelli's theorem is also proved. □

To prove Fubini's Theorem, let $f \in L^1(X \times Y, \mu \times \nu)$. Express $f = f^+ - f^-$ where $f^\pm$ are nonnegative and this is the canonical decomposition we learned about in Math 202A. Then $f^\pm \in L^1$. Apply Tonelli to $f^\pm$, and reap all of its conclusions.

Because $\int_X (\int_Y f^\pm(x,y)\, d\nu(y))\, d\mu(x) = \iint f^\pm\, d(\mu \times \nu)$, we conclude that

$$\int_X (\int_Y f^\pm(x,y)\, d\nu(y))\, d\mu(x) < \infty$$

and consequently that

$$\int_Y f^{\pm}(x,y)\,d\nu(y) < \infty \quad \text{for } \mu\text{–almost every } x \in X.$$

Thus for almost every x, f_x^+, f_x^- both belong to $L^1(Y,\nu)$, and therefore

$$f_x \in L^1(Y,\nu) \quad \text{for } \mu\text{–almost every } x \in X.$$

Therefore $F(x) = \int_Y f(x,y)\,d\nu(y)$ is well-defined and finite for almost every x. Moreover

$$F(x) = \int_Y f^+(x,y)\,d\nu(y) - \int_Y f^-(x,y)\,d\nu(y) = F^+(x) - F^-(x)$$

is a difference of two measurable functions, each of which belongs to $L^1(X,\mu)$ and in particular, is finite almost everywhere. Therefore F is measurable, $F \in L^1$, and

$$\begin{aligned}
\int_X F\,d\mu &= \int_X F^+\,d\mu - \int_X F^-\,d\mu \\
&= \iint_{X\times Y} f^+\,d(\mu\times\nu) - \iint_{X\times Y} f^-\,d(\mu\times\nu) \\
&= \iint_{X\times Y} f\,d(\mu\times\nu).
\end{aligned}$$

Note. We have used the theory $\big[$premeasures $\rightsquigarrow$ outer measures $\rightsquigarrow$ measures$\big]$ to construct $\mu\times\nu$, but that theory can be easily avoided. The above argument shows that for any $E \in \mathcal{M}\otimes\mathcal{N}$, $x \mapsto \nu(E_x)$ is a well-defined measurable function. Define $(\mu\times\nu)(E)$ to be $\int_X \nu(E_x)\,d\mu$. It is easy to verify that this is a measure. If the general theory is not to be used, then one needs to add a proof of equality with $\int_Y \mu(E^y)\,d\nu(y)$. But this is also a measure, and the collection $\mathcal{C}$ of all sets on which it agrees with $\mu\times\nu$, is a σ-algebra. Hence the two measures agree on $\mu\times\nu$.

Complete measure spaces, and products

Suppose that $(X,\mathcal{M},\nu)$ and $(Y,\mathcal{N},\nu)$ are complete measure spaces. What about the product space?

The product is rarely complete. Suppose $\nu(Y) > 0$, X contains some nonempty measurable null set A, and Y contains some nonmeasurable set B. This situation arises in the basic case where both measure spaces are $\mathbb{R}^1$ with the Lebesgue measurable sets and Lebesgue measure. The product set $E = A\times B$ is not $\mu\times\nu$ measurable, because for any $a \in A$, $E_a = B$ is not $\mathcal{N}$–measurable, violating a conclusion of Fubini's Theorem. But $E \subset A\times Y$, which is a measurable rectangle, so $(\mu\times\nu)(A\times Y) = \mu(A)\nu(Y) = 0$. Therefore $\mu\times\nu$ is not complete.

This is not a serious obstacle to a satisfactory theory. Let's just complete $(X\times Y, \mathcal{M}\otimes\mathcal{N}, \mu\times\lambda)$; let $(X\times Y, \mathcal{L}, \lambda)$ be its completion.

The following situation comes up in the statement of the theorem. Recall that if g is a function defined almost everywhere on a complete metric space $(X, \mathcal{M}, \mu)$, then if we define $G(x) = g(x)$ wherever g is defined, and $G(x) =$ any value at all otherwise, then G is measurable. Moreover, whether G belongs to L^1 depends only on g, not on the arbitrary choice made in defining G; and if $G \in L^1$ then $\int_X G\,d\mu$ depends only on g, not on those choices.

Theorem. Let $(X, \mathcal{M}, \nu)$ and $(Y, \mathcal{N}, \nu)$ be complete σ-finite measure spaces. Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. Then the conclusions of Fubini's/Tonelli's theorems hold for all $\mathcal{L}$–measurable functions f, with these changes: One can only conclude that f_x is measurable for μ–almost every x, and f^y is measurable for ν–almost every y. □

In this situation, the function $\int_Y f_x\,d\nu(y)$ is well-defined only μ–almost everywhere, but the other conclusions still make sense because of the considerations two paragraphs above.

This version of the Tonelli/Fubini theorem follows from the version developed above together with the following lemma. One reduces the statement for functions to a corresponding statement for sets. Since any $\mathcal{L}$–measurable set is the union of a $\mathcal{M} \otimes \mathcal{N}$–measurable set with a subset of a $\mathcal{M} \otimes \mathcal{N}$–null set, it suffices to prove the following.

Lemma. Let $E \subset X \times Y$ be $\mathcal{M} \otimes \mathcal{N}$–measurable and assume that $(\mu \times \nu)(E) = 0$. Let $E' \subset E$. Then for almost every $x \in X$, $E'_x \in \mathcal{N}$ and $\nu(E'_x) = 0$. □Therefore the function $x \mapsto \nu(E'_x)$ vanishes for almost every $x \in X$. Since μ is complete, this is a measurable function and its integral vanishes.

This is immediate from Tonelli's Theorem. For almost every $x \in X$, E'_x is a subset of the null set E_x. Since $(Y, \mathcal{N}, \nu)$ is complete, $E'_x \in \mathcal{N}$ for almost every x and $0 \le \nu(E'_x) \le \nu(E_x) = 0$. □

This is more or less the end of the Monday 1/27 lecture. The material on products of complete measure spaces was discussed only very sketchily. Students should study this subtopic in our text.

Comment — The Borel Hierarchy

Here is a fundamental point concerning the very nature of the Borel sets. Let $X = \mathbb{R}^1$. Let $\mathcal{G}$ be the class of all open sets. Form the classes $\mathcal{G}_\delta$ (all countable intersections of open sets; this includes all open set, all closed sets, and more), $\mathcal{G}_{\sigma\delta}$ (all countable unions of sets in $\mathcal{G}_\delta$), $\mathcal{G}_{\delta\sigma\delta}$ (all countable intersections of sets in $\mathcal{G}_{\sigma\delta}$), *and so forth.* This produces an ascending collection of subsets of the Borel σ-algebra $\mathcal{B} \subset \mathcal{P}(\mathbb{R}^1)$.

A natural way to try understand the Borel subsets of $\mathbb{R}^1$ is in terms of this hierarchy of sets. However, Lebesgue himself proved that this hierarchy is a bit complicated: No matter how many times one iterates this construction, it never stabilizes. This is a subtler statement than it may at first encounter appear to be. A precise formulation requires the theory of *ordinal numbers*, and in particular, of the countable ordinals. See §0.4 of our text.

Define $\mathcal{G}_1$ to be the collection of all open sets, and inductively define

$$\mathcal{G}_{n+1} = (\mathcal{G}_n)_{\delta\sigma}.$$

Lebesgue showed that $\mathcal{G}_{n+1}$ properly contains $\mathcal{G}_n$ for all n. So define

$$\mathcal{G}_\omega = \cup_n \mathcal{G}_n$$

in order to incorporate all of these. Is $\mathcal{G}_\omega$ closed under countable unions? If so, then (it is easily seen to be closed under complements) it would be the Borel σ-algebra.

Suppose that $E_n \in \mathcal{G}_n$. Form $E = \cup_n E_n$. Does E necessarily belong to $\mathcal{G}_\omega$? Lebesgue proved that in general, it does not. Therefore $\mathcal{B}$ also contains $(\mathcal{G}_\omega)_{\delta\sigma}$. Call this set $\mathcal{G}_{\omega+1}$. Inductively define $\mathcal{G}_{\omega+n+1}$ as above. Form $\mathcal{G}_{2\omega} = \cup_{n=1}^\infty \mathcal{G}_{\omega+n}$. Is $\mathcal{G}_{2\omega}$ closed under countable unions? Nope (says Lebesgue). Inductively form $\mathcal{G}_{3\omega}, \ldots \mathcal{G}_{n\omega}$ for all $n \in \mathbb{N}$. Go ahead and form $\mathcal{G}_{\omega\cdot\omega} = \cup_{n=1}^\infty \mathcal{G}_{n\omega}$. The total number of steps we have taken is still countable.

Lebesgue proved that iterating this process for any countable number of steps never produces a collection of sets that is closed under countable unions. More precisely: To each countable ordinal α is associated a set $\mathcal{G}_\alpha$ of this type. Lebesgue showed that for any countable ordinal α, $\mathcal{G}_{\alpha+1}$ strictly contains $\mathcal{G}_\alpha$.

Form the union of $\mathcal{G}_\alpha$ over all countable ordinals (this requires transfinite induction; see Folland). The good news is that this union does equal $\mathcal{B}$. The bad news is that there are uncountably many countable ordinals. Thus it takes uncountably many operations to construct all Borel sets from the open sets.

Does this complication arise in mathematical practice? It does arise, in that it forces us to proceed indirectly in discussing Borel σ-algebras.[3] Yet it also fails to arise, in the sense that sets encountered in any direct way tend never to be very high in this hierarchy; $\mathcal{G}_{\delta\sigma\delta\sigma\delta}$ probably suffices for any sets I've ever encountered in any direct way in the contexts of Fourier transform or partial differential equations. A simple example: Any Lebesgue measurable set can be expressed as $E = F \cup A$ where F is a countable union of closed sets, and A is a Lebesgue null set. For most purposes, E is indistinguishable from F, which is in $\mathcal{G}_{\delta\sigma}$. This is why one does not often find much discussion of the Borel hierarchy in introductory texts on measure and integration.

[3] Or to prove everything by means of transfinite induction.