Mathematics 202B, Spring 2014 — M. Christ
Final Examination Solutions

Except where otherwise indicated, problems are set in a general measure space $(X, \mathcal{A}, \mu)$. (I meant to say that $(X, \mathcal{A}, \mu)$ is always assumed to be σ–finite, but I forgot to write that.) d is any dimension ≥ 1, and m denotes Lebesgue measure on $\mathbb{R}^d$. $L^p = L^p(X, \mathcal{A}, \mu)$ is a collection of equivalence classes of functions under the equivalence relation of equality almost everywhere with respect to μ. $\|f\|_p = \|f\|_{L^p} = \|f\|_{L^p(X,\mathcal{A},\mu)}$.

(1a) State the Riesz Representation Theorem for bounded complex-valued linear functionals.

Solution. Let X be a locally compact Hausdorff space. Let $C_0(X)$ be equipped with the supremum norm. Let $\mathcal{M}(X)$ be the space of all complex Radon measures on X. Let $\ell : C_0(X) \to \mathbb{C}$ be a bounded (complex) linear functional. Then there exists a unique $\mu \in \mathcal{M}(X)$ such that $\ell(f) = \int_X f\,d\mu$ for all $f \in C_0(X)$. Moreover, $\|\mu\|_{\mathcal{M}(X)} = \|\ell\|_{C_0(X)^*}$. Finally, this correspondence between elements of $(C_0(X))^*$ and $\mathcal{M}(X)$ is a bijection. □

(One could also define Radon measures, but I did not insist on that. A complex Radon measure is a complex measure on $\mathcal{B}(X)$, the smallest σ–algebra generated by the open subsets of X, that is of the form $\mu = \nu_+ - \nu_- + i\lambda_+ - i\lambda_-$, where $\nu_\pm$ and $\lambda_\pm$ are finite positive Radon measures; they are outer regular on arbitrary Borel sets, and inner regular on all open sets.) □

(1b) In the proof of the Riesz Representation Theorem for positive linear functionals, one constructs a measure by first defining a set function on arbitrary sets, then proving that this function is a measure on a certain σ–algebra. What is the definition of this set function?

Solution. First, for any open set $\mathcal{O}$ define $\nu(\mathcal{O}) = \sup_{f \prec \mathcal{O}} \ell(f)$, where $f \prec \mathcal{O}$ means that $f : X \to [0, \infty)$ is continuous, $f(x) \leq 1$ for all $x \in X$, and the support of f is contained in $\mathcal{O}$. Second, for any set $E \subset X$ define $\rho(E) = \inf_{\mathcal{O} \supset E} \nu(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ containing E. This ρ is the set function in question. □

(1c) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. How is $\mathcal{A} \times \mathcal{B}$ defined? (I goofed; this is called $\mathcal{A} \otimes \mathcal{B}$ by some authors, including ours.)
How is $\mu \times \nu$ defined?

Solution. $\mathcal{A} \times \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$ is defined to be the smallest σ–algebra of subsets of $X \times Y$ that contains all measurable rectangles. A measurable rectangle is a subset of the form $A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

$\mu \times \nu$ is defined as follows: For any set $E \subset X \times Y$, define

$$\rho^*(E) = \inf_{\{(A_j, B_j)\}} \sum_j \mu(A_j)\nu(B_j)$$

where the infimum is taken over all countable families of measurable rectangles $A_j \times B_j$ whose union contains E.

One proves that ρ^* is an outer measure, and that every measurable rectangle is measurable in the sense of Caratheodory with respect to ρ^*. A theorem of Caratheodory associates to any outer measure a σ–algebra, such that the restriction of ρ^* to that σ–algebra is a measure. The restriction of ρ^* to $\mathcal{A} \otimes \mathcal{B}$ is defined to be $\mu \times \nu$. □

(1d) *Briefly outline* an example of a compact set $K \subset \mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $m(K) = 0$ but $m(f(K)) > 0$. (You need not prove that your example is correct.)

Solution. Let $\mathcal{C} \subset [0, 1]$ be the Cantor set defined by successively removing middle thirds of intervals. After n steps of the construction of $\mathcal{C}$, $[0, 1]$ is divided into 2^n open intervals I_j^n, where I_j^n lies to the left of I_{j+1}^n for each $j \in \{1, 2, \ldots, 2^n - 1\}$, together with the compact set $\mathcal{C}_n = [0, 1] \setminus \cup_{j=1}^{2^n} I_j^n$. Prove that there exists a unique nondecreasing function $g : [0, 1] \to [0, 1]$ that satisfies $g(x) \equiv j2^{-n}$ for all $x \in I_j^n$, and that g is continuous. Define $f(x) = g(x) + x$. Then $g([0, 1]) = [0, 2]$, g is strictly increasing, and $g([0, 1] \setminus \mathcal{C})$ can be shown to have Lebesgue measure 1, so $g(\mathcal{C})$ must have Lebesgue measure equal to 1. But of course $m(\mathcal{C}) = 0$. □

(1e) List two major theorems of this course whose proofs either directly relied on Zorn's Lemma or the Axiom of Choice, or used other results whose proofs relied directly on one of these.

Solution. (i) Alaoglu's Theorem relied on Tychonoff's theorem (about the compactness of arbitrary products of compact topological spaces), which in turn relied directly on the Axiom of Choice.

(ii) Our proof of the Hahn-Banach Theorem relied on Zorn's Lemma. □

(1f) Let X be a normed vector space and let X^* be its dual space. Define the weak* topology.

Solution. This is a topology on X^*. Open sets are all unions of finite intersections of sets

$$V_{\ell_0, g, \delta} = \{\ell \in X^* : |\ell(g) - \ell_0(g)| < \delta\}.$$

□

(1g) Give an example of a sequence of Radon measures μ_n on $\mathbb{R}$ such that $\mu_n \to 0$ vaguely, but $\|\mu_n\|_{\mathcal{M}} \geq 1$ for all n.

Solution. $\mu_n(E) = m(E \cap [n, n+1])$ for $n \in \mathbb{N}$. □

(1h) Define the Schwartz space $\mathcal{S}$. Define the (standard) topology on this space.
Solution. $\mathcal{S}$ is the set of all infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{C}$ with the property that for each α and each M there exists $C = C_{\alpha,M,f}$ such that $|\partial_x^\alpha f(x)| \le C(1 + |x|)^{-M}$ for all $x \in \mathbb{R}^d$. ($\mathcal{S}$ is given a vector space structure via pointwise addition and scalar multiplication.)

$\mathcal{S}$ has the topology defined by the countable family of seminorms

$$\|f\|_{k,M} = \sup_{x \in \mathbb{R}^d} \sum_{0 \le |\alpha| \le M} (1 + |x|)^M |\partial_x^\alpha f(x)|.$$

This is the smallest topology that contains all sets

$$V_{g,k,M,\delta} = \{f \in \mathcal{S} : \|f - g\|_{k,M} < \delta\}.$$

□

(1i) The proof of an important result of this course relied on the fact that the second derivative of $\varphi(x) = e^x$ is nonnegative. State that result.
Solution. Hölder's inequality: If $p \in [1, \infty]$ and $q = p' = p/(p-1)$ ($q = \infty$ if $p = 1$; $q = 1$ if $p = \infty$) then $fg \in L^1$ and $|\int fg\,d\mu| \le \|f\|_{L^p}\|g\|_{L^q}$. □

(The convexity of e^x is not needed to treat the cases $p = 1, \infty$ of Hölder's inequality. I didn't quibble about this distinction.)

(1j) Let $p \in [1, \infty)$ and let $f, g \in L^p(\mathbb{R})$. Define: g is a strong L^p derivative of f.
Solution. g is a strong L^p derivative of f if

$$y^{-1}(\tau_{-y} f - f) \to g \text{ in } L^p \text{ norm as } y \to 0.$$

□

(One can make this definition either for genuine functions or for equivalence classes of functions; changing f, g on sets of measure zero has no effect in the definition.)

(1k) State Chebyshev's inequality.
Solution. Let $p \in [1, \infty)$ and $f \in L^p = L^p(X, \mathcal{A}, \mu)$. Then for any $\alpha \in (0, \infty)$,

$$\mu(\{x : |f(x)| > \alpha\}) \le \alpha^{-p}\|f\|_p^p.$$

□

(1l) In this course we proved that the set of all infinitely differentiable functions with compact supports is dense in $L^p(\mathbb{R}^d)$ for $1 \le p < \infty$, by deducing this from the simpler fact that $C_c(\mathbb{R}^d)$ is dense in L^p. What technique or techniques was/were used in this deduction? (Short answer; a few words suffices.)
Solution. Convolution with C^∞ (compactly supported) functions. □

(One could also mention the procedure of forming the family $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$, but I view that as a subsidiary idea and I gave full credit for mention of convolution with C^∞ functions.)

(2) Show that convolution is associative. That is, for any $f, g, h \in L^1(\mathbb{R}^d, \mathcal{B}, m)$, $(f * g) * h = f * (g * h)$ almost everywhere.
∘ *I have simplified your task slightly by assuming Borel measurability.*
∘ *Recall that* $(f * g)(x) = \int f(x-y)g(y)\,dm(y)$.
∘ *You may use facts shown in the course concerning convolutions of two* L^1 *functions, such as that the integral defining such a convolution is absolutely convergent for almost every* $x \in \mathbb{R}^d$*, and defines an* L^1 *function of* x.
∘ *Hint: Use Fubini's Theorem.*

Solution. Choose representatives f, g, h of the associated equivalence classes. These are Borel measurable functions, defined at every point of $\mathbb{R}^d$. Let $x \in \mathbb{R}^d$. Consider the functions $\Phi_x : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ and $\Psi_x : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ defined to be

$$\Phi_x(y, z) = f(x - y - z)g(y)h(z) \quad \text{and} \quad \Psi_x(u, v) = f(x - u)g(u - v)h(v).$$

Each is a product of three factors, and each factor is a Borel measurable function, being the composition of a Borel measurable function with an invertible linear mapping. For instance, $(x, y, z) \mapsto f(x - y - z)$ is the composition of $F(u, v, w) = f(u)$ with the mapping $(u, v, w) \mapsto (u - v - w, v, w)$; F is measurable since it is of product form and f is measurable.

For almost every x, $\Phi_x \in L^1$. Indeed, we know by Tonelli's Theorem that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |\Phi_x(y, z)|\,dm(y, z) = \int (\int |\Phi_x(y, z)|\,dm(y))\,dm(z),$$

and that $y \mapsto \Phi_x(y, z)$ is Borel measurable for every z and that the inner integral defines a Borel measurable function of z. We have proved in the course that

$$\begin{aligned} \int (\int |\Phi_x(y, z)|\,dm(y))\,dm(z) &= \int (\int |f(x - z - y)| \cdot |g(y)|\,dm(y))|h(z)|\,dm(z) \\ &= \int (|f| * |g|)(x - z)|h(z)|\,dm(z) \end{aligned}$$

is finite for almost every x. Therefore for almost every $x \in \mathbb{R}^d$, $\Phi_x \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$.

Therefore by Fubini's Theorem, for any such x,

$$\int (\int f(x - z - y) \cdot g(y)\,dm(y))h(z)\,dm(z) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x - y - z)g(y)h(z)\,dm(y, z),$$

that is,

$$\big((f * g) * h\big)(x) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \Phi_x(y, z)\, dm(y, z).$$

Now for any $x \in \mathbb{R}^d$, $\Phi_x(y, z) = \Psi_x(u, v)$ where $(u, v) = (y + z, z)$. Because the mapping $(y, z) \mapsto (y + z, z)$ is an invertible linear transformation of $\mathbb{R}^d \times \mathbb{R}^d$ with Jacobian determinant $\equiv 1$,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \Phi_x(y, z)\, dm(y, z) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \Psi_x(u, v)\, dm(u, v).$$

For any x, $\Phi_x \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ if and only if $\Psi_x \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$; so $\Psi_x \in L^1$ for almost every x.

For any x for which $\Psi_x \in L^1$, a direct application of Fubini's Theorem gives

$$\begin{aligned}\iint_{\mathbb{R}^d \times \mathbb{R}^d} \Psi_x(u, v)\, dm(u, v) &= \int \big(\int \Psi_x(u, v)\, dm(v)\big)\, dm(u) \\ &= \int f(x - u)\big(\int g(u - v)h(v)\, dm(v)\big)\, dm(u) \\ &= \int f(x - u)(g * h)(u)\, dm(u) \\ &= \big(f * (g * h)\big)(x).\end{aligned}$$

We conclude that $\big((f * g) * h\big)(x) = \big(f * (g * h)\big)(x)$ for almost every $x \in \mathbb{R}^d$. □

(3) Let X be a normed vector space and let V be a subspace of X. Show that if V is norm-closed then V is weakly closed.

Solution. Suppose that V is a norm-closed subspace of X. Suppose that $x \notin V$. By a corollary to the Hahn-Banach Theorem, there exists $f \in X^*$ such that $f|_V \equiv 0$, and $f(x) = 1$. The set $\mathcal{O} = \{y \in X : |f(y) - 1| < 1\}$ is an open subset of X in the weak topology, which contains x but contains no element of V.

Thus we have shown that $X \setminus V$ is open in the weak topology; so V is weakly closed. □

(4) Let X be a locally compact Hausdorff space and $\mathcal{B}$ the σ–algebra of Borel subsets of X. Let μ be a (positive) Radon measure on $\mathcal{B}$. Prove that for any $p \in [1, \infty)$, $C_c(X)$ is dense in $L^p(X, \mathcal{B}, \mu)$.

Solution. Consider any Borel set $E \subset X$ satisfying $0 \le \mu(E) < \infty$. Let $\varepsilon > 0$. By inner and outer regularity of Radon measures (on sets of finite measures), there exist $K \subset E \subset \mathcal{O}$ a compact set and an open set, respectively, such that

$$\mu(E) - \varepsilon < \mu(K) \le \mu(E) \le \mu(\mathcal{O}) < \mu(E) + \varepsilon.$$

By Urysohn's Lemma there exists $\varphi \in C_c(X)$ such that $\varphi \equiv 1$ on K, φ is supported in $\mathcal{O}$, and $0 \le \varphi(x) \le 1$ for all $x \in X$. Therefore

$$|\mathbf{1}_E - \varphi| \le \mathbf{1}_{\mathcal{O}\setminus K}.$$

Therefore

$$\|\mathbf{1}_E - \varphi\|_p \le \mu(\mathcal{O} \setminus K)^{1/p} \le (2\varepsilon)^{1/p}.$$

So we have shown that $\mathbf{1}_E$ can be approximated arbitrarily closely in L^p norm by functions in $C_c(X)$.

Now consider any $f \in L^p$. Let $\varepsilon > 0$. There exists a simple function $g = \sum_{n=1}^N c_n \mathbf{1}_{E_n}$ such that $\|f - g\|_p < \varepsilon$, where each set E_n is Borel and $\mu(E_n) < \infty$. Choose $\varphi_n \in C_c(X)$ such that $\|\varphi_n - \mathbf{1}_{E_n}\|_p \le N^{-1}(1 + \max_n |c_n|)^{-1}\varepsilon$. Then $\psi = \sum_n c_n\varphi_n \in C_c(X)$, and ψ satisfies

$$\|\psi - g\|_p \le \sum_n |c_n| \|\varphi_n - \mathbf{1}_{E_n}\|_p \le \varepsilon,$$

using Minkowski's inequality (the triangle inequality for L^p norms). In all, $\|f - \psi\|_p \le 2\varepsilon$, again by Minkowski's inequality. □

(5) Let (K_n) be a sequence of functions in $L^1(\mathbb{R}^d)$ such that $\|K_n\|_{L^1}$ is uniformly bounded, $\int_{\mathbb{R}^d} K_n \, dm = 1$, and for any $\delta > 0$, $\int_{|x|\ge\delta} |K_n(x)| \, dm(x) \to 0$ as $n \to \infty$. Show that if $f : \mathbb{R}^d \to \mathbb{C}$ is bounded and uniformly continuous, then $f * K_n \to f$ uniformly on $\mathbb{R}^d$.

Solution. Let $A = \sup_n \|K_n\|_{L^1} < \infty$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that $|f(x) - f(x')| \le \varepsilon$ whenever $|x - x'| \le \delta$. I'll write $\|f\|_\infty$ as shorthand for $\sup_{z\in\mathbb{R}^d} |f(x)|$; this supremum is indeed equal to the L^∞ norm anyway, for any bounded continuous function.

For any n,

$$\begin{aligned} f * K_n(x) - f(x) &= \int f(x-y)K_n(y)\, dm(y) \;-\; f(x) \\ &= \int \big(f(x-y) - f(x)\big)K_n(y)\, dm(y) \end{aligned}$$

since $\int K_n = 1$, and consequently

$$\begin{aligned}\big|f * K_n(x) - f(x)\big| &\le \int \big|f(x-y) - f(x)\big| \cdot |K_n(y)|\,dm(y)\\ &= \int_{|y|\le\delta} \big|f(x-y) - f(x)\big| \cdot |K_n(y)|\,dm(y)\\ &\qquad + \int_{|y|>\delta} \big|f(x-y) - f(x)\big| \cdot |K_n(y)|\,dm(y)\\ &\le \int_{|y|\le\delta} \varepsilon |K_n(y)|\,dm(y) + 2\|f\|_\infty \int_{|y|>\delta} |K_n(y)|\,dm(y)\\ &\le \varepsilon A + 2\|f\|_\infty \int_{|y|>\delta} |K_n(y)|\,dm(y).\end{aligned}$$

Choose $N < \infty$ so that $\int_{|y|>\delta} |K_n(y)|\,dm(y) < (1 + \|f\|_\infty)^{-1}\varepsilon$ for all $n \ge N$. Then we have shown that

$$\big|(f * K_n)(x) - f(x)\big| < A\varepsilon + 2\varepsilon \quad \text{for all } n \ge N.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

(6) Let $(X, \mathcal{A}, \mu)$ be σ–finite. Let $p \in [1, \infty)$ and let $q = p' = p/(p-1)$ be the exponent conjugate to p; $q = \infty$ if $p = 1$. Let $T : L^p \to L^p$ be a linear mapping. Suppose that whenever $f_n, f \in L^p$ and $f_n \to f$ in L^p norm, $\int T(f_n)\overline{g}\,d\mu \to \int T(f)\overline{g}\,d\mu$ for all $g \in L^q$. Prove that T is bounded.

Solution. Consider the graph $G_T = \{(f, T(f)) : f \in L^p\} \subset L^p \times L^p$. To show that G_T is closed, consider any sequence $(f_n) \in L^p$ such that $f_n \to f \in L^p$ and $T(f_n) \to F \in L^p$ as $n \to \infty$, with convergence in L^p norm for both sequences. In order to apply the CGT, we must show that $F = T(f)$.

If $F \ne T(f)$, then there exists a bounded linear functional $\ell \in (L^p)^*$ such that $\ell(F) \ne \ell(T(f))$. The dual of L^p is canonically isomorphic to L^q; there exists $g \in L^q$ such that $\ell(h) = \int h\,g\,d\mu$ for all $h \in L^p$. Thus $\int F\,g\,d\mu \ne \int T(f)\,g\,d\mu$.

Since $T(f_n) \to F$ in L^p norm,

$$\int T(f_n)\,g\,d\mu = \ell(T(f_n)) \to \ell(F) = \int F\,g\,d\mu \quad \text{as } n \to \infty$$

because ℓ is continuous. On the other hand, it is a hypothesis of this problem that

$$\int T(f)\,g\,d\mu = \lim_{n\to\infty} \int T(f_n)\,d\mu.$$

Thus the assumption $F \ne T(f)$ has led to a contradiction.

L^p is a Banach space. Therefore T is a linear mapping from one Banach space to another Banach space, which has closed graph. Therefore by the Closed Graph Theorem, T is bounded. □

(Incidentally, one does not need the full strength of the L^p–L^q duality here. One only needs to know Hölder's inequality, and the fact that if $0 \neq h \in L^p$ then there exists $g \in L^q$ such that $\int hg \, d\mu \neq 0$. To construct such a function g is easy, even if X is not σ–finite. Just let g be the product of $\overline{h}$ with the indicator function of $\{x : N^{-1} \leq |h(x)| \leq N\}$, for sufficiently large $N \in \mathbb{N}$.)

(7) Let X be a Banach space of countable dimension. (That is: There exists a countable subset $\{x_n\}$ of X such that every element of X can be represented as a finite linear combination of these elements, and if $\sum_n c_n x_n = 0$ and $c_n = 0$ for all but finitely many n, then $c_n = 0$ for all n.) Show that X has finite dimension.

Solution. (This is perhaps a bit tricky, but it was on one of the problem sets.) If $\{x_n\}$ is finite there is nothing to prove, so suppose the contrary. I'll prove that X is spanned by a finite subset of $\{x_n\}$, which is a contradiction.

Enumerate the given basis using $\mathbb{N}$, so our basis is $\{x_n : n \in \mathbb{N}\}$. Let W_n be the collection of all finite linear combinations of these basis elements; W_n is the collection of all quantities $\sum_{n=1}^N c_n x_n$, where N ranges over all of $\mathbb{N}$.

W_n is a subspace of X. Since it is finite-dimensional, W_n is closed. Since $\cup_n W_n = X$, and since X is a complete metric space, by the Baire Category Theorem there must exist N such that the closure of W_N contains some open ball of positive radius.

Since W_N is closed, it contains a ball of positive radius. Therefore for any $y \in W_N$, $W_N - y$ contains an open ball centered at 0. Since W_N is a subspace, it contains all of X. □

(There was a silly and unintentional slip in the formulation of the problem on the exam, but it was actually correct: If X has countably infinite dimension then a contradiction is reached. Therefore any assertion can be proved; in particular, X has finite dimension.)

(8) Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Consider the measure space $(\mathbb{T}, \mathcal{A}, m)$ where $\mathcal{A}$ denotes the Lebesgue measurable sets and m is Lebesgue measure. Let $\alpha \in \mathbb{R}$ be irrational. Define $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $Tf(x) = f(x + \alpha) - f(x)$. Here addition is interpreted modulo $\mathbb{Z}$, so $x + \alpha \in \mathbb{T}$. Consider any $g \in L^2(\mathbb{T})$. Give a necessary and sufficient condition on g for there to exist a solution $f \in L^2(\mathbb{T})$ of the equation $Tf = g$.

Solution. $Tf = \tau_{-\alpha} f - f$. We know that for all $n \in \mathbb{Z}$ and all $h \in L^1(\mathbb{T})$,

$$\widehat{\tau_y h}(n) = \int h(x-y) e^{-2\pi i x n} \, dm(x) = e^{-2\pi i y n} \widehat{h}(n).$$

Recall that since $\mathbb{T}$ has finite measure, $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ by Hölder's inequality.

Thus

$$\widehat{Tf}(n) = \big(e^{2\pi i\alpha n} - 1\big)\widehat{f}(n) \;\forall\, n \in \mathbb{Z}.$$

Therefore if $g \in L^2$ and $Tf = g$, then

$$\big(e^{2\pi i\alpha n} - 1\big)\widehat{f}(n) = \widehat{g}(n) \;\forall\, n \in \mathbb{Z}.$$

The factor $\big(e^{2\pi i\alpha n} - 1\big)$ vanishes for *no* integer n, because α is irrational.

If $f \in L^2$ then $\widehat{f} \in L^2$. Therefore given $g \in L^2$, a necessary condition for there to exist a solution $f \in L^2$ is that

$$\sum_{n\in\mathbb{Z}} \big|e^{2\pi i\alpha n} - 1\big|^{-2} |\widehat{g}(n)|^2 < \infty. \tag{1}$$

Convesely, if $g \in L^2$ satisfies (1), then there exists $f \in L^2$ satisfying $Tf = g$. Indeed, the mapping $h \mapsto \widehat{h}$ is a bijection from $L^2(\mathbb{T})$ to $\ell^2(\mathbb{Z})$. Therefore if g satisfies (1) then there exists (a unique equivalence class of) $f \in L^2$ such that $\widehat{f}(n) = \big(e^{2\pi i\alpha n} - 1\big)^{-1}\widehat{g}(n)$ for every $n \in \mathbb{Z}$. Then $\widehat{Tf}(n) = \widehat{g}(n)$ for every n. Since the mapping $h \mapsto \widehat{h}$ from L^2 to ℓ^2 is injective, $Tf = g$. □