Math 1B — UCB, Spring 2013 — M. Christ

Summary of Lecture 2, 1/25/2013

Trigonometric Integrals

This lecture is based primarily on §7.2 of our text.

0. Basics. Be sure you know the basic formulas:

$$\sin' = \cos \qquad \cos' = -\sin \qquad \tan' = \sec^2 \qquad \sec' = \tan\sec$$

You should also be familiar with the use of simple substitutions like $t = 3x$, for example,

$$\int \cos(3x)\,dx = \int \cos(t)\tfrac{1}{3}\,dt = \tfrac{1}{3}\sin(t) + C = \tfrac{1}{3}\sin(3x) + C.$$

Another example:

$$\int \cos(2x)\,dx = \text{ (substituting } 2x = u \text{ with } du = 2dx\text{) } \int \cos(u)\cdot\tfrac{1}{2}\,du = \tfrac{1}{2}\sin(u)+C = \tfrac{1}{2}\sin(2x)+C.$$

I will sometimes do these calculations in a single step, without showing all of the intermediate steps.

1. Half-angle formulas. How to integrate higher powers of $\sin, \cos$? The first thing(s) to know are the two half-angle formulas:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}, \quad \text{and} \quad \sin^2(x) = \frac{1-\cos(2x)}{2}\ .$$

These can simplify integrals by reducing second powers of $\sin, \cos$ to first powers.

Application: Using a half-angle formula to evaluate $\int \cos^2(x)\,dx$:

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \tfrac{1}{2}(1+\cos(2x))\,dx \\ &= \tfrac{1}{2}\int (1+\cos(2x))\,dx \\ &= \tfrac{1}{2}\int 1\,dx + \tfrac{1}{2}\int \cos(2x)\,dx \\ &= \tfrac{1}{2}x + \tfrac{1}{2}\cdot\tfrac{1}{2}\sin(2x) + C \\ &= \tfrac{1}{2}x + \tfrac{1}{4}\sin(2x) + C.\end{aligned}$$

We can check our work:

$$\frac{d}{dx}\Big(\tfrac{1}{2}x + \tfrac{1}{4}\sin(2x)\Big) = \tfrac{1}{2}\cdot 1 + \tfrac{1}{4}\cdot\cos(2x)\cdot 2 = \tfrac{1}{2}(1+\cos(2x)) = \cos^2(x).$$

2. The integral of $\boxed{\sec(x)}$: a **diabolical trick**. How to evaluate $\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx$? Integration by parts (e.g. insert a factor of 1, let $1 = g'$ and $\sec(x) = f(x)$) leads to $\int x\sec(x)\tan(x)\,dx$ — which doesn't look like progress.

What about substitution? One natural thought is to get rid of the inverse trig function by substituting $x = \arccos(y)$. This leads to $-\int y^{-1}\sqrt{1-y^2}\,dy$, which is not at all encouraging. Indeed, as we will learn in a few days, the usual method for calculating integrals involving $\sqrt{1-y^2}$ is to substitute $y = \sin(\theta)$, which leads back to $\int \csc(x)\,dx$ — which is just as tricky as $\int \sec(x)\,dx$.

Luckily for us, long ago a very clever person found a solution: Multiply by 1 in the form

$$1 = \frac{\tan(x) + \sec(x)}{\tan(x) + \sec(x)}$$

to obtain

$$\int \sec(x)\,dx = \int \sec(x)\,\frac{\tan(x) + \sec(x)}{\tan(x) + \sec(x)}\,dx = \int \frac{\tan(x)\sec(x) + \sec^2(x)}{\tan(x) + \sec(x)}\,dx.$$

Substitute $u =$ the denominator, that is,

$$u = \tan(x) + \sec(x).$$

Then

$$du = \big(\sec^2(x) + \tan(x)\sec(x)\big)\,dx$$

and there's the numerator! (A miracle?)

The integral becomes

$$\int \frac{du}{u} = \ln(|u|) + C = \ln\big|\sec(x) + \tan(x)\big| + C.$$

Note the presence of the **absolute value signs** inside the natural logarithm. Nothing in the problem prevents $\sec(x) + \tan(x)$ from being negative (it is negative when $\frac{\pi}{2} < x < \pi$, for instance), so the absolute value signs are needed.

If you feel a bit frustrated at seeing this problem solved by such a trick, good! In science we always seek explanations, and while there's no arguing with the *correctness* of the reasoning above, the trick doesn't seem to *explain* anything. There is a general method, the **Weierstrass substitution**, which handles an enormous number of integrals involving trig functions in a systematic way, including $\int \sec$. See Exercise 59 of §7.4.[1]

Comment: In the text and worksheets are various reduction formulas. I want you to understand the general method behind their derivation, not to memorize them. You're unlikely to succeed in memorizing them all for an exam, so understanding the method is the way to go.

[1] That exercise doesn't explain how Professor Weierstrass invented his substitution, but there's an explanation for that, too ...

Comment: Formula sheets will be provided on exams, so that you can concentrate on learning methods rather than on memorizing formulas. You'll be told in advance which formulas will be provided.

3. $\int \sec^3$**.** (Recall that $\sec^3(x)$ means the same thing as $\sec(x)^3 = \big(\sec(x)\big)^3$.)

Now we've learned how to integrate $\cos^2(x)$ and $\frac{1}{\cos(x)} = \cos(x)^{-1}$. What about other powers? Bad news: $\int \sec^3(x)\,dx = \int \cos(x)^{-3}\,dx$ is still challenging.

The idea is to **reduce it to the first power of** $\sec$ by replacing $\sec^2$ by $1 + \tan^2$.

$$\int \sec^3(x)\,dx = \int \sec(x)(1 + \tan^2(x))\,dx = \int \sec(x)\,dx + \int \tan^2(x)\sec(x)\,dx.$$

The first integral was evaluated above. In the second, write the integrand as $\tan(x)\cdot\tan(x)\sec(x)$ and integrate by parts, with $f(x) = \tan(x)$ and $g(x) = \sec(x)$, so that $g'(x) = \tan(x)\sec(x)$ and $f'(x) = \sec^2(x)$. We get

$$\int \tan(x)\cdot\tan(x)\sec(x)\,dx = \tan(x)\sec(x) - \int \sec^2(x)\sec(x)\,dx.$$

Putting everything together:

$$\int \sec^3(x)\,dx = \int \sec(x)\,dx + \tan(x)\sec(x) - \int \sec^3(x)\,dx.$$

This is the **"circular reasoning"** situation which we encountered in Wednesday's lecture: Adding $\int \sec^3(x)\,dx$ to both sides of this equation and dividing by 2 gives

$$\begin{aligned}\int \sec^3(x)\,dx &= \tfrac{1}{2}\int \sec(x)\,dx + \tfrac{1}{2}\tan(x)\sec(x)\\ &= \tfrac{1}{2}\ln|\sec(x) + \tan(x)| + \tfrac{1}{2}\tan(x)\sec(x) + C.\end{aligned}$$

4. Even powers of $\sec$. Even powers of sec are easier to integrate than odd powers. The method is based on the identities

$$\tan^2 + 1 = \sec^2 \quad \text{and} \quad \tan' = \sec^2.$$

As a representative example, consider $\int \sec^6(x)\,dx$. The key is to **keep one factor of** $\sec^2$, and to express all of the remaining powers in terms of $\tan^2$:

$$\sec^6(x) = \sec^4(x)\sec^2(x) = (1 + \tau^2(x))^2\sec^2(x).$$

Then

$$\int \sec^6(x)\,dx = \int (\tan^2(x) + 1)^2\sec^2(x)\,dx.$$

Substitute $u = \tan(x)$. Then $du = \sec^2(x)\,dx$. (That's why we kept one factor of $\sec^2$, rather than rewriting it in terms of $\tan^2$.) We find

$$\int \sec^6(x)\,dx = \int (u^2+1)^2\,du.$$

Before finishing the problem, let's look at this and agree that we have definitely made progress: All trig functions have disappeared, and all we need to do is to integrate the polynomial $(u^2+1)^2 = u^4 + 2u^2 + 1$. So

$$\begin{aligned}\int \sec^6(x)\,dx &= \int (u^4 + 2u^2 + 1)\,du \\ &= \tfrac{1}{5}u^5 + \tfrac{2}{3}u^3 + u + C \\ &= \tfrac{1}{5}\tan(x)^5 + \tfrac{2}{3}\tan(x)^3 + \tan(x) + C.\end{aligned}$$

Some people prefer to write the answer as

$$\tfrac{1}{5}\tan^5(x) + \tfrac{2}{3}\tan^3(x) + \tan(x) + C;$$

same thing.

5 **Reduction formulas.** This is a slightly more advanced topic, which I don't expect to get to in Lecture 1.

Example: For any positive whole number n,

$$\int \sin(x)^n\,dx = -\tfrac{1}{n}\cos(x)\sin(x)^{n-1} + \tfrac{n-1}{n}\int \sin(x)^{n-2}\,dx.$$

"Reduction formula" is a general term for a formula which expresses a relatively complicated integral in terms of simpler, but possibly still quite complicated, ones, which can be further simplified by repeated use of the same reduction formula. [Such a process is also called recursion.]

This formula doesn't tell us how to integrate a power of sin right out, but it does *reduce* the problem to one of the same type, with a lower power. (See also our text, §7.1 Example 6.)

Here is a derivation of this reduction formula, using IBP: Let $f(x) = \sin(x)^{n-1}$ and $g'(x) = \sin(x)$. Then $g(x) = -\cos(x)$ and $f'(x) = (n-1)\sin(x)^{n-2}\cos(x)$, by the chain rule. Therefore

$$\begin{aligned}\int \sin(x)^n\,dx &= -\cos(x)\sin(x)^{n-1} - (n-1)\int \sin(x)^{n-2}\cos(x)\cdot(-\cos(x))\,dx \\ &= -\cos(x)\sin(x)^{n-1} + (n-1)\int \sin(x)^{n-2}\cos(x)^2\,dx.\end{aligned}$$

Now convert the factors of cos to factors of sin by using $\cos^2 = 1 - \sin^2$ to get

$$\begin{aligned}\int \sin(x)^n\,dx &= -\cos(x)\sin(x)^{n-1} + (n-1)\int \sin(x)^{n-2}\,dx - (n-1)\int \sin(x)^{n-2}\sin(x)^2\,dx \\ &= -\cos(x)\sin(x)^{n-1} + (n-1)\int \sin(x)^{n-2}\,dx - (n-1)\int \sin(x)^n\,dx.\end{aligned}$$

The integral which we started out with has reappeared, now multiplied by $-(n-1)$. By adding this quantity to both sides, we find that

$$n \int \sin(x)^n \, dx = -\cos(x)\sin(x)^{n-1} + (n-1)\int \sin(x)^{n-2}\, dx.$$

Dividing by both sides n gives the formula stated. (There's no need to write "$+C$" in the formula, since there's an implicit arbitrary constant in the integral on the right-hand side.)

Example: $\int \sec^n(x)\, dx$, where n is a positive whole number. We already know the cases $n = 1, 2, 3$. For larger n, use IBP with $f = \sec^{n-2}$ and $g' = \sec^2$, so that $f' = (n-2)\sec^{n-2}\tan$ and $g = \tan$. Thus

$$\begin{aligned}\int \sec^n(x)\, dx &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\, dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\big(\sec^2(x) - 1\big)\, dx \\ &= \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\, dx - (n-2)\int \sec^n(x)\, dx.\end{aligned}$$

The usual trick of adding $(n-2)\int \sec^n(x)\, dx$ to both sides and dividing through by $(n-1)$ gives

$$\int \sec^n(x)\, dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\, dx.$$

(This formula applies for whole numbers $n \geq 2$.)

With application of plenty of elbow grease, an integral like $\int \sec^8(x)\, dx$ or $\int \sec^9(x)\, dx$ can be reduced by this method to either $\int \sec^2(x)\, dx$ or $\int \sec(x)\, dx$, both of which we already know how to evaluate.

6. Integrating $\tan^m \sec^n$ This is easy if the power m of tan is *odd*. Write

$$\tan^m = \tan\cdot(\tan^2)^{(m-1)/2} = \tan\cdot(\sec^2 - 1)^{(m-1)/2}.$$

Expand the power of $(\sec^2 - 1)$ to get a sum of integrals, each of the simpler form

$$\int \sec^k(x)\cdot\sec(x)\tan(x)\, dx$$

for some nonnegative integer k. Now use the substitution $u = \sec(x)$, $du = \sec(x)\tan(x)\, dx$ to reduce matters to

$$\int u^k\, du = (k+1)^{-1}u^{k+1} + C.$$

A similar method applies if the power n of sec is even. See our text, §7.2.

7. Products of trig functions.

(I will not have time for this topic in class; please read our text.)

The identity

$$\sin(A)\cos(B) = \tfrac{1}{2}\big(\sin(A-B) + \sin(A+B)\big)$$

and its cousins — see end of §7.2 of Stewart. This identity is easy to justify (but harder to remember exactly): Expand $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$, expand $\sin(A+B)$ similarly, cancel everything that can be cancelled, and divide by 2.

Example: $\int \sin(3x)\cos(5x)\,dx$.

With the formula, such an integral is very easy:

$$\int \sin(3x)\cos(5x)\,dx = \tfrac{1}{2}\int \big(\sin(-2x) + \sin(8x)\big)\,dx$$

and this is now a routine integral which equals $\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C$.