Degree Spectra of Relations on a Cone

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Setting: $\mathcal{A}$ a computable structure, and $R \subseteq A^n$ an additional relation on $\mathcal{A}$ not in the signature of $\mathcal{A}$.

Suppose that $\mathcal{A}$ is a very “nice” structure.

OR

Consider behaviour on a cone.

Which sets of degrees can be the degree spectrum of such a relation?
All of our languages and structures will be countable.

**Definition**

A structure is computable if its atomic diagram is computable.

**Definition**

Let $\mathcal{A}$ be a structure and $R$ a relation on $\mathcal{A}$. $R$ is *invariant* if it is fixed by automorphisms of $\mathcal{A}$.

If $\mathcal{B} \cong \mathcal{A}$, we obtain a relation $R^B$ on $\mathcal{B}$ using the invariance of $R$. 
Let $\mathcal{A}$ be a computable structure and $R$ a relation on $\mathcal{A}$.

**Definition (Harizanov)**

The *degree spectrum* of $R$ is

$$\text{dgSp}(R) = \{ d(R^B) : B \text{ is a computable copy of } \mathcal{A} \}$$

Pathological examples:

- (Hirschfeldt) the degrees below a given c.e. degree.
- (Harizanov) $\{0, d\}$, $d$ is $\Delta^0_2$ but not a c.e. degree.
- (Hirschfeldt) $\{0, d\}$, $d$ is a c.e. degree.
Let $\mathcal{A}$ be a computable structure and $R$ a relation on $\mathcal{A}$.

**Theorem (Harizanov)**

Suppose that $R$ is computable. Suppose moreover that the property $(\ast)$ holds of $\mathcal{A}$ and $R$. Then

$$\text{dgSp}(R) \neq \{0\} \Rightarrow \text{dgSp}(R) \supseteq \text{c.e.}$$

$(\ast)$ For every $\bar{a}$, we can computably find $a \in R$ such that for all $\bar{b}$ and quantifier-free formulas $\theta(\bar{z}, x, \bar{y})$ such that $\mathcal{A} \models \theta(\bar{a}, a, \bar{b})$, there are $a' \notin R$ and $\bar{b}'$ such that $\mathcal{A} \models \theta(\bar{a}, a', \bar{b}')$.

On a cone, the effectiveness condition holds.
Let $\mathcal{A}$ be a computable structure and $R$ a relation on $\mathcal{A}$.

**Definition**

The *degree spectrum of $R$ below the degree $d$* is

$$\text{dgSp}(\mathcal{A}, R)_{\leq d} = \{ d(R^B) \oplus d : B \cong \mathcal{A} \text{ and } B \leq_T d \}$$

**Corollary (Harizanov)**

One of the following is true for all degrees $d$ on a cone:

1. $\text{dgSp}(\mathcal{A}, R)_{\leq d} = \{ d \}$, or
2. $\text{dgSp}(\mathcal{A}, R)_{\leq d} \supseteq$ degrees c.e. in and above $d$. 

Let $A$ and $B$ be structures and $R$ and $S$ relations on $A$ and $B$ respectively.

For any degree $d$, either $\text{dgSp}(A, R)_{\leq d}$ is equal to $\text{dgSp}(B, S)_{\leq d}$, one is strictly contained in the other, or they are incomparable. By Borel determinacy, exactly one of these happens on a cone.

**Definition (Montalbán)**

The degree spectrum of $(A, R)$ on a cone is equal to that of $(B, S)$ if we have equality on a cone, and similarly for containment and incomparability.
Two classes of degrees

**Definition**

A set $A$ is d.c.e. if it is of the form $B - C$ for some c.e. sets $B$ and $C$.

A set is $n$-c.e. if it has a computable approximation which is allowed $n$ alternations.

We omit the definition of $\alpha$-c.e.

**Definition**

A set $A$ is CEA in $B$ if $A$ is c.e. in $B$ and $A \geq_T B$.

$A$ is $n$-CEA if there are sets $A_1, A_2, \ldots, A_n = A$ such that $A_1$ is c.e., $A_2$ is CEA in $A_1$, and so on.

We omit the definition of $\alpha$-CEA.
Let $\Gamma$ be a natural class of degrees which relativises. For example, $\Gamma$ might be the $\Delta^0_\alpha$, $\Sigma^0_\alpha$, or $\Pi^0_\alpha$ degrees. We will also be interested in the $\alpha$-c.e. and $\alpha$-CEA degrees we just defined.

For any of these classes $\Gamma$ of degrees, there is a structure $\mathcal{A}$ and a relation $R$ such that, for each degree $d$,

$$dgSp_{\leq d}(\mathcal{A}, R) = \Gamma(d) \oplus d.$$ 

So we may talk, for example, about a degree spectrum being equal to the $\Sigma_\alpha$ degrees on a cone.
Harizanov’s result earlier showed that degree spectra on a cone behave nicely with respect to c.e. degrees.

Corollary (Harizanov)

*Any degree spectrum on a cone is either equal to $\Delta^0_1$ or contains $\Sigma^0_1$.*

Question

What are the possible degree spectra on a cone?
Theorem (H.)

There is a computable structure $\mathcal{A}$ and relatively intrinsically d.c.e. relations $R$ and $S$ on $\mathcal{A}$ with the following property:

for any degree $d$, $\text{dgSp}(\mathcal{A}, R)_{\leq d}$ and $\text{dgSp}(\mathcal{B}, S)_{\leq d}$ are incomparable.

Corollary (H.)

There are two degree spectra on a cone which are incomparable, each contained within the d.c.e. degrees and containing the c.e. degrees.
A question of Ash and Knight

Question (Ash-Knight)

Can one show (assuming some effectiveness condition) that any relation which is not intrinsically $\Delta^0_\alpha$ realises every $\alpha$-CEA degree?

Stated in terms of degree spectra on a cone, is it true that every degree spectrum on a cone is either contained in $\Delta^0_\alpha$, or contains $\alpha$-CEA?
Ash and Knight gave a result which goes towards answering this question.

**Theorem (Ash-Knight)**

Let $A$ be a computable structure with an additional computable relation $R$. Suppose that $R$ is not relatively intrinsically $\Delta^0_\alpha$.

Moreover, suppose that $A$ is $\alpha$-friendly and that for all $\bar{c}$, we can find $a \notin R$ which is $\alpha$-free over $\bar{c}$.

Then for any $\Sigma^0_\alpha$ set $C$, there is a computable copy $B$ of $A$ such that

$$R^B \oplus \Delta^0_\alpha \equiv_T C \oplus \Delta^0_\alpha$$

where $\Delta^0_\alpha$ is a $\Delta^0_\alpha$-complete set.
**Theorem (H.)**

Let $\mathcal{A}$ be a structure and $R$ a relation on $\mathcal{A}$. Then one of the following is true relative to all degrees on a cone:

1. $\text{dgSp}(\mathcal{A}, R) \subseteq \Delta^0_2$, or
2. $2\text{-CEA} \subseteq \text{dgSp}(\mathcal{A}, R)$. 

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Degree Spectra of Relations on a Cone
Unresolved questions

**Question**

What about $\alpha > 2$?

**Question**

Are there more than two degree spectra on a cone which are contained within the d.c.e. degrees but strictly contain the c.e. degrees?

**Question**

Are degree spectra on a cone closed under join?
Thanks!